Output-Feedback Control for a Class of Uncertain Nonlinear Systems with Linearly Unmeasured States Dependent Growth

SHANG Fang LIU Yun-Gang

Abstract This paper is devoted to the problem of global stabilization by output-feedback for a class of nonlinear systems with uncertain control coefficients, stable zero-dynamics and linearly unmeasured states dependent growth. By first introducing two kinds of appropriate state transformations, the original system is converted into the new system with deterministic virtual control coefficients and the separated zero-dynamics. Then, a suitable observer based on high-gain K-filters is constructed for the new system, and the backstepping design approach is successfully proposed to the output-feedback controller. It is shown that the global asymptotic stability of the closed-loop system can be guaranteed by the appropriate choice of the design parameters. A simulation example is also provided to show the correctness of the theoretical results and the effectiveness of the proposed approach.

Key words Nonlinear systems, uncertain control coefficient, high-gain K-filters, output-feedback, unmeasured states dependent growth, global asymptotic stability

Because that the system states are incompletely measurable, the problem of output-feedback control is more challenging and difficult than that of state-feedback control[1∼5]. Unlike linear systems, the separation principle is invalid for most nonlinear systems, and hence the observer and controller should be designed and analyzed together. The objective of control design based on observer is to guarantee not only the stability of the closed-loop system but also the convergence of the estimation error. Because the theory of observer develops slowly[6∼9], the output-feedback control based on observer is far from maturity, and there are still many problems remaining unsolved[2, 10]. For example, a foundational question is what the sufficient and necessary conditions are to output-feedback stabilize nonlinear systems. It has been pointed out in the remarkable paper [11] that if the power of the nonlinearity growth with respect to unmeasured states is greater than 2, there are counterexamples for which no output-feedback controls exist.

Recently, the problem of output-feedback control design has received much attention and been intensively investigated for a class of nonlinear systems with unmeasured states dependent growth[3, 12∼18]. For details, in [14], adaptive output-feedback tracking control was investigated for a class of nonlinear systems linearly depending on unmeasured states and in generalized output-feedback canonical form. In [12], exponentially stable output-feedback control was considered for a family of nonlinear systems that are dominated by a triangular system satisfying linear growth condition, and in [15, 19], the more general case was considered where the linear growth rate was an unknown constant. In [18], the results of [12] are generalized to the systems with unknown control coefficients by introducing the novel observer based on high-gain K-filters. Furthermore, in [3, 13, 16], output-feedback control was investigated for systems with output dependent growth rate, and by constructing norm estimators, an extension was obtained to the systems with unmeasured states dependent growth rate in [17].

This paper continues the investigation started in [18] and considers the output-feedback stabilizing control design for a more general class of nonlinear systems with uncertain control coefficients, hidden zero-dynamics and linearly unmeasured states dependent growth, and generalizes the relevant results distributed in [1, 12, 15, 18]. Mainly thanks to the presence of the unmeasured states dependent growth and the uncertain control coefficients, the output-feedback control problem of the systems under consideration is very hard and hence a very meaningful question. In addition, due to the existence of the hidden zero-dynamics, it is difficult to directly carry out the output-feedback control design. To achieve the control objective, two kinds of linear state transformations are firstly introduced. One is to lump the uncertain control coefficients together, and then a new system with deterministic virtual control coefficients is obtained. The other is to separate the zero-dynamics from the system. Then, enlightened by [18], we propose an appropriate high-gain K-filters based on which the state estimation is successfully constructed. Such kind of K-filters indeed play a central role in the output-feedback control design which will be realized by the backstepping approach. Moreover, by choosing the design parameters properly, the global asymptotic stability of the closed-loop system can be guaranteed.

The remainder of this paper is organized as follows. Section 2 describes the system to be considered and formulates the control problem to be solved. Section 3 provides two kinds of linear state transformations through which the original system can be converted into a new system that is convenient for observer design and observer-based output-feedback control design. Section 4 gives the main results of this paper, that is, the high-gain K-filters based observer is constructed and the output-feedback stabilizing control design is given using backstepping method. Section 5 is the necessary complementarity to Section 4, where further study on a second-order system is addressed to the case without zero-dynamics, and accordingly the numerical simulation is given to demonstrate the correctness of the theoretical results. Section 6 presents some concluding remarks. The paper ends with an appendix which provides rigorous proofs of a number of important propositions and lemmas.

Notations

Throughout this paper, I denotes identity matrix

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School of Control Science and Engineering, Shandong University, Jinan, 250061, P. R. China
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of appropriate dimension; for any $\boldsymbol{x} \in \mathbf{R}^n$, x_i denotes its ith element and $\boldsymbol{x}_{[i]}$ denotes $[x_1, \ldots, x_i]^{\mathrm{T}}$; $\mathbf{0}_{[i]} \in \mathbf{R}^i$ denotes the zero vector; $\boldsymbol{e}_{i,j} (i \le j)$ denotes $[\mathbf{0}_{[i-1]}^{\mathrm{T}}, 1, \mathbf{0}_{[j-i]}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbf{R}^j$; $\boldsymbol{b}'_{[i,0]}$ denotes $[b'_i, b'_{i-1}, \ldots, b'_0]^{\mathrm{T}} \in \mathbf{R}^{i+1}$; $\boldsymbol{c}_{i,[j]}$ denotes $[c_{i1}, \ldots, c_{ij}]^{\mathrm{T}} \in \mathbf{R}^j$ and $\boldsymbol{\Phi}_{i,[j,k]}$ denotes $[\Phi_{i,j}, \ldots, \Phi_{i,k}]^{\mathrm{T}} \in \mathbf{R}^{k-j+1}$. We use $\|\cdot\|_1$, $\|\cdot\|$(i.e., $\|\cdot\|_2$) and $\|\cdot\|_F$ to denote 1-norm, Euclidean norm (or 2-norm) and Frobenius norm for vectors respectively, and the corresponding induced norm for matrices. Specifically, for vector $\boldsymbol{x}$, $\|\boldsymbol{x}\| \le \|\boldsymbol{x}\|_1 \le \sqrt{n}\|\boldsymbol{x}\|$ and for matrix A, $\|A\| \le \|A\|_F$. Besides, for simplicity of expression, we sometimes drop the arguments of function when no confusion is caused.

1 System description and problem formulation

Consider the following class of single-input-single-output (SISO) nonlinear systems:

$$\begin{cases} \dot{\zeta}_i &= g_i\zeta_{i+1} + \phi_i(t, \boldsymbol{\zeta}, u), \quad i = 1, \ldots, \rho - 1, \\ \dot{\zeta}_j &= g_j\zeta_{j+1} + b_{n-j}u + \phi_j(t, \boldsymbol{\zeta}, u), \\ & \quad j = \rho, \rho+1, \ldots, n-1, \\ \dot{\zeta}_n &= g_n b_0 u + \phi_n(t, \boldsymbol{\zeta}, u), \\ y &= \zeta_1, \end{cases} \tag{1}$$

where $\boldsymbol{\zeta} = [\zeta_1, \ldots, \zeta_n]^{\mathrm{T}} \in \mathbf{R}^n$ is the system state with the initial condition $\boldsymbol{\zeta}(0) = \boldsymbol{\zeta}_0$; $u \in \mathbf{R}$ and $y \in \mathbf{R}$ are the control input and output, respectively; ρ is the relative degree of the system satisfying $\rho + m = n$; both g_i's and b_i's are unknown constants, called *uncertain control coefficients*, and specially, $g_i \neq 0$, $b_0 \neq 0$ and $b_m \neq 0$; functions $\phi_i : \mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$, $i = 1, \ldots, n$ are piecewise continuous in the first argument and locally Lipschitz in the rest arguments. In what follows, suppose that only the system output is measurable, and the relative degree $\rho > 1$.

Obviously, we can see that control u appears in the last $(m+1)$ equations of system (1). This means that hidden zero-dynamics exist in the system. When control coefficients are known, if $1 < \rho < n$, then we can construct K-filters and the output-feedback controller to stabilize system (1) by consulting Chapter 8 of [1] and if $\rho = n$, system (1) degenerates to a simpler case which has been extensively investigated based on the Luenberger-like observer[12∼15]. Moreover, when control coefficients are unknown, various theoretical results have been obtained under somewhat strong assumption that the nonlinearities are dominated by some known functions of measurable output y[1]. When the system nonlinearities inherently depend on unmeasured states, the output-feedback control design will become very hard. Faire recently, for the systems with unmeasured states dependent growth and without zero-dynamics, [12, 15, 16] and [18] consider the problem of output-feedback stabilization for the cases of exactly known control coefficients and uncertain control coefficients, respectively.

This paper is to investigate the problem of global output-feedback stabilization for system (1) with uncertain control coefficients, zero-dynamics and unmeasured states dependent growth under the following assumptions:

Assumption 1. For $i = 1, \ldots, n$ and any $t \in \mathbf{R}^+$, $\boldsymbol{\zeta} \in \mathbf{R}^n$, $u \in \mathbf{R}$, there exists a known constant $c > 0$, such that

$$|\phi_i(t, \boldsymbol{\zeta}, u)| \le c(|\zeta_1| + |\zeta_2| + \cdots + |\zeta_k|), \quad k = \min\{i, \rho\}. \tag{2}$$

Assumption 2. The signs of g_i, $i = 1, \ldots, \rho - 1$ are known, and there exist known positive constants $\underline{g}_i$ and $\overline{g}_i$ satisfying:

$$\underline{g}_i \le |g_i| \le \overline{g}_i, \qquad i = 1, \ldots, n. \tag{3}$$

Assumption 3. The sign of b_m is known, and there exist known constants $b_N > 0$ and $\overline{b}_i > 0$, $i = 0, 1, \ldots, m$ such that

$$|b_i| \le \overline{b}_i, \ i = 0, 1, \ldots, m-1, \quad b_N \le |b_m| \le \overline{b}_m. \tag{4}$$

Assumption 4. For the polynomial $p(s) = b'_m s^m + \cdots + b'_1 s + b'_0$, where $m \ge 1$, $b'_i = b_i \Pi_{k=1}^{n-i-1} g_k$, $i = 1, \ldots, m$ and $b'_0 = b_0 \Pi_{k=1}^{n} g_k$, there exists a known positive constant d such that the real part of each pole of the polynomial is not larger than $-\frac{d}{2}$.

For the aim of better comprehension, we would like to give further interpretation for the above four assumptions. Assumption 1 means that system (1) has linearly unmeasured states dependent growth. It seems to be stringent, but it is crucial to carry out the output-feedback control. This point can be seen from [3, 12 ∼ 15, 18, 20]. On the other hand, maybe due to the existence of hidden zero-dynamics and uncertainties in control coefficients, it is hard to relax Assumption 1 to a weaker one.

Assumptions 2 and 3 give some available information about the uncertain coefficients g_i's and b_i's. The known signs of b_m and g_i, $i = 1, \ldots, \rho - 1$ will play an important role in control design. Otherwise, one can not decide the direction along which the control operates, and the closed-loop system may be unstable. The boundary restrictions imposed on g_i's and b_i's look somewhat severe but cannot be removed as will be detailed later.

Assumption 4 means that the hidden zero-dynamics of the system possess the input-to-state stable property. This assumption on the zero-dynamics is commonly implicitly assumed in most of the work in robust/adaptive output-feedback control of nonlinear systems; see [1, 10, 13] and references therein.

According to Assumption 2, there exist known positive constants $g_N := \min\{\underline{g}_1, \underline{g}_1\underline{g}_2, \cdots, \Pi_{j=1}^n \underline{g}_j\}$ and $g_M := \max\{\overline{g}_1, \overline{g}_1\overline{g}_2, \ldots, \Pi_{j=1}^n \overline{g}_j\}$ such that $g_N \le |\Pi_{j=1}^i g_j| \le g_M$, $i = 1, \ldots, n$.

Similarly, by Assumption 3, we know that $b_M := \max\{\overline{b}_0, \overline{b}_1, \ldots, \overline{b}_m\}$ is a known positive constant such that $|b_i| \le b_M$, $i = 0, 1, \ldots, m-1$ and $b_N \le |b_m| \le b_M$. Moreover, by Assumptions 2 and 3, the sign of $b'_m = b_m \Pi_{k=1}^{\rho-1} g_k \neq 0$ is known.

The objective of the paper is to design a dynamic output-feedback controller for system (1), so that the closed-loop system is globally asymptotically stable. This will be carried out based on the methods of appropriate state transformation and observer-based backstepping as will be detailed in the next sections.

2 Linear state transformations

In this section, it can be shown that, through two kinds of linear state transformations, the system can be successfully transformed into a system with deterministic virtual control coefficients, and the hidden zero-dynamics can be separated out, and then the output-feedback control design becomes less difficult.

By carefully examining system (1), we find that the negative influence of uncertain control coefficient, g_i, disappears if the corresponding equation is multiplied by $g_1 \cdots g_{i-1}$

with the scaled state being defined as the new state. This simple observation motivates us to introduce the following linear state transformation to convert the original system into a new system in which there are no uncertain virtual control coefficients, and the original uncertain control coefficients are lumped into the actual control coefficients:

$$x_1 = \zeta_1, \quad x_2 = g_1\zeta_2, \quad \cdots, \quad x_n = g_1 g_2 \cdots g_{n-1}\zeta_n. \tag{5}$$

The dynamics of $\boldsymbol{x} = [x_1, \ldots, x_n]^{\mathrm{T}}$ are then given by

$$\begin{cases} \dot{\boldsymbol{x}} &= A_n\boldsymbol{x} + \boldsymbol{b}u + \boldsymbol{f}(t, \boldsymbol{x}, u), \\ y &= x_1, \end{cases} \tag{6}$$

where $\boldsymbol{x} \in \mathbf{R}^n$ is the state of the new system with the initial condition depending on $\boldsymbol{\zeta}_0$ and (5), and $\boldsymbol{f} = [f_1, \ldots, f_n]^{\mathrm{T}} = [\phi_1, g_1\phi_2, \ldots, g_1g_2\cdots g_{n-1}\phi_n]^{\mathrm{T}}$,

$$A_n = \begin{bmatrix} 0 & & & \\ \vdots & & I & \\ & & & \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} \mathbf{0}_{[\rho-1]} \\ \boldsymbol{b}'_{[m,0]} \end{bmatrix}.$$

It is worth pointing out that when $\rho = n$ (or $m = 0$), the above transformation is enough and output-feedback control design can be pursued since there are no zero-dynamics in this particular case. For the sake of integrality in the later development, system (6) with $\rho = n$ is rewritten as:

$$\begin{cases} \Sigma_{\boldsymbol{\chi}} : \dot{\boldsymbol{\chi}} &= A_\rho\boldsymbol{\chi} + b'_m\boldsymbol{e}_{\rho,\rho}u + \boldsymbol{\Phi}_{\rho,[\rho]}(t, \boldsymbol{x}, u); \\ \qquad \chi_1 &= \boldsymbol{e}_{1,\rho}^{\mathrm{T}}\boldsymbol{\chi} = y, \end{cases} \tag{7}$$

where $\boldsymbol{\chi} = \boldsymbol{x}$, $\boldsymbol{\Phi}_{\rho,[\rho]} = \boldsymbol{f}$.

However, when $\rho < n$, the foregoing linear transformation is still necessary but somewhat insufficient for the control design purpose. This is chiefly because that to develop an output-feedback stabilizing controller, certain closed-loop observer should be constructed. But due to the presence of hidden zero-dynamics in (6), it is very difficult to determine what kind of observer is appropriate and to analyze the closed-loop observer error performance. Mainly enlightened by the existing relevant works on the stabilization of the similar systems with hidden zero-dynamics (see e.g., [21, 22]), an effectual idea is to look for some kind of state transformation to separate the zero-dynamics from the system. To achieve this, let us next introduce the second kind of linear state transformation for system (6) with $\rho < n$, which will be defined in a step-by-step manner, and simultaneously, the effect and meaning for each step of the state transformation will be exhibited accordingly.

The effect of the first transformation T_1 (i.e., the linear transformation matrix) is to transform all the elements of B into zero except the ρth element. Let $\boldsymbol{\varsigma}_1 = T_1\boldsymbol{x}$, where T_1 and T_1^{-1} are the same as I except their ρth columns that are $[\mathbf{0}_{[\rho-1]}^{\mathrm{T}}, 1, -b'_{m-1}/b'_m, \ldots, -b'_0/b'_m]^{\mathrm{T}}$ and $[\mathbf{0}_{[\rho-1]}^{\mathrm{T}}, 1, b'_{m-1}/b'_m, \ldots, b'_0/b'_m]^{\mathrm{T}}$, respectively. Then, we can get the dynamics of $\boldsymbol{\varsigma}_1$ as follows:

$$\dot{\boldsymbol{\varsigma}}_1 \quad = \quad C_1\boldsymbol{\varsigma}_1 + \boldsymbol{B}u + \boldsymbol{\Phi}_1(t, \boldsymbol{x}, u), \tag{8}$$

where C_1 is the same as A_n except its ρth and $(\rho+1)$st columns, which are $[\mathbf{0}_{[\rho-2]}^{\mathrm{T}}, 1, \boldsymbol{c}_{1,\,[m+1]}^{\mathrm{T}}]^{\mathrm{T}}$ and $[\mathbf{0}_{[\rho-1]}^{\mathrm{T}}, 1, -\frac{1}{b'_m}\boldsymbol{b}'^{\mathrm{T}}_{[m-1,\,0]}]^{\mathrm{T}}$, respectively, and $\boldsymbol{B} = T_1\boldsymbol{b} = [\mathbf{0}_{[\rho-1]}^{\mathrm{T}}, b'_m, \mathbf{0}_{[m]}^{\mathrm{T}}]^{\mathrm{T}} = b'_m\boldsymbol{e}_{\rho,n}$, $\boldsymbol{\Phi}_1 = T_1\boldsymbol{f}(t, x, u)$.

The rest $\rho - 1$ transformations are designed mainly for matrix C_1 such that we can eventually get C_ρ which is the same as matrix I except the first and $(\rho+1)$st columns. Specifically, the second transformation is to transform the ρth column of C_1 into $\boldsymbol{e}_{\rho-1,n}$, i.e., $[\mathbf{0}_{[\rho-2]}^{\mathrm{T}}, 1, \mathbf{0}_{[m+1]}^{\mathrm{T}}]^{\mathrm{T}}$. Let $\boldsymbol{\varsigma}_2 = T_2\boldsymbol{\varsigma}_1$, where T_2 and T_2^{-1} are the same as I except their $(\rho-1)$st columns that are $[\mathbf{0}_{[\rho-2]}^{\mathrm{T}}, 1, -\boldsymbol{c}_{1,\,[m+1]}^{\mathrm{T}}]^{\mathrm{T}}$ and $[\mathbf{0}_{[\rho-2]}^{\mathrm{T}}, 1, \boldsymbol{c}_{1,\,[m+1]}^{\mathrm{T}}]^{\mathrm{T}}$, respectively. Noting that $\boldsymbol{B} = T_2\boldsymbol{B}$, the dynamics of $\boldsymbol{\varsigma}_2$ is:

$$\dot{\boldsymbol{\varsigma}}_2 \quad = \quad C_2\boldsymbol{\varsigma}_2 + \boldsymbol{B}u + \boldsymbol{\Phi}_2(t, \boldsymbol{x}, u), \tag{9}$$

where C_2 is the same as A_n except its $(\rho-1)$st and $(\rho+1)$st columns which are $[\mathbf{0}_{[\rho-3]}^{\mathrm{T}}, 1, \boldsymbol{c}_{2,\,[m+2]}^{\mathrm{T}}]^{\mathrm{T}}$ and $[\mathbf{0}_{[\rho-1]}^{\mathrm{T}}, 1, -\frac{1}{b'_m}\boldsymbol{b}'^{\mathrm{T}}_{[m-1,\,0]}]^{\mathrm{T}}$, respectively, and $\boldsymbol{\Phi}_2 = T_2\boldsymbol{\Phi}_1(t, \boldsymbol{x}, u)$.

Similarly, the ith ($i = 3, 4, \ldots, \rho-1$) linear transformation T_i can be constructed to transform the $(\rho-i+2)$nd column of C_{i-1} into $\boldsymbol{e}_{\rho-i+1,n}$. Let $\boldsymbol{\varsigma}_i = T_i\boldsymbol{\varsigma}_{i-1}$, where T_i and T_i^{-1} are the same as I except their $(\rho-i+1)$st columns, which are $[\mathbf{0}_{[\rho-i]}^{\mathrm{T}}, 1, -\boldsymbol{c}_{i-1,\,[m+i-1]}^{\mathrm{T}}]^{\mathrm{T}}$ and $[\mathbf{0}_{[\rho-i]}^{\mathrm{T}}, 1, \boldsymbol{c}_{i-1,\,[m+i-1]}^{\mathrm{T}}]^{\mathrm{T}}$, respectively. Noting that $\boldsymbol{B} = T_iT_{i-1}\cdots T_2\boldsymbol{B}$, the dynamics of $\boldsymbol{\varsigma}_i$ is:

$$\dot{\boldsymbol{\varsigma}}_i \quad = \quad C_i\boldsymbol{\varsigma}_i + \boldsymbol{B}u + \boldsymbol{\Phi}_i(t, \boldsymbol{x}, u), \tag{10}$$

where C_i is the same as A_n except its $(\rho-i+1)$st and $(\rho+1)$st columns which are $\left[\mathbf{0}_{[\rho-i-1]}^{\mathrm{T}}, 1, \boldsymbol{c}_{i,\,[m+i]}^{\mathrm{T}}\right]^{\mathrm{T}}$ and $\left[\mathbf{0}_{[\rho-1]}^{\mathrm{T}}, 1, -\frac{1}{b'_m}\boldsymbol{b}'^{\mathrm{T}}_{[m-1,\,0]}\right]^{\mathrm{T}}$, respectively, and $\boldsymbol{\Phi}_i = T_i\boldsymbol{\Phi}_{i-1}(t, \boldsymbol{x}, u)$.

Finally, the ρth linear transformation $\boldsymbol{\varsigma}_\rho = T_\rho\boldsymbol{\varsigma}_{\rho-1}$ can be constructed to transform the second column of $C_{\rho-1}$ into $\boldsymbol{e}_{1,n}$, where T_ρ and T_ρ^{-1} are the same as I except their first columns that are $\left[1, -\boldsymbol{c}_{\rho-1,\,[n-1]}^{\mathrm{T}}\right]^{\mathrm{T}}$ and $\left[1, \boldsymbol{c}_{\rho-1,\,[n-1]}^{\mathrm{T}}\right]^{\mathrm{T}}$, respectively. Noting that $\boldsymbol{B} = T_\rho T_{\rho-1}\cdots T_2\boldsymbol{B}$, the dynamics of $\boldsymbol{\varsigma}_\rho$ is:

$$\dot{\boldsymbol{\varsigma}}_\rho \quad = \quad C_\rho\boldsymbol{\varsigma}_\rho + \boldsymbol{B}u + \boldsymbol{\Phi}_\rho(t, \boldsymbol{x}, u), \tag{11}$$

where C_ρ is the same as A_n except its first and $(\rho+1)$st columns which are $\boldsymbol{c}_{\rho,[n]}$ and $\left[\mathbf{0}_{[\rho-1]}^{\mathrm{T}}, 1, -\frac{1}{b'_m}\boldsymbol{b}'^{\mathrm{T}}_{[m-1,\,0]}\right]^{\mathrm{T}}$, respectively, and $\boldsymbol{\Phi}_\rho = T_\rho\boldsymbol{\Phi}_{\rho-1} = [\Phi_{\rho,1}, \ldots, \Phi_{\rho,n}]^{\mathrm{T}}$.

Up to now, the two kinds of linear state transformations are completely introduced. For the transformed system (11), it is natural to design observer and observer-based output-feedback control, as will be seen from the later development.

First, we would like to give the 2-norm estimation of the unknown vectors $\boldsymbol{c}_{\rho,[\rho]}$ and $\boldsymbol{c}_{\rho,\,[\rho+1,n]}$ which can be deduced from the above state transformations though a bit complicatedly. In fact, from its definition, one can see that each element of $\boldsymbol{c}_{i,\,[m+i]}$ is in the form of a quotient whose denominator is a power function of b'_m and numerator a polynomial of the constants $b'_0, b'_1, \ldots, b'_m$. From this observation and Assumptions 2 and 3, we can easily verify the following proposition:

Proposition 1. For $\boldsymbol{c}_{\rho,\,[n]}$ introduced above, there exist known positive constants d_0 and d'_0 such that

$$\|\boldsymbol{c}_{\rho,[\rho]}\| \le d_0, \qquad \|\boldsymbol{c}_{\rho,[\rho+1,n]}\| \le d'_0. \tag{12}$$

Before designing high-gain K-filters based observer and output-feedback controller, we have to decompose system (11) appropriately. Let $\boldsymbol{\chi} = [\chi_1, \ldots, \chi_\rho]^{\mathrm{T}} = \boldsymbol{\varsigma}_{\rho,[1,\,\rho]}$ and $\boldsymbol{\eta} = [\eta_1, \ldots, \eta_m]^{\mathrm{T}} = \boldsymbol{\varsigma}_{\rho,[\rho+1,\,n]}$. Then system (11) can be divided into the following form:

$$\begin{cases}\Sigma_{\boldsymbol{\eta}}:\dot{\boldsymbol{\eta}} &= D\boldsymbol{\eta}+\boldsymbol{c}_{\rho,[\rho+1,n]}\chi_1+\boldsymbol{\Phi}_{\rho,[\rho+1,n]}(t,\boldsymbol{x},u);\\ \Sigma_{\boldsymbol{\chi}}:\dot{\boldsymbol{\chi}} &= A_\rho\boldsymbol{\chi}+b'_m\boldsymbol{e}_{\rho,\rho}u+\boldsymbol{e}_{\rho,\rho}\eta_1+\boldsymbol{c}_{\rho,[\rho]}\chi_1\\ &\quad+\boldsymbol{\Phi}_{\rho,[\rho]}(t,\boldsymbol{x},u);\\ \chi_1 &= \boldsymbol{e}_{1,\rho}^{\rm T}\boldsymbol{\chi}=y,\end{cases} \tag{13}$$

where $D=\left[-\frac{\boldsymbol{b}'_{[m-1,0]}}{b'_m},[I,\boldsymbol{0}_{[m-1]}]^{\rm T}\right]$ is Hurwitz ensured by Assumption 4. So there is a positive definite matrix

$$Q=Q^{\rm T}=\int_0^\infty \exp(D^{\rm T}t)\exp(Dt){\rm d}t, \tag{14}$$

which satisfies the Lyapunov equation: $D^{\rm T}Q+QD=-I$.

We have the following lemma for the 2-norm of Q, whose proof is given in Appendix.

Lemma 1. For any Hurwitz matrix D with ${\rm Re}(\lambda(D))\le -\frac{d}{2}<0$, the matrix Q defined by (14) satisfies the following inequality: $\|Q\|\le\frac{n}{d}$.

The following lemma is natural and can be easily proved with Assumptions 1, 2 and 3.

Lemma 2. For $\forall(t,\boldsymbol{x},u)\in\mathbf{R}^+\times\mathbf{R}^n\times\mathbf{R}$ and $\boldsymbol{\chi}=[I\ \ 0_{\rho\times m}]T_\rho\cdots T_1\boldsymbol{x}\in\mathbf{R}^\rho$, there is a known positive constant c' such that the unknown nonlinearities $\Phi_{\rho,i}:\ \mathbf{R}^+\times\mathbf{R}^n\times\mathbf{R}\to\mathbf{R}$, $i=1,\ldots,n$ of system (7) or system (13) satisfy

$$|\Phi_{\rho,i}(t,\boldsymbol{x},u)|\le c'\big(|\chi_1|+\cdots+|\chi_k|\big),\ k=\min\{i,\rho\}. \tag{15}$$

The next section is the core of present paper, in which we will firstly construct the appropriate high-gain K-filters based observer, and then design the output-feedback controller by the backstepping method.

3 Output-feedback stabilizing control design

This section turns to output-feedback control design for system (7) ($\rho=n$) or system (13) ($\rho<n$). Since they are equivalent to system (1), the global asymptotic stability of the transformed and original systems can be achieved simultaneously. This section includes three parts: in Subsection 4.1, a novel approach for observer design will be proposed based on high-gain K-filters for system (7) or the subsystem $\Sigma_{\boldsymbol{\chi}}$ of system (13); Subsection 4.2 is restricted to the case of $\rho<n$, where a globally asymptotically stabilizing output-feedback controller will be constructed by the traditional backstepping method[1], and the case of $\rho=n$ is simple which will be studied in the next section by an example; in Subsection 4.3, the main result of the paper is summarized, which shows that the global asymptotic stability of the closed-loop system can be guaranteed by the appropriate choice of the design parameters.

3.1 High-gain K-filters and state estimation

For system (7) or the subsystem $\Sigma_{\boldsymbol{\chi}}$ of system (13), by [18], we introduce the following high-gain K-filters:

$$\begin{cases}\dot{\boldsymbol{\xi}}=A_{\boldsymbol{l}_\varepsilon}\boldsymbol{\xi}-\boldsymbol{l}_\varepsilon y,\\ \dot{\boldsymbol{\lambda}}=A_{\boldsymbol{l}_\varepsilon}\boldsymbol{\lambda}+\boldsymbol{e}_{\rho,\rho}u,\end{cases} \tag{16}$$

where $\varepsilon\in(0,\ 1)$ is a constant, $\boldsymbol{l}_\varepsilon=[\frac{l_1}{\varepsilon},\frac{l_2}{\varepsilon^2},\ldots,\frac{l_\rho}{\varepsilon^\rho}]^{\rm T}$, and $A_{\boldsymbol{l}_\varepsilon}=A_\rho+\boldsymbol{l}_\varepsilon\boldsymbol{e}_{1,\rho}^{\rm T}$.

The constant vector $\boldsymbol{l}=[l_1,\ldots,l_\rho]^{\rm T}$ is chosen to ensure that matrix $A_{\boldsymbol{l}}=A_\rho+\boldsymbol{l}\boldsymbol{e}_{1,\rho}^{\rm T}$ is Hurwitz. Then there is a unique symmetric positive definite matrix $P_{\boldsymbol{l}}$ satisfying $A_{\boldsymbol{l}}^{\rm T}P_{\boldsymbol{l}}+P_{\boldsymbol{l}}A_{\boldsymbol{l}}=-I$. Define $I_\varepsilon={\rm diag}[1,\varepsilon,\cdots,\varepsilon^{\rho-1}]$. It can be shown that $\varepsilon A_{\boldsymbol{l}_\varepsilon}=I_\varepsilon^{-1}A_{\boldsymbol{l}}I_\varepsilon$. So matrix $P_{\boldsymbol{l}_\varepsilon}=I_\varepsilon P_{\boldsymbol{l}}I_\varepsilon$ is symmetric positive definite, and satisfies

$$A_{\boldsymbol{l}_\varepsilon}^{\rm T}P_{\boldsymbol{l}_\varepsilon}+P_{\boldsymbol{l}_\varepsilon}A_{\boldsymbol{l}_\varepsilon}=-\varepsilon^{-1}I_\varepsilon^2. \tag{17}$$

This means that $A_{\boldsymbol{l}_\varepsilon}$ is also a Hurwitz matrix.

Define the state estimate of state $\boldsymbol{\chi}$ as $\hat{\boldsymbol{\chi}}=\boldsymbol{\xi}+b'_m\boldsymbol{\lambda}$, and the state estimation error as $\tilde{\boldsymbol{\chi}}=\boldsymbol{\chi}-\hat{\boldsymbol{\chi}}$.

In the next development, we will consider the properties of the state estimation error for the cases of $\rho<n$ and $\rho=n$, respectively.

Case 1: When $\rho<n$, the state estimation error $\tilde{\boldsymbol{\chi}}$ satisfies

$$\dot{\tilde{\boldsymbol{\chi}}}=A_{\boldsymbol{l}_\varepsilon}\tilde{\boldsymbol{\chi}}+\boldsymbol{e}_{\rho,\rho}\eta_1+\boldsymbol{c}_{\rho,[\rho]}\chi_1+\boldsymbol{\Phi}_{\rho,[\rho]}. \tag{18}$$

To prepare for the backstepping procedure in the next subsection, we have to rewrite the equation of $\dot{y}$. Since the state χ_2 is not available, we need to replace χ_2 by $\tilde{\chi}_2+\xi_2+b'_m\lambda_2$. Then, we have

$$\dot{y}=b'_m\lambda_2+c_{\rho,1}y+\tilde{\chi}_2+\xi_2+\Phi_{\rho,1}. \tag{19}$$

Up to now, the whole system for control design is obtained

$$\begin{cases}\dot{\tilde{\boldsymbol{\chi}}}=A_{\boldsymbol{l}_\varepsilon}\tilde{\boldsymbol{\chi}}+\boldsymbol{e}_{\rho,\rho}\eta_1+\boldsymbol{c}_{\rho,[\rho]}\chi_1+\boldsymbol{\Phi}_{\rho,[\rho]},\\ \dot{\boldsymbol{\xi}}=A_{\boldsymbol{l}_\varepsilon}\boldsymbol{\xi}-\boldsymbol{l}_\varepsilon y,\\ \dot{y}=b'_m\lambda_2+c_{\rho,1}y+\tilde{\chi}_2+\xi_2+\Phi_{\rho,1},\\ \dot{\lambda}_i=\frac{l_i}{\varepsilon^i}\lambda_1+\lambda_{i+1},\quad i=2,3,\ldots,\rho-1,\\ \dot{\lambda}_\rho=\frac{l_\rho}{\varepsilon^\rho}\lambda_1+u.\end{cases} \tag{20}$$

Obviously, the study of the global asymptotic stabilization of the above system is equivalent to that of system (13), as well as the original system (1). So, it is enough to consider system (20).

Let's next study the property of the subsystem $\Sigma_{\boldsymbol{\eta}}$. The following proposition shows that this subsystem is input-to-state stable (ISS) and can be regarded as the zero-dynamics of the whole system (13). The proof of the proposition is provided in Appendix.

Proposition 2. For the subsystem $\Sigma_{\boldsymbol{\eta}}$ of system (13), let $V_{\boldsymbol{\eta}}=\boldsymbol{\eta}^{\rm T}Q\boldsymbol{\eta}$. Then there are known positive constants $d_{0,1}$, $d_{0,2}$ and $d_{0,3}$, independent of ε, such that

$$\begin{aligned}\dot{V}_{\boldsymbol{\eta}} \le\ & -\tfrac{3}{8}\|\boldsymbol{\eta}\|^2+d_{0,1}\left(d_0'^2+m^2c'^2\varepsilon^{-(2\rho-2)}\right)y^2\\ &+\varepsilon^{-(2\rho-2)}d_{0,2}\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2+\varepsilon^{-(2\rho-2)}d_{0,2}\|I_\varepsilon\boldsymbol{\xi}\|^2\\ &+\varepsilon^{-(2\rho-2)}d_{0,3}\sum_{i=2}^{\rho}\varepsilon^{2i-2}\lambda_i^2,\end{aligned} \tag{21}$$

for any $\tilde{\boldsymbol{\chi}},\boldsymbol{\xi},\boldsymbol{\eta},y,\lambda_i$, $i=2,3,\ldots,\rho$, where d_0' has been specified in Proposition 1.

The following two propositions play an important role in control design in the sequent subsections. Specifically, Propositions 3 and 4 characterize the ISS-like properties of $\tilde{\boldsymbol{\chi}}$ and $\boldsymbol{\xi}$ of system (20) respectively. Besides, for the sake of compactness, the proof of Proposition 3 is provided in Appendix.

Proposition 3. For the subsystem $\tilde{\boldsymbol{\chi}}$ of system (20), let $V_{\tilde{\boldsymbol{\chi}}}=\tilde{\boldsymbol{\chi}}^{\rm T}P_{\boldsymbol{l}_\varepsilon}\tilde{\boldsymbol{\chi}}$. Then there are known positive constants $d_{0,4}\sim d_{0,7}$ independent of ε, such that

$$\begin{aligned}\dot{V}_{\tilde{\boldsymbol{\chi}}}\le &-\left(\tfrac{1}{2\varepsilon}-d_{0,4}\right)\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2+d_{0,5}\|I_\varepsilon\boldsymbol{\xi}\|^2+2\varepsilon^{2\rho-1}\|P_l\|^2\|\boldsymbol{\eta}\|^2\\ &+d_{0,6}y^2+d_{0,7}\sum_{i=2}^{\rho}\varepsilon^{2i-2}\lambda_i^2,\end{aligned} \tag{22}$$

for any $\tilde{\boldsymbol{\chi}}$, $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, y, λ_i, $i=2,3,\ldots,\rho$.

Proposition 4. For the subsystem $\boldsymbol{\xi}$ of system (20), let $V_{\boldsymbol{\xi}}=\boldsymbol{\xi}^{\rm T}P_{\boldsymbol{l}_\varepsilon}\boldsymbol{\xi}$. Then, for any $\boldsymbol{\xi}$, y, the following inequality holds:

$$\dot V_{\boldsymbol{\xi}} \quad\le\quad -\tfrac{1}{2\varepsilon}\|I_\varepsilon\boldsymbol{\xi}\|^2+\tfrac{2\|P_{\boldsymbol{l}}\|^2\|\boldsymbol{l}\|^2}{\varepsilon}y^2. \tag{23}$$

Proof. Noting that $\boldsymbol{l}_\varepsilon=\frac{1}{\varepsilon}I_\varepsilon^{-1}\boldsymbol{l}$ and $P_{\boldsymbol{l}_\varepsilon}=I_\varepsilon P_{\boldsymbol{l}}I_\varepsilon$, we can show that the time derivative of $V_{\boldsymbol{\xi}}$ satisfies

$$\dot V_{\boldsymbol{\xi}}=-\tfrac{\|I_\varepsilon\boldsymbol{\xi}\|^2}{\varepsilon}-2\boldsymbol{\xi}^{\rm T}I_\varepsilon P_{\boldsymbol{l}}I_\varepsilon\boldsymbol{l}_\varepsilon y=-\tfrac{\|I_\varepsilon\boldsymbol{\xi}\|^2}{\varepsilon}-\tfrac{2}{\varepsilon}\boldsymbol{\xi}^{\rm T}I_\varepsilon P_{\boldsymbol{l}}\boldsymbol{l}y.$$

Then, by the method of completing square, we have

$$\begin{aligned}\dot V_{\boldsymbol{\xi}}\quad&\le\quad -\tfrac{\|I_\varepsilon\boldsymbol{\xi}\|^2}{\varepsilon}+\tfrac{\|I_\varepsilon\boldsymbol{\xi}\|^2}{2\varepsilon}+\tfrac{2}{\varepsilon}\|P_{\boldsymbol{l}}\|^2\|\boldsymbol{l}\|^2y^2\\ &=\quad -\tfrac{1}{2\varepsilon}\|I_\varepsilon\boldsymbol{\xi}\|^2+\tfrac{2\|P_{\boldsymbol{l}}\|^2\|\boldsymbol{l}\|^2}{\varepsilon}y^2,\end{aligned}$$

for any $\boldsymbol{\xi}$ and y. □

Case 2: When $\rho=n$, the state estimation error $\tilde{\boldsymbol{\chi}}$ satisfies

$$\dot{\tilde{\boldsymbol{\chi}}}=A_{\boldsymbol{l}_\varepsilon}\tilde{\boldsymbol{\chi}}+\boldsymbol{\Phi}_{\rho,[\rho]}. \tag{24}$$

Similarly, we rewrite the equation of $\dot y=\chi_2+\Phi_{\rho,1}$ as $\dot y=b'_m\lambda_2+\tilde\chi_2+\xi_2+\Phi_{\rho,1}$ and thus obtain the whole system for control design:

$$\begin{cases}\dot{\tilde{\boldsymbol{\chi}}}=A_{\boldsymbol{l}_\varepsilon}\tilde{\boldsymbol{\chi}}+\boldsymbol{\Phi}_{\rho,[\rho]},\\ \dot{\boldsymbol{\xi}}=A_{\boldsymbol{l}_\varepsilon}\boldsymbol{\xi}-\boldsymbol{l}_\varepsilon y,\\ \dot y=b'_m\lambda_2+\tilde\chi_2+\xi_2+\Phi_{\rho,1},\\ \dot\lambda_i=\frac{l_i}{\varepsilon^i}\lambda_1+\lambda_{i+1},\quad i=2,3,\ldots,\rho-1,\\ \dot\lambda_\rho=\frac{l_\rho}{\varepsilon^\rho}\lambda_1+u.\end{cases} \tag{25}$$

3.2 Output-feedback control design

This subsection is devoted to the constructive design of output-feedback control for system (20) with $\rho<n$ by the traditional backstepping method, which is presented in a step-by-step manner.

Step 1. Define $V_0=\varepsilon^{2\rho-2}V_{\boldsymbol{\eta}}+V_{\tilde{\boldsymbol{\chi}}}+V_{\boldsymbol{\xi}}$, where $V_{\boldsymbol{\eta}}, V_{\tilde{\boldsymbol{\chi}}}$ and $V_{\boldsymbol{\xi}}$ have been defined respectively in Propositions 2, 3 and 4. Let $V_1=V_0+\frac{1}{2}y^2$ be the Lyapunov function candidate for this step. Then, from Propositions 2, 3 and 4, and (19), it follows that

$$\begin{aligned}\dot V_1\le&-\varepsilon^{2\rho-1}\left(\tfrac{3}{8\varepsilon}-2\|P_{\boldsymbol{l}}\|^2\right)\|\boldsymbol{\eta}\|^2-\left(\tfrac{1}{2\varepsilon}-d_{0,2}-d_{0,4}\right)\\ &\cdot\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2-\left(\tfrac{1}{2\varepsilon}-d_{0,2}-d_{0,5}\right)\|I_\varepsilon\boldsymbol{\xi}\|^2+\Big(d_0'^2d_{0,1}\varepsilon^{2\rho-2}\\ &+m^2c'^2d_{0,1}+d_{0,6}+\frac{2\|P_{\boldsymbol{l}}\|^2\|\boldsymbol{l}\|^2}{\varepsilon}\Big)y^2+(d_{0,3}+d_{0,7})\\ &\cdot\sum_{i=2}^{\rho}\varepsilon^{2i-2}\lambda_i^2+y(b'_m\lambda_2+c_{\rho,1}y+\tilde\chi_2+\xi_2+\Phi_{\rho,1}).\end{aligned}\tag{26}$$

Let us first handle the last term on the right-hand side of (26). From Proposition 1, it follows that $|c_{\rho,1}|\le\|\boldsymbol{c}_{\rho,[\rho]}\|\le d_0$. Then, for any $\boldsymbol{\xi}$, y, $\tilde{\boldsymbol{\chi}}$, it is evident that

$$\begin{aligned}&c_{\rho,1}y^2\le d_0y^2, && y\tilde\chi_2\le\tfrac{1}{4\varepsilon}\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2+\tfrac{1}{\varepsilon}y^2,\\ &y\xi_2\le\tfrac{1}{4\varepsilon}\|I_\varepsilon\boldsymbol{\xi}\|^2+\tfrac{1}{\varepsilon}y^2, && y\Phi_{\rho,1}\le c'y^2.\end{aligned}\tag{27}$$

By substituting (27) into (26) and letting $d'_{1,1}=d_{0,2}+d_{0,4}$, $d'_{1,2}=d_{0,2}+d_{0,5}$ and $d_{\boldsymbol{\lambda}}=d_{0,3}+d_{0,7}$, we have

$$\begin{aligned}\dot V_1\quad\le\quad&-\varepsilon^{2\rho-1}\left(\tfrac{3}{8\varepsilon}-2\|P_{\boldsymbol{l}}\|^2\right)\|\boldsymbol{\eta}\|^2-\left(\tfrac{1}{4\varepsilon}-d'_{1,1}\right)\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2\\ &-\left(\tfrac{1}{4\varepsilon}-d'_{1,2}\right)\|I_\varepsilon\boldsymbol{\xi}\|^2+\Big(d_0'^2d_{0,1}\varepsilon^{2\rho-2}+m^2c'^2d_{0,1}\\ &+d_{0,6}+c'+d_0+\tfrac{2\|P_{\boldsymbol{l}}\|^2\|\boldsymbol{l}\|^2+2}{\varepsilon}\Big)y^2\\ &+d_{\boldsymbol{\lambda}}\sum_{i=3}^{\rho}\varepsilon^{2i-2}\lambda_i^2+d_{\boldsymbol{\lambda}}\varepsilon^2\lambda_2^2+b'_my\lambda_2.\end{aligned}\tag{28}$$

Choose the virtual controller as

$$\lambda_2^*=-\tfrac{1}{\varepsilon}\Big(\tfrac{{\rm sign}(b'_m)}{g_Nb_N}L_1+\bar\alpha_1\Big)y=:-\tfrac{\alpha_1(L_1)}{\varepsilon}y,\tag{29}$$

where $\bar\alpha_1=\frac{{\rm sign}(b'_m)}{g_Nb_N}\big(d_0'^2d_{0,1}+m^2c'^2d_{0,1}+d_{0,6}+c'+d_0+2\|P_{\boldsymbol{l}}\|^2\|\boldsymbol{l}\|^2+2\big)$ is clearly a known constant, and $L_1\ge1$ is a constant to be determined later.

Define $z_1=y$ and $z_2=\lambda_2-\lambda_2^*$. Observe that

$$\begin{aligned}d_{\boldsymbol{\lambda}}\varepsilon^2\lambda_2^2\ \le&\ 5d_{\boldsymbol{\lambda}}\varepsilon^2z_2^2+\mu_1^{z_1}(L_1)y^2\\ &+\mu_1^{\tilde{\boldsymbol{\chi}}}\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2+\mu_1^{\boldsymbol{\xi}}\|I_\varepsilon\boldsymbol{\xi}\|^2,\end{aligned}\tag{30}$$

where $\mu_1^{z_1}(L_1)=2d_{\boldsymbol{\lambda}}\alpha_1^2(L_1)$, and $\mu_1^{\tilde{\boldsymbol{\chi}}}=\mu_1^{\boldsymbol{\xi}}=0$ for the initial assignment of the forthcoming inductive step. Then, substituting (29) and (30) into (28) results in

$$\begin{aligned}\dot V_1\le&-\varepsilon^{2\rho-1}\left(\tfrac{3}{8\varepsilon}-2\|P_{\boldsymbol{l}}\|^2\right)\|\boldsymbol{\eta}\|^2-\left(\tfrac{1}{4\varepsilon}-d_{1,1}\right)\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2\\ &-\left(\tfrac{1}{4\varepsilon}-d_{1,2}\right)\|I_\varepsilon\boldsymbol{\xi}\|^2-\left(\tfrac{L_1}{\varepsilon}-d_{1,3}(L_1)\right)y^2\\ &+d_{\boldsymbol{\lambda}}\sum_{i=3}^{\rho}\varepsilon^{2i-2}\lambda_i^2+5d_{\boldsymbol{\lambda}}\varepsilon^2z_2^2+b'_myz_2,\end{aligned}\tag{31}$$

where $d_{1,1}=d'_{1,1}+\mu_1^{\tilde{\boldsymbol{\chi}}}$, $d_{1,2}=d'_{1,2}+\mu_1^{\boldsymbol{\xi}}$, $d_{1,3}(L_1)=\mu_1^{z_1}(L_1)$.

Step 2. Let $V_2=V_1+\frac{1}{2}\varepsilon^2z_2^2$. Then, by (31), the time derivative of V_2 satisfies

$$\dot V_2\quad\le\quad\dot V_1+\varepsilon^2z_2\dot z_2.\tag{32}$$

By the definition of z_2, we have

$$\begin{aligned}\dot z_2=&\lambda_3+\sum_{j=1}^{2}\gamma_2^{\lambda_j}(L_1)\varepsilon^{-(3-j)}\lambda_j+\sum_{j=1}^{2}\gamma_2^{\xi_j}(L_1)\varepsilon^{-(3-j)}\\ &\cdot\xi_j+\gamma_2^{y}(L_1)\varepsilon^{-2}y+\gamma_2^{\Delta}(L_1)\varepsilon^{-1}(c_{\rho,1}y+b'_m\lambda_2\\ &+\tilde\chi_2+\Phi_{\rho,1})\end{aligned}\tag{33}$$

with specified $\gamma_2^{\lambda_2}(L_1)=0$, $\gamma_2^{\xi_1}(L_1)=0$, $\gamma_2^{y}(L_1)=0$ for the initial assignment of the forthcoming inductive step.

Before deriving the virtual controller λ_3^*, we should eliminate the "undesired" effect of z_2 in (32). For this purpose, by the method of completing square, we have

$$b'_myz_2\le b_Mg_M|y||z_2|\le\tfrac{1}{\varepsilon}y^2+\tfrac{1}{4}\max\{1,b_M^2g_M^2\}\varepsilon z_2^2,\tag{34}$$

and the following inequalities

$$\begin{cases}\varepsilon\gamma_2^{\Delta}(L_1)c_{\rho,1}yz_2 &\le\ \tfrac{1}{4}d_0^2(\gamma_2^{\Delta}(L_1))^2\varepsilon^2z_2^2+y^2,\\ b'_m\varepsilon\gamma_2^{\Delta}(L_1)\lambda_2z_2 &\le\ \tfrac{1}{\varepsilon}y^2+\Big(b_Mg_M|\gamma_2^{\Delta}(L_1)|\\ &\qquad+\tfrac{1}{4}b_M^2g_M^2\alpha_1^2(L_1)(\gamma_2^{\Delta}(L_1))^2\Big)\varepsilon z_2^2,\\ \varepsilon\gamma_2^{\Delta}(L_1)\tilde\chi_2z_2 &\le\ \tfrac{1}{8\varepsilon}\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2+2(\gamma_2^{\Delta}(L_1))^2\varepsilon z_2^2,\\ \varepsilon\gamma_2^{\Delta}(L_1)\Phi_{\rho,1}z_2 &\le\ \tfrac{1}{4}c'^2(\gamma_2^{\Delta}(L_1))^2\varepsilon^2z_2^2+y^2.\end{cases}\tag{35}$$

Substituting (33), (34) and (35) into (32), we have

$$\begin{aligned}\dot V_2\quad\le\quad&-\varepsilon^{2\rho-1}\big(\tfrac{3}{8\varepsilon}-2\|P_{\boldsymbol{l}}\|^2\big)\|\boldsymbol{\eta}\|^2-\big(\tfrac{1}{8\varepsilon}-d_{1,1}\big)\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2\\ &-\big(\tfrac{1}{4\varepsilon}-d_{1,2}\big)\|I_\varepsilon\boldsymbol{\xi}\|^2-\Big(\tfrac{L_1-2}{\varepsilon}-d_{1,3}(L_1)-2\Big)y^2\\ &+d_{\boldsymbol{\lambda}}\sum_{i=4}^{\rho}\varepsilon^{2i-2}\lambda_i^2+\varepsilon^2z_2\Big(\sum_{j=1}^{2}\gamma_2^{\lambda_j}(L_1)\varepsilon^{-(3-j)}\lambda_j\end{aligned}$$

$$\begin{aligned}&+\sum_{j=1}^{2}\gamma_2^{\xi_j}(L_1)\varepsilon^{-(3-j)}\xi_j+\gamma_2^{y}(L_1)\varepsilon^{-2}y\Big)\\&+\bar{\alpha}_2(L_1)z_2+d_{\boldsymbol{\lambda}}\varepsilon^4\lambda_3^2+\varepsilon^2 z_2\lambda_3.\end{aligned}\tag{36}$$

where $\bar{\alpha}_2(L_1)=5d_{\boldsymbol{\lambda}}+\frac{1}{4}(d_0^2+c'^2)(\gamma_2^{\Delta}(L_1))^2+b_Mg_M|\gamma_2^{\Delta}(L_1)|+2(\gamma_2^{\Delta}(L_1))^2+\frac{1}{4}\big(b_M^2g_M^2\alpha_1^2(L_1)(\gamma_2^{\Delta}(L_1))^2+\max\{1,b_M^2g_M^2\}\big)$. Thus, we can choose the virtual controller as

$$\begin{aligned}\lambda_3^* \ =\ & -\varepsilon^{-1}\alpha_2(\boldsymbol{L}_{[2]})z_2-\sum_{j=1}^{2}\gamma_2^{\lambda_j}(L_1)\varepsilon^{-(3-j)}\lambda_j\\&-\sum_{j=1}^{2}\gamma_2^{\xi_j}(L_1)\varepsilon^{-(3-j)}\xi_j-\gamma_2^{y}(L_1)\varepsilon^{-2}y,\end{aligned}\tag{37}$$

where $\alpha_2(\boldsymbol{L}_{[2]})=L_2+\bar{\alpha}_2(L_1)$, and $L_2\geq 1$ is a constant to be determined later.

Define $z_3=\lambda_3-\lambda_3^*$. Similar to Step 1, we should estimate $d_{\boldsymbol{\lambda}}\varepsilon^4\lambda_3^2$ by an appropriate function of z_1, z_2, z_3, $\boldsymbol{\xi}$ and $\tilde{\boldsymbol{\chi}}$. Since $\lambda_1=\frac{1}{b'_m}\big(y-\tilde{\chi}_1-\xi_1\big)$, we have $\lambda_1^2\leq\frac{3}{b_N^2g_N^2}\big(y^2+\tilde{\chi}_1^2+\xi_1^2\big)$. By this and (30), we have

$$\begin{aligned}d_{\boldsymbol{\lambda}}\varepsilon^4\lambda_3^2\ \leq\ & 7d_{\boldsymbol{\lambda}}\varepsilon^4z_3^2+\sum_{j=1}^{2}\mu_2^{z_j}(\boldsymbol{L}_{[2]})\varepsilon^{2j-2}z_j^2\\&+\mu_2^{\tilde{\boldsymbol{\chi}}}(L_1)\|I_{\varepsilon}\tilde{\boldsymbol{\chi}}\|^2+\mu_2^{\boldsymbol{\xi}}(L_1)\|I_{\varepsilon}\boldsymbol{\xi}\|^2,\end{aligned}$$

where $\mu_2^{z_1}(\boldsymbol{L}_{[2]})=\frac{21d_{\boldsymbol{\lambda}}}{b_N^2g_N^2}(\gamma_2^{\lambda_1}(L_1))^2+7(\gamma_2^{\lambda_2}(L_1))^2\mu_1^{z_1}(L_1)+7d_{\boldsymbol{\lambda}}(\gamma_2^{y}(L_1))^2$, $\mu_2^{z_2}(\boldsymbol{L}_{[2]})=7d_{\boldsymbol{\lambda}}\alpha_2^2(\boldsymbol{L}_{[2]})+35d_{\boldsymbol{\lambda}}(\gamma_2^{\lambda_2}(L_1))^2$, $\mu_2^{\tilde{\boldsymbol{\chi}}}(L_1)=\frac{21d_{\boldsymbol{\lambda}}}{b_N^2g_N^2}(\gamma_2^{\lambda_1}(L_1))^2+7(\gamma_2^{\lambda_2}(L_1))^2\mu_1^{\tilde{\boldsymbol{\chi}}}$, $\mu_2^{\boldsymbol{\xi}}(L_1)=\frac{21d_{\boldsymbol{\lambda}}}{b_N^2g_N^2}(\gamma_2^{\lambda_1}(L_1))^2+7(\gamma_2^{\lambda_2}(L_1))^2\mu_1^{\boldsymbol{\xi}}+\max\{7d_{\boldsymbol{\lambda}}(\gamma_2^{\xi_j}(L_1))^2,\ j=1,2\}$.

By substituting this and (37) into (36), we have

$$\begin{aligned}\dot{V}_2\ \leq\ & -\varepsilon^{2\rho-1}\left(\tfrac{3}{8\varepsilon}-2\|P_{\boldsymbol{l}}\|^2\right)\|\boldsymbol{\eta}\|^2-\left(\tfrac{1}{8\varepsilon}-d_{2,1}(L_1)\right)\\&\cdot\|I_{\varepsilon}\tilde{\boldsymbol{\chi}}\|^2-\left(\tfrac{1}{4\varepsilon}-d_{2,2}(L_1)\right)\|I_{\varepsilon}\boldsymbol{\xi}\|^2-\Big(\tfrac{L_1-2}{\varepsilon}\\&-d_{2,3}(\boldsymbol{L}_{[2]})\Big)z_1^2-\left(\frac{L_2}{\varepsilon}-d_{2,4}(\boldsymbol{L}_{[2]})\right)\varepsilon^2z_2^2\\&+d_{\boldsymbol{\lambda}}\sum_{i=4}^{\rho}\varepsilon^{2i-2}\lambda_i^2+7d_{\boldsymbol{\lambda}}\varepsilon^4z_3^2+\varepsilon^2z_2z_3,\end{aligned}\tag{38}$$

where $d_{2,1}(L_1)=d_{1,1}+\mu_2^{\tilde{\boldsymbol{\chi}}}(L_1)$, $d_{2,2}(L_1)=d_{1,2}+\mu_2^{\boldsymbol{\xi}}(L_1)$, $d_{2,3}(\boldsymbol{L}_{[2]})=d_{1,3}(L_1)+2+\mu_2^{z_1}(\boldsymbol{L}_{[2]})$, and $d_{2,4}(\boldsymbol{L}_{[2]})=\mu_2^{z_2}(\boldsymbol{L}_{[2]})$.

Inductive Step. Suppose at step $k-1(k=3,\,4,\,\ldots,\,\rho)$, there exists a smooth, positive definite and proper function $V_{k-1}(\boldsymbol{\eta},\,\tilde{\boldsymbol{\chi}},\,\boldsymbol{\xi},\,z_1,\,\ldots,\,z_{k-1})$ whose time derivative satisfies

$$\begin{aligned}\dot{V}_{k-1}\leq & -\varepsilon^{2\rho-1}\left(\frac{3}{8\varepsilon}-2\|P_{\boldsymbol{l}}\|^2\right)\|\boldsymbol{\eta}\|^2-\Big(\frac{1}{2^k\varepsilon}-d_{k-1,1}\\&(\boldsymbol{L}_{[k-2]})\Big)\|I_{\varepsilon}\tilde{\boldsymbol{\chi}}\|^2-\left(\tfrac{1}{4\varepsilon}-d_{k-1,2}(\boldsymbol{L}_{[k-2]})\right)\|I_{\varepsilon}\boldsymbol{\xi}\|^2\\&-\sum_{j=1}^{2}\left(\frac{L_j-(k-j)}{\varepsilon}-d_{k-1,j+2}(\boldsymbol{L}_{[k-1]})\right)\varepsilon^{2j-2}z_j^2\\&-\sum_{j=3}^{k-2}\left(\frac{L_j-1}{\varepsilon}-d_{k-1,j+2}(\boldsymbol{L}_{[k-1]})\right)\varepsilon^{2j-2}z_j^2\\&-\Big(\frac{L_{k-1}}{\varepsilon}-d_{k-1,k+1}(\boldsymbol{L}_{[k-1]})\Big)\varepsilon^{2k-4}z_{k-1}^2\\&+d_{\boldsymbol{\lambda}}\sum_{i=k+1}^{\rho}\varepsilon^{2i-2}\lambda_i^2+(2k+1)d_{\boldsymbol{\lambda}}\varepsilon^{2k-2}z_k^2\\&+\varepsilon^{2k-4}z_{k-1}z_k,\end{aligned}\tag{39}$$

where $z_1=y$, $z_i=\lambda_i-\lambda_i^*$, $i=2,\,3,\,\ldots,\,k$, and λ_i^*'s are a set of virtual controllers in the following forms:

$$\begin{aligned}\lambda_i^*=&-\varepsilon^{-1}\alpha_{i-1}(\boldsymbol{L}_{[i-1]})z_{i-1}-\sum_{j=1}^{i-1}\gamma_{i-1}^{\lambda_j}(\boldsymbol{L}_{[i-2]})\varepsilon^{-(i-j)}\lambda_j\\&-\sum_{j=1}^{i-1}\gamma_{i-1}^{\xi_j}(\boldsymbol{L}_{[i-2]})\varepsilon^{-(i-j)}\xi_j-\gamma_{i-1}^{y}(L_1)\varepsilon^{-(i-1)}y.\end{aligned}\tag{40}$$

From (40), we can find nonnegative functions $\mu_{i-1}^{z_j}(\cdot)$, $\mu_{i-1}^{\tilde{\boldsymbol{\chi}}}(\cdot)$, $\mu_{i-1}^{\boldsymbol{\xi}}(\cdot)$, $i=2,\,3,\,\ldots,\,k$, $j=1,\,\ldots,\,i-1$, such that

$$\begin{aligned}d_{\boldsymbol{\lambda}}\varepsilon^{2i-2}\lambda_i^2\leq&(2i+1)d_{\boldsymbol{\lambda}}\varepsilon^{2i-2}z_i^2+\sum_{j=1}^{i-1}\mu_{i-1}^{z_j}(\boldsymbol{L}_{[i-1]})\varepsilon^{2j-2}z_j^2\\&+\mu_{i-1}^{\tilde{\boldsymbol{\chi}}}(\boldsymbol{L}_{[i-2]})\|I_{\varepsilon}\tilde{\boldsymbol{\chi}}\|^2+\mu_{i-1}^{\boldsymbol{\xi}}(\boldsymbol{L}_{[i-2]})\|I_{\varepsilon}\boldsymbol{\xi}\|^2.\end{aligned}\tag{41}$$

The dynamics of variable $z_i\,(i=2,\,3,\,\ldots,\,k-1)$ can be immediately computed from (40):

$$\begin{aligned}\dot{z}_i\ =\ &\lambda_{i+1}+\sum_{j=1}^{i}\gamma_i^{\lambda_j}(\boldsymbol{L}_{[i-1]})\varepsilon^{-(i+1-j)}\lambda_j\\&+\sum_{j=1}^{i}\gamma_i^{\xi_j}(\boldsymbol{L}_{[i-1]})\varepsilon^{-(i+1-j)}\xi_j+\gamma_i^{y}(L_1)\varepsilon^{-i}y\\&+\gamma_i^{\Delta}(\boldsymbol{L}_{[i-1]})\varepsilon^{-(i-1)}(c_{\rho,1}y+b'_m\lambda_2+\tilde{\chi}_2+\Phi_{\rho,1}).\end{aligned}$$

In what follows, we will show that the above statements still hold at step k. For this aim, choose $V_k=V_{k-1}+\frac{1}{2}\varepsilon^{2k-2}z_k^2$, where $z_k=\lambda_k-\lambda_k^*$ and the virtual controller λ_k^* are smooth functions. For notational convenience and consistency, let $\lambda_{\rho+1}=u$. Then, computing the time derivative of V_k, we have

$$\dot{V}_k\ \leq\ \dot{V}_{k-1}+\varepsilon^{2k-2}z_k\dot{z}_k.\tag{42}$$

By the definition of z_k, we have

$$\begin{aligned}\dot{z}_k=&\lambda_{k+1}+\sum_{j=1}^{k}\gamma_k^{\lambda_j}(\boldsymbol{L}_{[k-1]})\varepsilon^{-(k+1-j)}\lambda_j\\&+\sum_{j=1}^{k}\gamma_k^{\xi_j}(\boldsymbol{L}_{[k-1]})\varepsilon^{-(k+1-j)}\xi_j+\gamma_k^{y}(L_1)\varepsilon^{-k}y\\&+\gamma_k^{\Delta}(\boldsymbol{L}_{[k-1]})\varepsilon^{-(k-1)}(c_{\rho,1}y+b'_m\lambda_2+\tilde{\chi}_2+\Phi_{\rho,1}),\end{aligned}\tag{43}$$

where $\gamma_k^{\lambda_j}(\boldsymbol{L}_{[k-1]}),\gamma_k^{\xi_j}(\boldsymbol{L}_{[k-1]}),\gamma_k^{y}(L_1)$ and $\gamma_k^{\Delta}(\boldsymbol{L}_{[k-1]}),j=1,\ldots,k$ are defined as

$$\begin{aligned}\gamma_k^{\lambda_1}(\boldsymbol{L}_{[k-1]})=&\,l_k+\sum_{j=1}^{k-1}l_j\gamma_{k-1}^{\lambda_j}(\boldsymbol{L}_{[k-2]})\\&+\alpha_{k-1}(\boldsymbol{L}_{[k-1]})\gamma_{k-1}^{\lambda_1}(\boldsymbol{L}_{[k-2]}),\\\gamma_k^{\lambda_j}(\boldsymbol{L}_{[k-1]})=&\,\alpha_{k-1}(\boldsymbol{L}_{[k-1]})\gamma_{k-1}^{\lambda_j}(\boldsymbol{L}_{[k-2]})\\&+\gamma_{k-1}^{\lambda_{j-1}}(\boldsymbol{L}_{[k-2]}),\ j=2,3,\ldots,k-1,\\\gamma_k^{\lambda_k}(\boldsymbol{L}_{[k-1]})=&\,\gamma_{k-1}^{\lambda_{k-1}}(\boldsymbol{L}_{[k-2]})+\alpha_{k-1}(\boldsymbol{L}_{[k-1]}),\\\gamma_k^{\xi_1}(\boldsymbol{L}_{[k-1]})=&\sum_{j=1}^{k-1}l_j\gamma_{k-1}^{\xi_j}(\boldsymbol{L}_{[k-2]})\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad +\alpha_{k-1}(\boldsymbol{L}_{[k-1]})\gamma_{k-1}^{\xi_1}(\boldsymbol{L}_{[k-2]}),\\
\gamma_k^{\xi_2}(\boldsymbol{L}_{[k-1]}) &= \alpha_{k-1}(\boldsymbol{L}_{[k-1]})\gamma_{k-1}^{\xi_2}(\boldsymbol{L}_{[k-2]})\\
&\quad +\gamma_{k-1}^{y}(L_1)+\gamma_{k-1}^{\xi_1}(\boldsymbol{L}_{[k-2]}),\\
\gamma_k^{\xi_j}(\boldsymbol{L}_{[k-1]}) &= \alpha_{k-1}(\boldsymbol{L}_{[k-1]})\gamma_{k-1}^{\xi_j}(\boldsymbol{L}_{[k-2]})\\
&\quad +\gamma_{k-1}^{\xi_{j-1}}(\boldsymbol{L}_{[k-2]}),\ j=2,3,\dots,k-1,\\
\gamma_k^{\xi_k}(\boldsymbol{L}_{[k-1]}) &= \gamma_{k-1}^{\xi_{k-1}}(\boldsymbol{L}_{[k-2]}),\\
\gamma_k^{y}(L_1) &= \alpha_{k-1}(\boldsymbol{L}_{[k-1]})\gamma_{k-1}^{y}(L_1)-\sum_{j=1}^{k-1} l_j\gamma_{k-1}^{\xi_j}(\boldsymbol{L}_{[k-2]}),\\
\gamma_k^{\triangle}(\boldsymbol{L}_{[k-1]}) &= \alpha_{k-1}(\boldsymbol{L}_{[k-1]})\gamma_{k-1}^{\triangle}(\boldsymbol{L}_{[k-2]})+\gamma_{k-1}^{y}(L_1).
\end{aligned}
$$

From the above equations, we can see that $\gamma_k^{\lambda_j}(\cdot),\gamma_k^{\xi_j}(\cdot),\gamma_k^{y}(\cdot)$ and $\gamma_k^{\triangle}(\cdot), j=1,\dots,k$ are continuous functions which can be derived recursively from their initial assignment in Step 2, and become constant once the value of $\boldsymbol{L}_{[k-1]}$ is specified.

Before deriving the virtual controller λ_{k+1}^*, we should eliminate the "undesired" effect of z_k in (42). For this purpose, by the method of completing square, we have

$$
\begin{aligned}
\varepsilon^{2k-4}z_{k-1}z_k &\le \varepsilon^{2k-5}z_{k-1}^2+\tfrac{1}{4}\varepsilon^{2k-3}z_k^2\\
&\le \varepsilon^{2k-5}z_{k-1}^2+\tfrac{1}{4}\max\{1,b_M^2g_M^2\}\varepsilon^{2k-3}z_k^2, \quad (44)
\end{aligned}
$$

and the following inequalities

$$
\begin{cases}
\varepsilon^{k-1}\gamma_k^{\triangle}(\boldsymbol{L}_{[k-1]})c_{\rho,1}yz_k \le \frac{1}{4}d_0^2(\gamma_k^{\triangle}(\boldsymbol{L}_{[k-1]}))^2\varepsilon^{2k-2}z_k^2+y^2,\\
b_m'\varepsilon^{k-1}\gamma_k^{\triangle}(\boldsymbol{L}_{[k-1]})\lambda_2 z_k \le \frac{1}{4}b_M^2g_M^2(\gamma_k^{\triangle}(\boldsymbol{L}_{[k-1]}))^2\varepsilon^{2k-3}z_k^2\\
\qquad +\varepsilon z_2^2+\frac{1}{\varepsilon}y^2+\frac{1}{4}b_M^2g_M^2\alpha_1^2(L_1)\\
\qquad \cdot(\gamma_k^{\triangle}(\boldsymbol{L}_{[k-1]}))^2\varepsilon^{2k-3}z_k^2,\\
\varepsilon^{k-1}\gamma_k^{\triangle}(\boldsymbol{L}_{[k-1]})\tilde{\chi}_2 z_k \le 2^{k-1}(\gamma_k^{\triangle}(\boldsymbol{L}_{[k-1]}))^2\varepsilon^{2k-3}z_k^2\\
\qquad +\frac{1}{2^{k+1}\varepsilon}\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2,\\
\varepsilon^{k-1}\gamma_k^{\triangle}(\boldsymbol{L}_{[k-1]})\Phi_{\rho,1}z_k \le \frac{1}{4}c'^2(\gamma_k^{\triangle}(\boldsymbol{L}_{[k-1]}))^2\varepsilon^{2k-2}z_k^2+y^2.
\end{cases}
\quad (45)
$$

Choose the virtual controller as

$$
\begin{aligned}
\lambda_{k+1}^* = &-\varepsilon^{-1}\alpha_k(\boldsymbol{L}_{[k]})z_k-\sum_{j=1}^{k}\gamma_k^{\lambda_j}(\boldsymbol{L}_{[k-1]})\varepsilon^{-(k+1-j)}\lambda_j\\
&-\sum_{j=1}^{k}\gamma_k^{\xi_j}(\boldsymbol{L}_{[k-1]})\varepsilon^{-(k+1-j)}\xi_j-\gamma_k^{y}(L_1)\varepsilon^{-k}y, \quad (46)
\end{aligned}
$$

where $\alpha_k(\boldsymbol{L}_{[k]})=L_k+\bar{\alpha}_k(\boldsymbol{L}_{[k-1]})$, $\bar{\alpha}_k(\boldsymbol{L}_{[k-1]})=(2k+1)d_{\boldsymbol{\lambda}}+\frac{1}{4}\max\{1,b_M^2g_M^2\}+2^{k-1}(\gamma_k^{\triangle}(\boldsymbol{L}_{[k-1]}))^2+\frac{1}{4}(d_0^2+c'^2)(\gamma_k^{\triangle}(\boldsymbol{L}_{[k-1]}))^2+\frac{1}{4}b_M^2g_M^2(\alpha_1^2(L_1)+1)(\gamma_k^{\triangle}(\boldsymbol{L}_{[k-1]}))^2$, and $L_k\ge 1$ is a constant to be determined later. Note that once the value of $\boldsymbol{L}_{[k]}$ is specified, $\alpha_k(\boldsymbol{L}_{[k]})$ is a known positive constant.

Define $z_{k+1}=\lambda_{k+1}-\lambda_{k+1}^*$ when $k<\rho$ and $z_{\rho+1}=0$. Then, similarly, we have

$$
\begin{aligned}
d_{\boldsymbol{\lambda}}\varepsilon^{2k}\lambda_{k+1}^2 \le &(2k+3)d_{\boldsymbol{\lambda}}\varepsilon^{2k}z_{k+1}^2+(2k+3)d_{\boldsymbol{\lambda}}(\gamma_k^{y}(L_1))^2y^2\\
&+(2k+3)d_{\boldsymbol{\lambda}}\alpha_k^2(\boldsymbol{L}_{[k]})\varepsilon^{2k-2}z_k^2+(2k+3)d_{\boldsymbol{\lambda}}\\
&\cdot\max\left\{(\gamma_k^{\xi_j}(\boldsymbol{L}_{[k-1]}))^2,\ j=1,\dots,k\right\}\|I_\varepsilon\boldsymbol{\xi}\|^2\\
&+(2k+3)d_{\boldsymbol{\lambda}}\sum_{j=1}^{k}(\gamma_k^{\lambda_j}(\boldsymbol{L}_{[k-1]}))^2\varepsilon^{2j-2}\lambda_j^2.
\end{aligned}
\quad (47)
$$

Clearly, the last term on the right-hand side of the above inequality is undesirable and has to be handled. By (41), we have the following inequality

$$
\begin{aligned}
&(2k+3)d_{\boldsymbol{\lambda}}\sum_{j=1}^{k}(\gamma_k^{\lambda_j}(\boldsymbol{L}_{[k-1]}))^2\varepsilon^{2j-2}\lambda_j^2\\
\le &(2k+3)\sum_{j=1}^{k}(\gamma_k^{\lambda_j}(\boldsymbol{L}_{[k-1]}))^2(2j+1)d_{\boldsymbol{\lambda}}\varepsilon^{2j-2}z_j^2\\
&+(2k+3)\sum_{j=1}^{k}(\gamma_k^{\lambda_j}(\boldsymbol{L}_{[k-1]}))^2\sum_{i=1}^{j-1}\mu_{j-1}^{z_i}(\boldsymbol{L}_{[j-1]})\varepsilon^{2i-2}z_i^2\\
&+(2k+3)\sum_{j=1}^{k}(\gamma_k^{\lambda_j}(\boldsymbol{L}_{[k-1]}))^2\mu_{j-1}^{\tilde{\boldsymbol{\chi}}}(\boldsymbol{L}_{[i-2]})\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2\\
&+(2k+3)\sum_{j=1}^{k}(\gamma_k^{\lambda_j}(\boldsymbol{L}_{[k-1]}))^2\mu_{j-1}^{\boldsymbol{\xi}}(\boldsymbol{L}_{[i-2]})\|I_\varepsilon\boldsymbol{\xi}\|^2. \quad (48)
\end{aligned}
$$

Substituting this into (47), we have for $k<\rho$

$$
\begin{aligned}
d_{\boldsymbol{\lambda}}\varepsilon^{2k}\lambda_{k+1}^2 \le &(2k+3)d_{\boldsymbol{\lambda}}\varepsilon^{2k}z_{k+1}^2+\sum_{j=1}^{k}\mu_k^{z_j}(\boldsymbol{L}_{[k]})\varepsilon^{2j-2}z_j^2\\
&+\mu_k^{\tilde{\boldsymbol{\chi}}}(\boldsymbol{L}_{[k-1]})\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2+\mu_k^{\boldsymbol{\xi}}(\boldsymbol{L}_{[k-1]})\|I_\varepsilon\boldsymbol{\xi}\|^2.
\end{aligned}
\quad (49)
$$

Specially, one can see that since $d_{\boldsymbol{\lambda}}\sum_{i=\rho+1}^{\rho}\varepsilon^{2i-2}\lambda_i^2=0$, it is unnecessary for Step ρ to take the computation similar to (49). However, for the sake of the integrality of the inductive steps, we let $\mu_\rho^{z_j}(\boldsymbol{L}_{[\rho]})=0$, $j=1,\dots,\rho$, $\mu_\rho^{\tilde{\boldsymbol{\chi}}}(\boldsymbol{L}_{[\rho-1]})=0$ and $\mu_\rho^{\boldsymbol{\xi}}(\boldsymbol{L}_{[\rho-1]})=0$.

Substituting (43), (44), (45), (46) and (49) into (42), we obtain

$$
\begin{aligned}
\dot{V}_k \le &-\varepsilon^{2\rho-1}\Big(\frac{3}{8\varepsilon}-2\|P_l\|^2\Big)\|\boldsymbol{\eta}\|^2-\Big(\frac{1}{2^{k+1}\varepsilon}-d_{k,1}(\boldsymbol{L}_{[k-1]})\Big)\\
&\cdot\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2-\Big(\frac{1}{4\varepsilon}-d_{k,2}(\boldsymbol{L}_{[k-1]})\Big)\|I_\varepsilon\boldsymbol{\xi}\|^2-\sum_{j=1}^{2}\Big(\frac{L_j}{\varepsilon}\\
&-\frac{(k+1-j)}{\varepsilon}-d_{k,j+2}(\boldsymbol{L}_{[k]})\Big)\varepsilon^{2j-2}z_j^2-\sum_{j=3}^{k-1}\Big(\frac{L_j-1}{\varepsilon}\\
&-d_{k,j+2}(\boldsymbol{L}_{[k]})\Big)\varepsilon^{2j-2}z_j^2-\Big(\frac{L_k}{\varepsilon}-d_{k,k+2}(\boldsymbol{L}_{[k]})\Big)\varepsilon^{2k-2}\\
&\cdot z_k^2+d_{\boldsymbol{\lambda}}\sum_{i=k+2}^{\rho}\varepsilon^{2i-2}\lambda_i^2+(2k+3)d_{\boldsymbol{\lambda}}\varepsilon^{2k}z_{k+1}^2\\
&+\varepsilon^{2k-2}z_kz_{k+1},
\end{aligned}
\quad (50)
$$

where

$$
\begin{cases}
d_{k,1}(\boldsymbol{L}_{[k-1]}) = d_{k-1,1}(\boldsymbol{L}_{[k-2]})+\mu_k^{\tilde{\boldsymbol{\chi}}}(\boldsymbol{L}_{[k-1]}),\\
d_{k,2}(\boldsymbol{L}_{[k-1]}) = d_{k-1,2}(\boldsymbol{L}_{[k-2]})+\mu_k^{\boldsymbol{\xi}}(\boldsymbol{L}_{[k-1]}),\\
d_{k,3}(\boldsymbol{L}_{[k]}) = d_{k-1,3}(\boldsymbol{L}_{[k-1]})+2+\mu_k^{z_1}(\boldsymbol{L}_{[k]}),\\
d_{k,4}(\boldsymbol{L}_{[k]}) = d_{k-1,4}(\boldsymbol{L}_{[k-1]})+\mu_k^{z_2}(\boldsymbol{L}_{[k]}),\\
d_{k,j+2}(\boldsymbol{L}_{[k]}) = d_{k-1,j+2}(\boldsymbol{L}_{[k-1]})+\mu_k^{z_j}(\boldsymbol{L}_{[k]}),\\
\qquad j=3,4,\dots,k-1,\\
d_{k,k+2}(\boldsymbol{L}_{[k]}) = \mu_k^{z_k}(\boldsymbol{L}_{[k]}).
\end{cases}
$$

From the above equations, we can see that $d_{k,i}(\cdot)$, $k=3,4,\dots,\rho$, $i=1,\dots,k+2$ are continuous functions which can be derived recursively from their initial assignment in step 2, and are undoubtedly constant once the value of $\boldsymbol{L}_{[k]}$ is specified. By more detailed analysis, we can see that they are all positive except $d_{\rho,\rho+2}(\boldsymbol{L}_{[\rho]})=0$.

At the last step, the design of the controller u is slightly different from that of other steps, because all the junk terms, i.e., $\lambda_i^2, 1 \le i \le \rho$ in (39), have already been canceled at step $\rho - 1$. Using the inductive procedure and letting $k = \rho$ in (46), we can design the controller as follows:

$$u = \lambda_{\rho+1} = \lambda_{\rho+1}^*. \tag{51}$$

Accordingly, letting $k = \rho$ in (50) and noting that $d_{\boldsymbol{\lambda}} \sum_{i=\rho+1}^{\rho} \varepsilon^{2i-2}\lambda_i^2 = 0$, $z_{\rho+1} = 0$, we have

$$\begin{aligned}\dot{V}_\rho \le & -\varepsilon^{2\rho-1}\big(\frac{3}{8\varepsilon} - 2\|P_{\boldsymbol{l}}\|^2\big)\|\boldsymbol{\eta}\|^2 - \Big(\frac{1}{2^{\rho+1}\varepsilon} - d_{\rho,1}(\boldsymbol{L}_{[\rho-1]})\Big) \\ & \cdot \|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2 - \Big(\frac{1}{4\varepsilon} - d_{\rho,2}(\boldsymbol{L}_{[\rho-1]})\Big)\|I_\varepsilon\boldsymbol{\xi}\|^2 - \sum_{j=1}^{2}\Big(\frac{L_j}{\varepsilon} \\ & - \frac{(\rho+1-j)}{\varepsilon} - d_{\rho,j+2}(\boldsymbol{L}_{[\rho]})\Big)\varepsilon^{2j-2}z_j^2 - \sum_{j=3}^{\rho-1}\Big(\frac{L_j-1}{\varepsilon}\Big) \\ & - d_{\rho,j+2}(\boldsymbol{L}_{[\rho]})\varepsilon^{2j-2}z_j^2 - \Big(\frac{L_\rho}{\varepsilon} - d_{\rho,\rho+2}(\boldsymbol{L}_{[\rho]})\Big)\varepsilon^{2\rho-2}z_\rho^2,\end{aligned} \tag{52}$$

where $V_\rho(\boldsymbol{\eta}, \tilde{\boldsymbol{\chi}}, \boldsymbol{\xi}, z_1, \dots, z_\rho)$ is a positive definite and proper function defined by

$$V_\rho = V_0 + \frac{1}{2}\sum_{i=1}^{\rho}\varepsilon^{2i-2}z_i^2. \tag{53}$$

3.3 Main results

From (52), we can see that to realize stabilization of the closed-loop system, the controller designed should ensure the negative definiteness of the time derivative of V_ρ. This can be guaranteed by choosing positive constants ε and L_i's such that

$$\begin{cases} \frac{3}{8\varepsilon} - 2\|P_{\boldsymbol{l}}\|^2 > 0,\ \frac{1}{2^{\rho+1}\varepsilon} - d_{\rho,1}(\boldsymbol{L}_{[\rho-1]}) > 0, \\ \frac{1}{4\varepsilon} - d_{\rho,2}(\boldsymbol{L}_{[\rho-1]}) > 0, \\ \frac{L_i-(\rho+1-i)}{\varepsilon} - d_{\rho,i+2}(\boldsymbol{L}_{[\rho]}) > 0, \\ \frac{L_j-1}{\varepsilon} - d_{\rho,j+2}(\boldsymbol{L}_{[\rho]}) > 0,\ \frac{L_\rho}{\varepsilon} - d_{\rho,\rho+2}(\boldsymbol{L}_{[\rho]}) > 0, \end{cases} \tag{54}$$

for $i = 1, 2$ and $j = 3, 4, \dots, \rho - 1$.

The following lemma shows the existence of positive constants ε and L_i's satisfying (54). Besides, the choice of such constants is discussed in the proof of the lemma.

Lemma 3. There are always positive constants $L_i \ge 1$, $i = 1, \dots, \rho$ and $0 < \varepsilon < 1$ satisfying inequalities (54).

Proof. We prove this lemma by construction.

Firstly, for any specified constants $L_i^* > 0$, $i = 1, \dots, \rho - 1$ and $L_\rho^* \ge 1$, choose

$$\begin{cases} L_i \ \ge\ L_i^* + \rho + 1 - i, \quad i = 1, 2, \\ L_j \ \ge\ L_j^* + 1, \quad j = 3, 4, \dots, \rho - 1, \\ L_\rho \ \ge\ L_\rho^*. \end{cases} \tag{55}$$

Secondly, choose ε such that

$$\begin{aligned} 0 < \varepsilon < \varepsilon^* = \min\Big\{ & 1, \frac{3}{16\|P_{\boldsymbol{l}}\|^2}, \frac{1}{2^{\rho+1}d_{\rho,1}(\boldsymbol{L}_{[\rho-1]})}, \\ & \frac{1}{4d_{\rho,2}(\boldsymbol{L}_{[\rho-1]})}, \frac{L_i^*}{d_{\rho,i+2}(\boldsymbol{L}_{[\rho]})}, i = 1, \dots, \rho\Big\}. \end{aligned} \tag{56}$$

It is easy to verify that any L_i's and ε determined by (55) and (56) satisfy inequalities (54). □

The main result of the paper is summarized as follows:

Theorem 1. Consider the output-feedback control problem of system (1) with $\rho < n$. Suppose the system satisfies Assumptions 1∼4. If $\boldsymbol{l} = [l_1, \dots, l_\rho]^{\mathrm{T}}$ is chosen such that matrix $A_{\boldsymbol{l}} = A + \boldsymbol{l}\boldsymbol{e}_1^{\mathrm{T}}$ is Hurwitz, then the closed-loop system is globally asymptotically stable under the dynamic output-feedback control (51) with positive constants ε and L_i's satisfying (55) and (56).

Proof. First of all, observing that ε and L_i's satisfy (55) and (56), it is straightforward to deduce from (53) and (52) that there is a positive constant β such that $\dot{V}_\rho \le -\beta V_\rho$, which implies that $\boldsymbol{\eta}$, $\varepsilon^{i-1}\tilde{\chi}_i$, $\varepsilon^{i-1}\xi_i$, and $\varepsilon^{2i-2}z_i$ are globally asymptotically stable for $i = 1, \dots, \rho$, and so are $\boldsymbol{\eta}$, $\tilde{\boldsymbol{\chi}}$, $\boldsymbol{\xi}$, and z_i's since ε is a positive constant. This together with the fact $y = z_1 = \chi_1 = x_1$ concludes that the global asymptotic stability of y, χ_1 and x_1, and hence λ_1 since $\lambda_1 = \frac{1}{b_m'}(y - \tilde{\chi}_1 - \xi_1)$. Then from (29) and $\lambda_2 = z_2 + \lambda_2^*$, it follows that λ_2^* and λ_2 are globally asymptotically stable. Continuing in the same fashion, (40) and $\lambda_i = z_i + \lambda_i^*$ for $i = 3, 4, \dots, \rho$ recursively establish that $\boldsymbol{\lambda}$ is globally asymptotically stable. By the global asymptotic stability of $\boldsymbol{\xi}$, $\boldsymbol{\lambda}$ and $\tilde{\boldsymbol{\chi}}$, and $\hat{\boldsymbol{\chi}} = \boldsymbol{\xi} + b_m'\boldsymbol{\lambda}$, we know that $\hat{\boldsymbol{\chi}}$ and hence $\boldsymbol{\chi}$ $(= \hat{\boldsymbol{\chi}} + \tilde{\boldsymbol{\chi}})$ are globally asymptotically stable as well.

Finally, from $\boldsymbol{\varsigma}_{\rho,[1,\rho]} = \boldsymbol{\chi}$, $\boldsymbol{\varsigma}_{\rho,[\rho+1,n]} = \boldsymbol{\eta}$, and the equivalent transformations defined above, we conclude that $\boldsymbol{\varsigma}$, $\boldsymbol{x}$ and $\boldsymbol{\zeta}$ are all globally asymptotically stable. □

4 Further study and numerical simulation

It is easily seen that the design procedure given in the previous section cannot be unchangeably applied to the case $\rho = n$. In fact, there are minor differences between the control design procedures of the cases $\rho = n$ and $\rho < n$. Therefore, we would like to study a simple example, rather than a general system, which can adequately demonstrate the major characteristics of output-feedback control design for system (1) without zero-dynamics.

Consider the following second-order nonlinear system

$$\dot{\zeta}_1 = g_1\zeta_2 + 0.1\zeta_1\sin\zeta_2, \quad \dot{\zeta}_2 = g_2b_0u, \quad y = \zeta_1, \tag{57}$$

which satisfies Assumptions 1∼3 with $c = 0.1$, $0.05 \le |g_1| \le 1$, $1 \le |g_2| \le 2$, $0.5 \le |b_0| \le 1$, and the signs of g_1, g_2 and b_0 are assumed to be positive.

As discussed earlier, it is enough to apply the first kind of transformation given in Section 3 to system (57) since no zero-dynamics exist in this case. After transformation and defining $\boldsymbol{\chi} = \boldsymbol{x}$, we obtain the following new system:

$$\dot{\boldsymbol{\chi}} = A_2\boldsymbol{\chi} + b_0'\boldsymbol{e}_{2,2}u + \boldsymbol{\Phi}_{2,[2]}(t, \boldsymbol{x}, u),\ \chi_1 = \boldsymbol{e}_{1,\rho}^{\mathrm{T}}\boldsymbol{\chi} = y, \tag{58}$$

where $b_0' = b_0g_1g_2$ with positive sign, and $A_2 = \big[[0, 0]^{\mathrm{T}}, [1, 0]^{\mathrm{T}}\big]$, $\boldsymbol{\Phi}_{2,[2]} = \big[0.1\chi_1\sin\frac{\chi_2}{g_1}, 0\big]^{\mathrm{T}}$.

According to Subsection 4.1, we obtain the whole system for control design (25) with $\rho = n = 2$. Choose $\boldsymbol{l} = [-1, -0.8]^{\mathrm{T}}$ such that matrix $A_{\boldsymbol{l}}$ is Hurwitz.

Let $V_{\tilde{\boldsymbol{\chi}}} = \tilde{\boldsymbol{\chi}}^{\mathrm{T}}P_{\boldsymbol{l}_\varepsilon}\tilde{\boldsymbol{\chi}}$. Then, the time derivative of $V_{\tilde{\boldsymbol{\chi}}}$ along the trajectories of $\dot{\tilde{\boldsymbol{\chi}}}$ satisfies:

$$\dot{V}_{\tilde{\boldsymbol{\chi}}} \le -\tfrac{1}{\varepsilon}\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2 + 2\|I_\varepsilon\tilde{\boldsymbol{\chi}}\| \cdot \|P_{\boldsymbol{l}}\| \cdot \|I_\varepsilon\boldsymbol{\Phi}_{2,[2]}\|_1.$$

As before, noting that $\chi_1 = y$, we have $2\|I_\varepsilon\tilde{\boldsymbol{\chi}}\| \cdot \|P_{\boldsymbol{l}}\| \cdot \|I_\varepsilon\boldsymbol{\Phi}_{2,[2]}\|_1 \le \frac{1}{4}\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2 + 4c^2\|P_{\boldsymbol{l}}\|^2y^2$. Then we have

$$\dot{V}_{\tilde{\boldsymbol{\chi}}} \le -\big(\tfrac{1}{\varepsilon} - \tfrac{1}{4}\big)\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2 + 4c^2\|P_{\boldsymbol{l}}\|^2y^2. \tag{59}$$

Let's next turn to the output-feedback control design which will be completed in two steps.

Step 1. Let $V_1 = V_{\tilde{\boldsymbol{\chi}}} + V_{\boldsymbol{\xi}} + \frac{1}{2}y^2$ for this design step, where $V_{\boldsymbol{\xi}} = \boldsymbol{\xi}^{\rm T} P_{\boldsymbol{l}_\varepsilon}\boldsymbol{\xi}$. Then, by Proposition 4 and (59), we have

$$\begin{aligned}\dot{V}_1 \ \ = \ \ & -\left(\tfrac{1}{\varepsilon}-1\right)\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2 - \tfrac{1}{2\varepsilon}\|I_\varepsilon\boldsymbol{\xi}\|^2 + \Big(c^2\|P_{\boldsymbol{l}}\|^2 \\ & + \tfrac{2\|P_{\boldsymbol{l}}\|^2\|\boldsymbol{l}\|^2}{\varepsilon}\Big)y^2 + y(b_0'\lambda_2 + \tilde{\chi}_2 + \xi_2 + \Phi_{2,1}). \quad (60)\end{aligned}$$

Notice that $y\tilde{\chi}_2 \le \frac{1}{2\varepsilon}\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2 + \frac{y^2}{2\varepsilon}$, $y\xi_2 \le \frac{1}{4\varepsilon}\|I_\varepsilon\boldsymbol{\xi}\|^2 + \frac{y^2}{\varepsilon}$, $y\Phi_{2,1} \le cy^2$, for any $\boldsymbol{\xi}$, y, $\tilde{\boldsymbol{\chi}}$. Substituting this into (60) results in

$$\begin{aligned}\dot{V}_1 \ \ \le \ \ & -\left(\tfrac{1}{2\varepsilon}-1\right)\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2 - \tfrac{1}{4\varepsilon}\|I_\varepsilon\boldsymbol{\xi}\|^2 + \Big(c^2\|P_{\boldsymbol{l}}\|^2 \\ & + c + \tfrac{3+4\|P_{\boldsymbol{l}}\|^2\|\boldsymbol{l}\|^2}{2\varepsilon}\Big)y^2 + b_0' y\lambda_2. \qquad (61)\end{aligned}$$

Noting $\mathrm{sign}(b_0') = 1$, we choose the virtual controller as

$$\lambda_2^* = -\tfrac{L_1+\bar{\alpha}_1}{\varepsilon}y =: -\tfrac{\alpha_1(L_1)}{\varepsilon}y, \qquad (62)$$

where $\bar{\alpha}_1 = c^2\|P_{\boldsymbol{l}}\|^2 + c + \frac{3}{2} + 2\|P_{\boldsymbol{l}}\|^2\|\boldsymbol{l}\|^2$ is clearly a known constant, and $L_1 \ge 1$ is a constant to be determined later.

Define $z_1 = y$ and $z_2 = \lambda_2 - \lambda_2^*$. Then, substituting (62) into (61) results in

$$\dot{V}_1 \le -\left(\tfrac{1}{2\varepsilon}-1\right)\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2 - \tfrac{1}{4\varepsilon}\|I_\varepsilon\boldsymbol{\xi}\|^2 - \tfrac{L_1}{\varepsilon}y^2 + b_0' y z_2. \quad (63)$$

Step 2. Let $V_2 = V_1 + \frac{1}{2}\varepsilon^2 z_2^2$. Then, by (63), the time derivative of V_2 satisfies

$$\begin{aligned}\dot{V}_2 \ \ \le \ \ & -\left(\tfrac{1}{2\varepsilon}-1\right)\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2 - \tfrac{1}{4\varepsilon}\|I_\varepsilon\boldsymbol{\xi}\|^2 \\ & -\tfrac{L_1}{\varepsilon}y^2 + b_0' y z_2 + \varepsilon^2 z_2\dot{z}_2. \qquad (64)\end{aligned}$$

By the definition of z_2, we have

$$\dot{z}_2 = u + l_2\varepsilon^{-2}\lambda_1 + \alpha_1(L_1)\varepsilon^{-1}(b_0'\lambda_2 + \tilde{\chi}_2 + \xi_2 + \Phi_{2,1}). \ (65)$$

As before, we have

$$\begin{cases} b_0' y z_2 & \le \ \frac{1}{4}\varepsilon^{-1}y^2 + b_M^2 g_M^2 \varepsilon z_2^2, \\ b_0'\varepsilon\alpha_1(L_1)\lambda_2 z_2 & \le \ \big(b_M g_M|\alpha_1(L_1)| + \frac{1}{2}b_M^2 g_M^2\alpha_1^4(L_1)\big) \\ & \quad \cdot\varepsilon z_2^2 + \frac{1}{2}\varepsilon^{-1}y^2, \\ \varepsilon\alpha_1(L_1)\tilde{\chi}_2 z_2 & \le \ \frac{1}{4\varepsilon}\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2 + \alpha_1^2(L_1)\varepsilon z_2^2, \\ \varepsilon\alpha_1(L_1)\Phi_{2,1}z_2 & \le \ y^2 + \frac{c^2}{4}\alpha_1^2(L_1)\varepsilon^2 z_2^2. \end{cases}$$

Substituting this and (65) into (64), we have

$$\begin{aligned}\dot{V}_2 \ \ \le \ \ & -\left(\tfrac{1}{4\varepsilon}-1\right)\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2 - \tfrac{1}{4\varepsilon}\|I_\varepsilon\boldsymbol{\xi}\|^2 - \Big(\tfrac{L_1-\frac{3}{4}}{\varepsilon}-1\Big)y^2 \\ & + \varepsilon^2 z_2\Big(l_2\varepsilon^{-2}\lambda_1 + \alpha_1(L_1)\varepsilon^{-1}\xi_2 + \big(\varepsilon^{-1}(b_M g_M \\ & \cdot|\alpha_1(L_1)| + \alpha_1^2(L_1) + \tfrac{1}{2}b_M^2 g_M^2(\alpha_1^4(L_1)+2)) \\ & + \frac{c^2}{4}\alpha_1^2(L_1)\big)z_2\Big) + \varepsilon^2 z_2 u. \qquad (66)\end{aligned}$$

Thus, we can choose the controller as

$$u = -\varepsilon^{-1}\alpha_2(\boldsymbol{L}_{[2]})z_2 - l_2\varepsilon^{-2}\lambda_1 - \alpha_1(L_1)\varepsilon^{-1}\xi_2, \qquad (67)$$

where $\alpha_2(\boldsymbol{L}_{[2]}) = L_2 + \bar{\alpha}_2(L_1)$, $\bar{\alpha}_2(L_1) = b_M g_M|\alpha_1(L_1)| + \alpha_1^2(L_1) + \frac{1}{2}b_M^2 g_M^2(\alpha_1^4(L_1)+2) + \frac{c^2}{4}\alpha_1^2(L_1)$, and $L_2 \ge 1$ is a constant to be determined later.

Substituting (67) into (66) concludes that

$$\begin{aligned}\dot{V}_2 \ \ \le \ \ & -\left(\tfrac{1}{4\varepsilon}-1\right)\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2 - \tfrac{1}{4\varepsilon}\|I_\varepsilon\boldsymbol{\xi}\|^2 \\ & -\Big(\tfrac{L_1-\frac{3}{4}}{\varepsilon}-1\Big)y^2 - L_2\varepsilon z_2^2. \qquad (68)\end{aligned}$$

Choosing $L_1 \ge 1$, $L_2 \ge 1$ and $0 < \varepsilon < \varepsilon^* = \min\left\{1, \frac{1}{4}, L_1 - \frac{3}{4}\right\}$, we can see that $\dot{V}_2$ is negative definite, and hence the closed-loop system is globally asymptotically stable. With respect to system (57) and the output-feedback controller designed above, we next give the corresponding numerical simulation to show the effectiveness of the methods.

Let the initial value of the state be $\boldsymbol{\zeta}_0 = [0, 1]^{\rm T}$, $\boldsymbol{\xi}(0) = 0$ and $\boldsymbol{\lambda}(0) = 0$, and the system parameters be $g_1 = 0.05$, $g_2 = 1$ and $b_0 = 1$. Choosing $L_1 = L_2 = 1$ and $\varepsilon = 0.249$, we obtain Fig.1$\sim$Fig.4. These figures show that all the closed-loop states are indeed asymptotically stable.

5 Concluding Remarks

In this paper, the output-feedback stabilization has been investigated for a class of uncertain nonlinear systems. After introducing two kinds of linear state transformations, the control design becomes much natural since the converted system has known virtual control coefficients and separated stable zero-dynamics. For the new transformed system, an appropriate high-gain observer based on K-filters is introduced, and then by the backstepping approach, the output-feedback controller is successfully designed. It is shown that the global asymptotic stability of the closed-loop system can be guaranteed by the appropriate choice of the design parameters.

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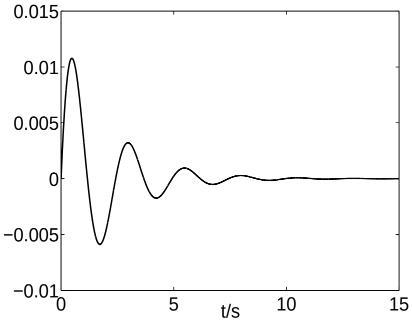

Fig. 1 System state ζ_1

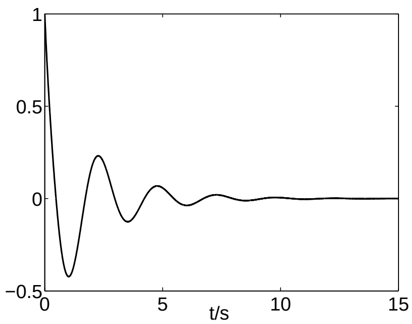

Fig. 2 System state ζ_2

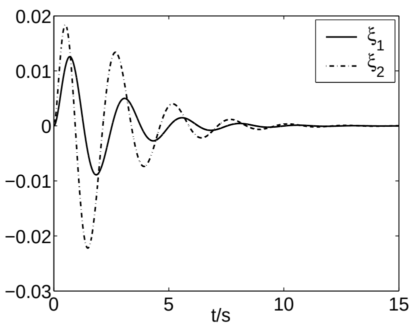

Fig. 3 State $\boldsymbol{\xi}$ of high-gain K-filters

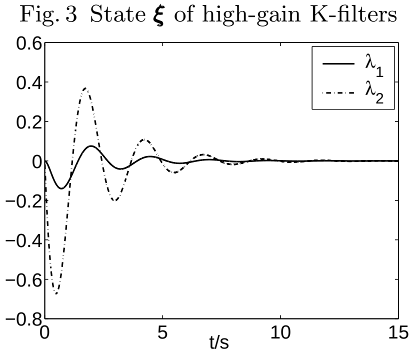

Fig. 4 State $\boldsymbol{\lambda}$ of high-gain K-filters

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SHANG Fang Ph.D candidate in the School of Control Science and Engineering at Shandong University. Her current research mainly includes nonlinear output feedback control, adaptive control theory, etc.
E-mail: shangfang929@163.com

LIU Yun-Gang Professor with the School of Control Science and Engineering, Shandong University. His research interests include stochastic control, nonlinear system analysis and adaptive control, etc. Corresponding author of this paper.
E-mail: lygfr@sdu.edu.cn

Appendix

In this appendix, we give proofs of Lemma 1, Propositions 2 and 3, respectively.

A.1 Proof of Lemma 1

It is clear that Q defined by (14) exists since it is the unique solution of the Lyapunov function: $D^{\mathrm{T}}Q + QD = -I$.

Because that D is Hurwitz and $\mathrm{Re}(\lambda(D)) \leq -\frac{d}{2}$, we can conclude that $D^{\mathrm{T}} + D \leq -dI$. Let $\lambda_i,\ i = 1, \dots, n$ be the eigenvalues of $D^{\mathrm{T}} + D$. Then there exists an orthogonal matrix P (i.e., $P^{-1} = P^T$) such that for $\forall t \in \mathbf{R}^+$,

$$\exp(D^{\mathrm{T}}t + Dt) \quad = \quad P\exp(\mathrm{diag}[\lambda_1, \cdots, \lambda_n]t)P^{-1}$$

$$= P\text{diag}[\exp(\lambda_1 t), \cdots, \exp(\lambda_n t)]P^{-1}.$$

By $\text{Tr}(D^{\text{T}}) = \text{Tr}(D)$, $\text{Tr}(D^{\text{T}}D) = \text{Tr}(DD^{\text{T}})$ and the expansion in series of the exponential matrix, it is easy to show that for $\forall t \in \mathbf{R}^+$, $\text{Tr}\left(\exp(D^{\text{T}}t)\exp(Dt)\right) = \text{Tr}\left(\exp(D^{\text{T}}t + Dt)\right)$. From this, it follows that

$$\begin{aligned}\text{Tr}(Q) &= \text{Tr}\left(\int_0^\infty \exp(D^{\text{T}}t)\exp(Dt)\text{d}t\right)\\ &= \int_0^\infty \text{Tr}\left(\exp(D^{\text{T}}t)\exp(Dt)\right)\text{d}t\\ &= \int_0^\infty \text{Tr}\left(\exp(D^{\text{T}}t + Dt)\right)\text{d}t\\ &= \int_0^\infty \text{Tr}\left(P\text{diag}[\exp(\lambda_1 t), \cdots, \exp(\lambda_n t)]P^{-1}\right)\text{d}t\\ &= \int_0^\infty \sum_{i=1}^n \exp(\lambda_i t)\text{d}t \ \le\ \frac{n}{d}.\end{aligned}$$

This together with $Q > 0$ concludes that $\|Q\| \le \|Q\|_F = \sqrt{\text{Tr}(Q^{\text{T}}Q)} = \sqrt{\text{Tr}(Q^2)} \le \text{Tr}(Q) \le \frac{n}{d}$. □

A.2 Proof of Proposition 2

Along the trajectories of the subsystem $\Sigma_{\boldsymbol{\eta}}$, the time derivative of $V_{\boldsymbol{\eta}}$ satisfies

$$\begin{aligned}\dot{V}_{\boldsymbol{\eta}} \ \le\ & -\|\boldsymbol{\eta}\|^2 + 2\|Q\|\cdot\|\boldsymbol{c}_{\rho,[\rho+1,n]}\|\cdot\|\boldsymbol{\eta}\|\cdot|\chi_1|\\ & +2\|\boldsymbol{\eta}\|\cdot\|Q\|\cdot\|\boldsymbol{\Phi}_{\rho,[\rho+1,n]}\|_1.\end{aligned} \tag{A1}$$

For the second term on the right-hand side of the above inequality, by Lemma 1 and Proposition 1, we have

$$2\|Q\|\cdot\|\boldsymbol{c}_{\rho,[\rho+1,n]}\|\cdot\|\boldsymbol{\eta}\|\cdot|\chi_1| \le \tfrac{1}{8}\|\boldsymbol{\eta}\|^2 + \tfrac{8n^2}{d^2}d_0'^2y^2. \tag{A2}$$

By the fact $0 < \varepsilon < 1$, and Lemmas 1 and 2, we know that

$$\begin{aligned}2\|\boldsymbol{\eta}\|\cdot\|Q\|\cdot\|\boldsymbol{\Phi}_{\rho,[\rho+1,n]}\|_1 \le\ & \tfrac{2mc'n}{d}\varepsilon^{-(\rho-1)}\|\boldsymbol{\eta}\|\big(|\chi_1| + |\varepsilon\chi_2|\\ & + \cdots + |\varepsilon^{\rho-1}\chi_\rho|\big).\end{aligned}$$

From this and $\chi_1 = y$, $\chi_i = \tilde{\chi}_i + \xi_i + b_m'\lambda_i$, $i = 2, 3, \ldots, \rho$, it follows that

$$\begin{aligned}& 2\|\boldsymbol{\eta}\|\cdot\|Q\|\cdot\|\boldsymbol{\Phi}_{\rho,[\rho+1,n]}\|_1\\ \le\ & \tfrac{1}{2}\|\boldsymbol{\eta}\|^2 + \tfrac{8m^2c'^2n^2}{d^2}\varepsilon^{-(2\rho-2)}y^2 + \tfrac{8m^2c'^2n^2\rho}{d^2}\varepsilon^{-(2\rho-2)}\\ & \cdot\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2 + \tfrac{8m^2c'^2n^2\rho}{d^2}\varepsilon^{-(2\rho-2)}\|I_\varepsilon\boldsymbol{\xi}\|^2\\ & + \frac{8m^2c'^2n^2}{d^2}\rho b_M^2 g_M^2\varepsilon^{-(2\rho-2)}\sum_{i=2}^\rho \varepsilon^{2i-2}\lambda_i^2.\end{aligned} \tag{A3}$$

Substituting (A2) and (A3) into (A1), we can get

$$\begin{aligned}\dot{V}_\eta \ \le\ & -\tfrac{3}{8}\|\boldsymbol{\eta}\|^2 + d_{0,1}\left(d_0'^2 + m^2c'^2\varepsilon^{-(2\rho-2)}\right)y^2\\ & +\varepsilon^{-(2\rho-2)}d_{0,2}\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2 + \varepsilon^{-(2\rho-2)}d_{0,2}\|I_\varepsilon\boldsymbol{\xi}\|^2\\ & +\varepsilon^{-(2\rho-2)}d_{0,3}\sum_{i=2}^\rho \varepsilon^{2i-2}\lambda_i^2,\end{aligned} \tag{A4}$$

where $d_{0,1} = \frac{8n^2}{d^2}$, $d_{0,2} = \frac{8m^2c'^2n^2\rho}{d^2}$, $d_{0,3} = \frac{8m^2c'^2n^2}{d^2}\rho b_M^2 g_M^2$. This leads to (21). □

A.3 Proof of Proposition 3

The time derivative of $V_{\tilde{\boldsymbol{\chi}}}$ along (18) satisfies

$$\begin{aligned}\dot{V}_{\tilde{\boldsymbol{\chi}}} \le & -\tfrac{\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2}{\varepsilon} + 2\varepsilon^{\rho-1}\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|\cdot\|P_{\boldsymbol{l}}\|\cdot\|\boldsymbol{\eta}\| + 2\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|\\ & \cdot\|P_{\boldsymbol{l}}\|\cdot\|\boldsymbol{c}_{\rho,[\rho]}\||y| + 2\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|\cdot\|P_{\boldsymbol{l}}\|\cdot\|I_\varepsilon\boldsymbol{\Phi}_{\rho,[\rho]}\|_1.\end{aligned} \tag{A5}$$

We will deal with the last three terms on the right-hand side of the above inequality. Firstly, by Proposition 1, the second term and the third term satisfy respectively

$$\begin{aligned}2\varepsilon^{\rho-1}\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|\cdot\|P_{\boldsymbol{l}}\|\cdot\|\boldsymbol{\eta}\| \ &\le\ \tfrac{\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2}{2\varepsilon} + 2\varepsilon^{2\rho-1}\|P_{\boldsymbol{l}}\|^2\|\boldsymbol{\eta}\|^2,\\ 2\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|\cdot\|P_{\boldsymbol{l}}\|\cdot\|\boldsymbol{c}_{\rho,[\rho]}\|\cdot|y| \ &\le\ \|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2 + d_0^2\|P_{\boldsymbol{l}}\|^2y^2.\end{aligned} \tag{A6}$$

For the last term, it is easy to show that $\|I_\varepsilon\boldsymbol{\Phi}_{\rho,[\rho]}\|_1 \le \rho c'\left(|\chi_1| + \varepsilon|\chi_2| + \cdots + \varepsilon^{\rho-1}|\chi_\rho|\right)$. Then, noting that $\chi_1 = y$ and $\chi_i = \tilde{\chi}_i + \xi_i + b_m'\lambda_i$, $i = 2, 3, \ldots, \rho$, we have

$$\begin{aligned}& 2\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|\cdot\|P_{\boldsymbol{l}}\|\cdot\|I_\varepsilon\boldsymbol{\Phi}_{\rho,[\rho]}\|_1\\ \le\ & (1 + 2\rho^{\frac{3}{2}}c'\|P_{\boldsymbol{l}}\|)\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2 + 3\rho^3c'^2\|P_{\boldsymbol{l}}\|^2\|I_\varepsilon\boldsymbol{\xi}\|^2\\ & +3\rho^2c'^2\|P_{\boldsymbol{l}}\|^2y^2 + 3\rho^3c'^2b_M^2g_M^2\|P_{\boldsymbol{l}}\|^2\sum_{i=2}^\rho \varepsilon^{2i-2}\lambda_i^2.\end{aligned} \tag{A7}$$

Substituting (A6) and (A7) into (A5), we have

$$\begin{aligned}\dot{V}_{\tilde{\boldsymbol{\chi}}} \le & -\left(\tfrac{1}{2\varepsilon} - d_{0,4}\right)\|I_\varepsilon\tilde{\boldsymbol{\chi}}\|^2 + d_{0,5}\|I_\varepsilon\boldsymbol{\xi}\|^2\\ & +2\varepsilon^{2\rho-1}\|P_{\boldsymbol{l}}\|^2\|\boldsymbol{\eta}\|^2 + d_{0,6}y^2 + d_{0,7}\sum_{i=2}^\rho \varepsilon^{2i-2}\lambda_i^2,\end{aligned} \tag{A8}$$

where $d_{0,4} = 2 + 2\rho^{\frac{3}{2}}c'\|P_{\boldsymbol{l}}\|$, $d_{0,5} = 3\rho^3c'^2\|P_{\boldsymbol{l}}\|^2$, $d_{0,6} = \left(d_0^2 + 3\rho^2c'^2\right)\|P_{\boldsymbol{l}}\|^2$, $d_{0,7} = 3\rho^3c'^2b_M^2g_M^2\|P_{\boldsymbol{l}}\|^2$ are known positive constants independent of ε. □