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A Note on Partial Isometries on Hilbert Space

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Abstract The semi-equivalence for partial isometries is introduced, and it is proved that for any infinite-dimensional Hilbert space H, there exist two partial isometries which are semi-equivalent, but not unitarily equivalent.

Key words Hilbert space; partial isometry; semi-equivalence; unitary equivalence

Let H be a Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. For $v \in B(H)$, v is said to be a partial isometry if $\| vh \| = \| h \|$ for any $h \in (Kerv)^{\perp}$, or equivalently, v^*v is a projection. For any partial isometry v, we write

$$\rho(v) = \dim(ranv),\ \rho'(v) = \dim(ranv)^{\perp}, \text{ and } v(v) = \dim Kerv.$$

These three cardinal numbers are usually called the rank, the co-rank, and the nullity of v respectively. Recall that if v is a partial isometry, then so is v^*. Since the initial space of v^* is the final space of v, and vice versa. It follows that

$$v(v^*) = \rho'(v) \text{ and } \rho'(v^*) = v(v).$$

Definition Let v_1, v_2 be two partial isometries. v_1 is said to be semi-equivalent to v_2, denoted by $v_1 \sim v_2$, if there exist two unitaries μ and w such that $\mu v_1^n w = v_2^n$ for all $n \in N$.

Lemma 1 $v_1 \sim v_2$ if and only if the following hold:

(1) $\rho(v_1) = \rho(v_2)$, $\rho'(v_1) = \rho'(v_2)$ and $v(v_1) = v(v_2)$.

(2) There exists a unitary $\mu = \mu_1 \oplus \mu_2$ with μ_1 maps isometrically from $Kerv_1$ onto $Kerv_2$, and μ_2 from $(Kerv_1)^{\perp}$ onto $(Kerv_2)^{\perp}$, such that $v_2 \mu v_1 = v_2^2 \mu$.

Proof "$\Leftarrow$". Suppose the conditions (1) and (2) hold. Let w be a linear transformation that maps $(ranv_1)^{\perp}$ isometrically onto $(ranv_2)^{\perp}$. For any $f \in ranv_1$, we define $wf = v_2 \mu v_1^* f$, then as shown in [1, Solution 131], w is a unitary and $wv_1 = v_2 \mu$. Since $v_2 \mu v_1 = v_2^2 \mu$, it is easy to show by induction that

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$$wv_1^n = v_2^n\mu \text{ for all } n \in N,$$

so
$$v_1 \sim v_2.$$

"⇒". If there exist two unitaries μ and w such that

$$wv_1^n = v_2^n\mu \text{ for all } n \in N,$$

in particular $wv_1\mu^* = v_2$. Therefore,

$$\dim Kerv_1 = \dim Kerv_2,\ \dim Kerv_1^* = \dim Kerv_2^*,\ \dim(Kerv_1)^\perp = \dim(Kerv_2)^\perp$$

$$\text{i. e. } v(v_1) = v(v_2),\ \rho'(v_1) = \rho'(v_2) \text{ and } \rho(v_1) = \rho(v_2),$$

so the condition (1) holds. Since for any $f \in H$, $f \in (Kerv_1)^\perp$ if and only if $\| v_1 f \| = \| f \|$, it follows that the unitary μ will satisfy the condition (2).

Remark Let v_1 and v_2 be two partial isometries such that

$$\dim Kerv_1 = \dim Kerv_2 \text{ and } \dim(Kerv_1)^\perp = \dim(Kerv_2)^\perp.$$

Let u be any unitary which maps $Kerv_1$ onto $Kerv_2$ and $(Kerv_1)^\perp$ onto $(Kerv_2)^\perp$. Then since $v_1 \sim uv_1u^*$, it follows that $v_1 \sim v_2$ if and only if $uv_1u^* \sim v_2$, so in the following we always assume that $Kerv_1 = Kerv_2$.

Now let $p = v_1^*v_1 = v_2^*v_2$, then with the decomposition $H = pH \oplus (1-p)H$, we know that

$$v_1 = \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} y_1 & 0 \\ y_2 & 0 \end{pmatrix},$$

where $x_1 = pv_1p$, $x_2 = (1-p)v_1p$, $y_1 = pv_2p$ and $y_2 = (1-p)v_2p$.

Since $y_1^*y_1 + y_2^*y_2 = p$, by Lemma 1, we have the following Lemma:

Lemma 2 $v_1 \sim v_2$ if and only if there is a unitary u on pH such that

$$x_1 \sim_u y_1 \text{ and } \dim Kerv_2^* = \dim Kerv_2^*.$$

Lemma 3 $v_1 \sim_u v_2$ if and only if $v_1 \sim v_2$ and $\dim Kerx_2^* = \dim Kery_2^*$. (Here we take x_2^* and y_2^* as bounded linear operators from $(1-p)H$ to pH).

Proof "⇒". Suppose that there exists a unitary w such that $wv_1w^* = v_2$. Set

$$w = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}, \text{ then } \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}\begin{pmatrix} x_1 & 0 \\ x_2 & 0 \end{pmatrix} = \begin{pmatrix} y_1 & 0 \\ y_2 & 0 \end{pmatrix}\begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$$

$$w^*w = ww^* = 1 \text{ and } x_1^*x_1 + x_2^*x_2 = p = y_1^*y_1 + y_2^*y_2.$$

It follows that

$$w_{11}x_1 = y_1w_{11},\ w_{21}x_1 + w_{22}x_2 = y_2w_{11}, \tag{1}$$

$$w_{12} = 0,\ w_{11}^*w_{11} + w_{21}^*w_{21} = p, \tag{2}$$

$$w_{21}^*w_{22} = 0,\ w_{22}^*w_{22} = 1 - p, \tag{3}$$

$$w_{11}w_{11}^* = p,\ w_{11}w_{21}^* = 0, \tag{4}$$

$$w_{21}w_{21}^* + w_{22}w_{22}^* = 1 - p, \tag{5}$$

By (1) and (3), we know that

$$x_2 = w_{22}^*y_2w_{11}, \text{ so } x_2^* = w_{11}^*y_2^*w_{22}.$$

By (3) and (4), we know that $\dim Kerx_2^* \leqslant \dim Kery_2^*$, which implies by symmetry that $\dim Kerx_2^* = \dim Kery_2^*$.

"$\Leftarrow$". If $v_1 \sim v_2$ and $\dim Kerx_2^* = \dim Kery_2^*$, then by Lemma 2, there is a unitary u_1 on pH such that $u_1 x_1 u_1^* = y_1$.

Define $w_1: \overline{ranx_2} \to \overline{rany_2}$, $w_1(x_2 u_1^* \zeta) = y_2 \zeta$ for $\zeta \in pH$. Since

$$x_1^* x_1 + x_2^* x_2 = p = y_1^* y_1 + y_2^* y_2$$

and

$$u_1 x_1 u_1^* = y_1,$$

we know that

$$y_2^* y_2 = u_1(x_2^* x_2) u_1^*,$$

which implies that w_1 maps isometrically from $\overline{ranx_2}$ onto $\overline{rany_2}$, and since $\dim Kerx_2^* = \dim Kery_2^*$, we know that w_1 canbe extended as a unitary on $(1-p)H$. Let $w = u_1 \oplus w_2$. Then it is easy to show that

$$w v_1 w^* = v_2.$$

Theorem If H is an infinite-dimensional Hilbert space, then there exist two partial isometries v_1 and v_2 such that $v_1 \sim v_2$ but $v_1 \not\sim_u v_2$. If however H is finite-dimensional, then $v_1 \sim v_2$ if and only if $v_1 \sim_u v_2$.

Proof (1) H is infinite-dimensional. Without loss of generality, we may assume that H is countably infinite-dimensional. Let $\{e_1, e_2, e_3, \cdots\}$ be a basis for H. Let H_0($resp.$ H_1) be the closed subspace of H generated by $\{e_2, e_4, e_6, \cdots\}$($resp.$ $\{e_4, e_8, e_{12}, \cdots\}$); its projection is denoted by p($resp.$ p_1).

Define $x_2, y_2: pH \to (1-p)H$ as follows:

$$x_2 e_{4k+2} = e_{4k+3}, \quad k = 0, 1, 2, \cdots$$
$$x_2 e_{4k} = 0, \quad k = 1, 2, 3, \cdots$$
$$\text{and } y_2 e_{4k+2} = e_{2k+1}, \quad k = 0, 1, 2, \cdots$$
$$y_2 e_{4k} = 0, \quad k = 1, 2, 3, \cdots$$

Let

$$v_1 = \begin{pmatrix} p_1 & 0 \\ x_2 & 0 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} p_1 & 0 \\ y_2 & 0 \end{pmatrix},$$

then

$$v_1^* v_1 = p = v_2^* v_2.$$

Since the closed subspace generated by $\{e_{4k+2} | k = 0, 1, 2, \cdots\}$ is contained in $Kerv_1^* \cap Kerv_2^*$, we know that $\dim Kerv_1^* = +\infty = \dim Kerv_2^*$. By Lemma 2, we know that $v_1 \sim v_2$, and since

$$\dim Kerx_2^* = +\infty \neq 0 = \dim Kery_2^*,$$

it follows from Lemma 3 that v_1 is not unitarily equivalent to v_2.

(2) In the case when H is finite-dimensional, if $v_1 \sim v_2$, then by Lemma 2 we know that there exists a unitary u on pH such that $u x_1 u^* = y_1$. Since

$$x_1^* x_1 + x_2^* x_2 = p = y_1^* y_1 + y_2^* y_2,$$

it follows that

$$y_2^* y_2 = u(x_2^* x_2)u^*,$$

so $\dim Ker y_2 = \dim Ker x_2$, which implies $\dim Ker y_2^* = \dim Ker x_2^*$. By Lemma 3, we know that v_1 is unitarily equivalent to v_2.

References

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关于Hilbert空间中部分等距算子的一点注记

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提 要 引进Hilbert空间中部分等距算子的半等价关系，证明了在任意的无限维Hilbert空间中，存在两个部分等距算子，它们为半等价，但不为酉等价.

关键词 Hilbert 空间；部分等距；半等价；酉等价