Asymptotically disjoint quantum states*

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Abstract

A clarification of the heuristic concept of decoherence requires a consistent description of the classical behavior of some quantum systems. We adopt algebraic quantum mechanics since it includes not only classical physics, but also permits a judicious concept of a classical mixture and explains the possibility of the emergence of a classical behavior of quantum systems. A nonpure quantum state can be interpreted as a classical mixture if and only if its components are disjoint. Here, two pure quantum states are called disjoint if there exists an element of the center of the algebra of observables such that its expectation values with respect to these states are different. An appropriate automorphic dynamics can transform a factor state into a classical mixture of asymptotically disjoint final states. Such asymptotically disjoint quantum states lead to regular decision problems while exactly disjoint states evoke singular problems which engineers reject as improperly posed.

1 On the classical behavior of quantum systems

Since the first years of quantum mechanics the relation between quantum-theoretical and classical descriptions has been controversial. According to Niels Bohr measuring instruments must be described classically. He justified his view by the remark that a prerequisite of any communication is the possibility of a description of facts in a classical language.[1] Every reasonable statement about an experimental fact has to be either true or false. This condition requires a domain of discourse which has a classical *Boolean* description. In a theoretical description the Boolean character is characterized by the absence of coherent superpositions of states describing this domain. Bohr's requirement reflects the actual scientific practice: *every experiment ever performed in physics, chemistry and biology has a classical operational description.* In the early years of quantum mechanics it has not been realized that quantum systems are capable of

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[1] Compare for example Bohr (1949), p.209.

developing classical structures so that these two reasonable requirements have often been considered as contradictory.[2]

To discuss the emergence of classical behavior of quantum systems we need a single theory which allows a coherent description of quantum physics, classical mechanics, electrodynamics, and engineering physics in the same mathematical language. A convenient framework for such a unified description is the formalism of *algebraic quantum mechanics*. Algebraic quantum mechanics is nothing but a mathematically precise and complete codification of the heuristic ideas of quantum mechanics of the pioneer days. It is a general representation theory of the basic kinematical symmetry group and the associated canonical commutation relations. It is valid for microscopic, mesoscopic and macroscopic systems with finitely or infinitely many degrees of freedom. They can be either purely quantal, purely classical or mixed quantal/classical.

A quantum system is called *classical* if its algebra of observables is commutative. In this case, we speak of a *classical quantum system*. By construction, a classical quantum system never contradicts the Heisenberg inequality for noncommuting observables. The behavior of a classical quantum system depends on the physical value of Planck's constant $\hbar$. The still widely held view that classical mechanics is the limiting case for vanishing Planck's constant $\hbar$ is untenable. Since the fictitious limit $\hbar \to 0$ does not exist in the norm topology, there is no universal classical limit of quantum mechanics.

2 Individual and statistical descriptions

Many discussions of the emergence of classical behavior of particular quantum systems are flawed by severe category mistakes. Philosophers speak of a *category mistake* when a term that belongs to one category is treated as if it belonged to another. In quantum physics a popular category mistake is the confusion of individual and statistical descriptions. A related category mistake occurs when one gives an ontic answer to an epistemic question, or an epistemic answer to an ontic question. The problem of the ontology of a scientific theory refers to the problem of the existence of the postulated entities. Ontological realism is the doctrine that at least part of nature is independent of human beings. An *ontic description* refers to the intrinsic properties of an individual object system, irrespective of whether we know them or not, and irrespective of observational arrangements. An *epistemic description* refers to our knowledge of the properties or modes of reactions of observed systems. Ontic descriptions reflect some "laws of nature", while epistemic descriptions summarize the results of observations and experiments.

As it is well known from classical mechanics, the mathematical formalism

[2]For example, Bohr argued that the description of a measuring instrument cannot be included in the realm of quantum mechanics. Most clearly Bohr stated his view in a letter of October 26, 1935 to Schrödinger: "Das Argument ist ja dabei vor allem, dass die Messinstrumente, wenn sie als solche dienen sollen, nicht in den eigentlichen Anwendungsbereich der Quantenmechanik einbezogen werden können." Quoted on p.510 in Kalckar (1996).

required for an individual description is different from the formalism required for a statistical description. A basic postulate for a statistical description is MACKEY'S axiom IX which calls for the commutativity of the operation of mixing with the time evolution.[3] This postulate implies the linearity of the dynamics of any statistical theory. Well-known examples are KOOPMAN'S Hilbert-space formalism (which rephrases the *nonlinear* Hamiltonian equations of motion of classical point mechanics in terms of *linear* equations of motion for classical statistical mechanics), the equivalence of *nonlinear* stochastic differential equations in the sense of ITÔ (which provide a stochastic individual description) with the *linear* Fokker-Planck equations (which give a statistical ensemble description). This connection suggests that the linear Schrödinger equation refers to a *statistical* description and that any suggested nonlinear generalization of the Schrödinger equation is inappropriate for a statistical description.

On the other hand, there are no arguments against a nonlinearity of the Schrödinger equation for an *individual* description. The linearity of the dynamics of statistical quantum mechanics has absolutely nothing to do with the quantum-mechanical superposition principle. The superposition principle just says that one can construct a completely new pure state from any two different pure states. Nonlinear equations for the dynamics of *individual* states do not violate any fundamental laws of quantum mechanics but can be derived from the interaction between quantum and classical systems.

The so-called "wave function collapse" is an example for a category mistake which confuses individual and statistical descriptions. In the individual description the time evolution transforms an individual state into an individual state, never into a mixture of individual states. If one refers to a collapse of a coherent superposition of individual states into a statistical mixture of individual states, one commits a category mistake. If we describe a measurement process *statistically*, we have to use *linear* equations of motion. The resulting final state describes a statistical ensemble, never a particular outcome. On the other hand, an *individual* description of the measuring process requires a *nonlinear* stochastic dynamics whose ensemble average results in a *linear* dynamics and corresponds to the usual equations of motion for the statistical state.

3 C*-algebraic description of individual systems

Among all the abstractions of classical science the idea of an isolated system is central. Yet, *on the fundamental level there are no isolated system.* Every physical system is interacting and entangled with the rest of the world which we call its *environment*. Long ago, ARTHUR EDDINGTON stressed that *"the environment must never be left out of consideration"*.[4] This implies that the evolution of any object system is not governed by intrinsic laws only. Since the environment of every material object includes the electromagnetic field, a

[3] MACKEY (1963), p.81.
[4] EDDINGTON (1946), p.13.

proper discussion of finite open systems requires a theory which can handle infinite systems.[5]

Ontic and epistemic descriptions have not only categorially different referents but they require also different mathematical tools. Ontic descriptions are intended to have quite universal validity and should give – as far as possible or reasonable – a *context-independent* description of the material world. On the other hand, every epistemic description depends on the *observational context*: it refers to empirical observations obtained via our cognitive apparatus or by instruments used by the experimentalist. Typical for epistemic descriptions are phenomenological Hamiltonians and phenomenological laws which depend on contextual parameters like relaxation times or diffusion constants.

Relative to a fixed universe of discourse a context-independent *ontic* description of an individual physical system can be given in terms of an *abstract C*-algebra* $\mathfrak{A}$.[6] A *state* of the system is represented by a positive linear functional ρ on $\mathfrak{A}$, $\rho(\mathbf{1}) = 1$, $\rho(A^*A) \geq 0$ for all $A \in \mathfrak{A}$. The set $\mathfrak{S}(\mathfrak{A})$ of all state functionals on $\mathfrak{A}$ is a convex and weak* compact subset of the dual $\mathfrak{A}^*$ of $\mathfrak{A}$, so that by the Krein–Milman theorem the state space $\mathfrak{S}(\mathfrak{A})$ is the weak* closed convex hull of the set $\mathfrak{P}(\mathfrak{A})$ of its extreme points. The elements of $\mathfrak{P}(\mathfrak{A})$ are called *pure state functionals* since they cannot be decomposed into different state functionals. More precisely: A state functional $\rho \in \mathfrak{A}^*$ is pure if and only if $\rho = p\rho_1 + (1-p)\rho_2$ with $\rho_1, \rho_2 \in \mathfrak{S}(\mathfrak{A})$ and $0 < p < 1$ implies $\rho = \rho_1 = \rho_2$.

If a system with the C*-algebra $\mathfrak{A}$ is described by the state functional ρ, then the *reduced state functional* of a subsystem described by the C*-algebra $\mathfrak{B} \subset \mathfrak{A}$ is given by the restriction $\rho_{\mathfrak{B}}$ of ρ to $\mathfrak{B}$,

$$\rho_{\mathfrak{B}}(B) := \rho(B) \quad \text{for all} \quad B \in \mathfrak{B} \quad , \quad \rho_{\mathfrak{B}} \in \mathfrak{B}^* \quad .$$

As a rule, the restriction of a pure state functional is not pure. The fact that a system consisting of an object system and its environment may be in a pure state without the object system being in a pure state is a typical quantum phenomenon reflecting the nonseparability of quantum systems. A reduced quantum state functional allows the evaluation of all expectation values of the observables of the object system but gives no information about the holistic correlations between the object system and its environment. Nonpure states are usually called "mixed states" but we shall avoid this term since it is conceptually misleading. *As a rule, nonpure quantum states cannot be interpreted as mixtures of individual states.*

A C*-algebra is a topological algebra with the extraordinary property that its topology (the so-called *norm topology*) is determined algebraically. Hence the topology of a C*-algebra is intrinsic and does not depend on any experimental

[5] The uniqueness theorem by Stone (1930) and von Neumann (1931) implies that traditional quantum mechanics is valid only for systems with finitely many degrees of freedom.

[6] A *-algebra $\mathfrak{A}$ is a collection of mathematical objects $A, B, C, \ldots$ that can be combined linearly, multiplied in a bilinear and associative way, and mapped by the conjugate linear *-operation $A \to A^*$ which satisfies $A^{**} = A$ and $(AB)^* = B^*A^*$. If a *-algebra $\mathfrak{A}$ is endowed with a Banach-space norm $\|\cdot\|$ with the properties $\|AB\| \leq \|A\|\|B\|$ and $\|A^*A\| = \|A\|^2$, then $\mathfrak{A}$ is called a C*-algebra.

context. The chosen C*-algebra $\mathfrak{A}$ characterizes the chosen universe of discourse. Relative to this universe the *context-independent intrinsic properties* can be represented by the selfadjoint elements of the C*-algebra. They describe what is taken to be real independently of any observation.

4 Inequivalent epistemic descriptions

A most interesting feature of algebraic quantum mechanics is that it provides the mathematical tools for the construction of *contextual descriptions.* A new *coarser*, contextually selected topology can be introduced by picking out a particular *reference state*, given by a positive linear state functional ρ on the context-independent abstract C*-algebra $\mathfrak{A}$ of intrinsic properties. The so-called GNS-construction (according to Gelfand, Naimark and Segal) allows the construction of a context-dependent Hilbert space $\mathcal{H}_\rho$ and an associated faithful representation $\pi_\rho(\mathfrak{A})$ of the C*-algebra $\mathfrak{A}$ acting on $\mathcal{H}_\rho$.[7] The closure of $\pi_\rho(\mathfrak{A})$ of the C*-algebra $\mathfrak{A}$ in the weak topology of the algebra $\mathfrak{B}(\mathcal{H}_\rho)$ of all bounded operators acting on $\mathcal{H}_\rho$ is a context-dependent W*-algebra $\mathfrak{M}_\rho$, $\pi_\rho(\mathfrak{A}) \subset \mathfrak{M}_\rho \subset \mathfrak{B}(\mathcal{H}_\rho)$, called the *algebra of contextual properties.*[8] Every W*-algebra is a C*-algebra, but not every C*-algebra is closed in the coarser W*-topology. The new contextual topology on $\mathfrak{A}$ which is induced by the reference state functional ρ corresponds to the weak operator topology on $\mathfrak{B}(\mathcal{H}_\rho)$.

The σ-weak topology induced by the reference state functional ρ is of crucial importance for the representation of *statistical states.* It is a continuity requirement necessary for a continuous representation of the contingent initial conditions. While all state functionals of the basic C*-algebra $\mathfrak{A}$ represent individual states, not all states on the C*-algebra $\mathfrak{A}$ of intrinsic properties are admissible states for a contextual statistical description in terms of the W*-algebra $\mathfrak{M}_\rho$ of contextual properties. In analogy to the concept of additivity of a measure in classical probability theory a linear positive functional φ on a W*-algebra $\mathfrak{M}_\rho$ is said to be completely additive if it satisfies $\varphi(\vee F_n) = \sum \varphi(F_n)$ for every set $\{F_n\}$ of pairwise orthogonal projections in $\mathfrak{M}_\rho$, $F_n F_m = 0$ for $n \neq m$. In measure theory the additivity of a measure implies Lebesgue's monotone convergence theorem. In analogy, a linear positive functional φ is said to be normal when $\varphi(M_n) \uparrow \varphi(M)$ for each monotonically increasing net $\{M_n\}$ of operators M_n in $\mathfrak{M}_\rho$ with least upper bound M. Statistical states are represented by *normal* state functionals; they are elements of the predual $(\mathfrak{M}_\rho)_* \subset (\mathfrak{M}_\rho)^*$ of the W*-algebra $\mathfrak{M}_\rho$ of contextual observables. Note that in a particular representation π_ρ only a small portion of the state functionals on the algebra $\mathfrak{A}$ of intrinsic properties corresponds to operationally accessible σ-additive statistical state functionals. The contextually selected topology is characterized by the fact that in the representation π_ρ the reference state functional ρ is the restriction of an operationally accessible statistical state functional. *Statistical states*

[7] For all mathematical questions we refer to Takesaki (1979), chapter I, section 9.

[8] A W*-algebra $\mathfrak{M}$ is a C*-algebra which is the dual of some Banach space $\mathfrak{M}_*$, called the predual of $\mathfrak{M}$, $\mathfrak{M} = (\mathfrak{M}_*)^*$.

are epistemic states: they refer to our knowledge of the ontic state as it appears in the context-independent C-formalism.*

A contextual W*-algebra $\mathfrak{M}_\rho$ is strictly larger than the faithful representation $\pi_\rho(\mathfrak{A})$ of the C*-algebra $\mathfrak{A}$ of intrinsic properties. That is, all intrinsic properties appear also as contextual properties, but in addition there are new properties which are not intrinsic. The elements in the W*-algebra $\mathfrak{M}_\rho$ of contextual properties which are not in the faithful representation $\pi_\rho(\mathfrak{A})$ of the C*-algebra $\mathfrak{A}$ of intrinsic properties are called *emergent properties*. They represent properties which are novel in the sense that they are absent in the context-independent C*-algebraic description. *The emergence of novelty in contextual descriptions is a compelling consequence of algebraic quantum theory.*

With the only exception of von Neumann's codification of traditional quantum mechanics (where the basic C*-algebra is the algebra of compact operators acting one a separable Hilbert space), there are always *infinitely many physically inequivalent W*-representations* of the underlying basic C*-algebra of intrinsic properties. Different inequivalent representations represent physically inequivalent contextual descriptions of one and the same C*-system. Even if the algebra $\mathfrak{A}$ of intrinsic observables (hence also $\pi_\rho(\mathfrak{A})$) has no center, the contextually constructed W*-algebra $\mathfrak{M}_\rho$ usually has a large center $\mathfrak{Z}_\rho(\mathfrak{M}_\rho)$,

$$\mathfrak{Z}_\rho(\mathfrak{M}_\rho) := \{Z | Z \in \mathfrak{M}_\rho \, , ZM = MZ \;\text{ for every } M \in \mathfrak{M}_\rho\} \quad .$$

The center $\mathfrak{Z}_\rho$ is a *commutative* W*-algebra. It represents the *classical part* of the system. The center is said to be trivial if it consists of the multiples of the identity element only. A W*-algebra with a trivial center is called a *factor*. Nontrivial selfadjoint elements of the center are called *classical observables*. They commute with all elements of the W*-algebra $\mathfrak{M}_\rho$.

Most classical observables are emergent – they are elements of $\mathfrak{Z}_\rho$ but not elements of $\pi_\rho(\mathfrak{A})$. Such emergent classical observables (like temperature or order parameters describing phase transitions) are neither intrinsic observables nor are they functions of the intrinsic observables. Nevertheless, they are *generated* by the basic C*-algebra $\mathfrak{A}$ of intrinsic observables together with a context which selects a particular representation. *In the algebraic approach emergent properties are not postulated but derived from contextual conditions which are necessary to describe physical systems besides the natural law.*

Von Neumann's irreducibility postulate[9] – the assumption that all selfadjoint operators acting on the Hilbert space of state vectors are observables – implies that the center of the algebra of observables is trivial so that in the traditional formulation there are no classical observables. However, we know empirically that von Neumann's irreducibility postulate is not universally valid.

5 Disjoint states and classical mixtures

The central part of a quantum system allows an important classification of states. The *support* S_φ of a normal state functional φ of a W*-algebra $\mathfrak{M}_\rho$ is

[9] Neumann (1932).

defined as the smallest projection operator $S \in \mathfrak{M}_\rho$ such that $\varphi(S) = 1$. The *central support* C_φ of a state functional φ is defined as the smallest projection operator $C \in \mathfrak{Z}_\rho(\mathfrak{M}_\rho)$ such that $\varphi(C) = 1$. With this definitions we get the following classification of state functionals on a W*-algebra $\mathfrak{M}_\rho$ with the center $\mathfrak{Z}_\rho(\mathfrak{M}_\rho)$:

- A state functional φ on the W*-algebra $\mathfrak{M}_\rho$ is *pure* if its support S_φ is an *atom*.
- Two state functionals φ' and φ'' are called *orthogonal* if their supports $S_{\varphi'}$ and $S_{\varphi''}$ are orthogonal, $S_{\varphi'}S_{\varphi''} = 0$.
- Two state functionals φ' and φ'' are called *disjoint* if their *central* supports $C_{\varphi'}$ and $C_{\varphi''}$ are orthogonal, $C_{\varphi'}C_{\varphi''} = 0$.
- Two state functionals φ' and φ'' are called *classically equivalent* if their central supports $C_{\varphi'}$ and $C_{\varphi''}$ are equal, $C_{\varphi'} = C_{\varphi''}$.
- A state functional φ which is dispersion-free with respect to every classical observable, $\varphi(Z^2) = \{\varphi(Z)\}^2$ for every $Z = Z^* \in \mathfrak{Z}_\rho(\mathfrak{M}_\rho)$ is called a *factor state functional*.

Disjointness implies orthogonality, but only in commutative algebras orthogonality implies disjointness. Disjointness is a much stronger condition than orthogonality. Two pure states are disjoint if and only if there exists a classical observable such that the expectation values with respect to these states are different. Therefore, mutually disjoint states can be distinguished and classified in a classical manner.

Every state functional can be decomposed uniquely into a sum or an integral of disjoint factor state functionals.[10] In classical theories the convex set of all state functionals is a simplex[11] so that every nonpure state can be decomposed uniquely into a mixture of pure states. Such a classical mixture allows an *ignorance interpretation*. In contrast to the classical case the convex set of state functionals of a nonclassical quantum system is not a simplex. A nonpure factor state functional allows infinitely many different decompositions into a convex sum of pure states. The nonpurity of factor states is always due to Einstein–Podolsky–Rosen correlations of the open system with its environment. It can *never* be interpreted as some kind of mixing.

If we think of a classical mixture of two components (like a mixture of water and alcohol), then we tacitly presuppose that we can *distinguish* operationally between the two components. That is, *it must be possible to label every component of a proper mixture so that the components can be distinguished*. Since such a label must be determinable together with any other property of the component, it has to be characterized by a value of a *classical observable*. More precisely: *Nonpure quantum states can be interpreted in terms of a classical*

[10] For details, compare for example Takesaki (1979), chap.IV.6.

[11] The state space is a simplex if and only if the C*-algebra is commutative. Compare Takesaki (1979), p.251.

mixture of factor states if and only if these factor states are mutually disjoint. Since in VON NEUMANN'S codification there are no disjoint states, a straightforward ignorance interpretation is not possible in traditional quantum mechanics.

Disjoint states are of crucial importance as final states in *any* processes – natural processes or measurement processes – which produce facts. The measurement problem is not, as often asserted, the problem how a pure statistical state can be transformed into a nonpure state, or how the density operator can become diagonal in a preferred basis. This is a trivial task – appropriate dynamical linear semigroups and their Hamiltonian dilations can describe such a decoherence mechanism. The same is true for the so-called "stochastic unraveling of dynamical semigroups of statistical descriptions": in the framework of traditional quantum mechanics there are always infinitely many stochastic differential equations for pure states which in the statistical average result in one and the same statistical dynamical semigroup. *A proper statistical description of the measurement process requires a dynamics which transforms factor states into a classical mixture of disjoint factor states.*

Classical observables and disjoint states exist only if the joint system consisting of the object and its environment has infinitely many degrees of freedom. However, a dynamical description of the emergence of new classical observables is not straightforward. A general result due to KLAUS HEPP shows that automorphisms preserve the disjointness of states:[12]

> If φ' and φ'' are two disjoint state functionals on a C*-algebra and if α is an automorphism of this C*-algebra, then the transformed state functionals $\varphi' \circ \alpha$ and $\varphi'' \circ \alpha$ are disjoint.

This theorem is not at all surprising since an automorphism is a symmetry which transforms a description into a fully equivalent description. The fact that an automorphic time evolution on any C*-algebra cannot generate new disjoint states is not a "no-go theorem" for the possibility of a theoretical description of measurement-type processes. First, the postulate of an automorphic dynamics has no sound physical basis. It is not acceptable simply to *postulate* that the time evolution should be an automorphism – we have to *derive* the dynamics from the known interactions. It is true that the dynamics of group-theoretically defined elementary systems is usually automorphic, but this fact does not imply that the dynamics of interacting systems is automorphic. Many physically reasonable C*-algebraic systems without an automorphic dynamics are known. But since at present no general theory is available for nonautomorphic time evolutions, I confine myself to the simpler case of an automorphic dynamics.

For automorphic time evolutions Hepp showed that there exist quantum systems with an automorphic dynamics $\{\alpha_t | t \in \mathbb{R}\}$ such that for equivalent initial state functionals φ' and φ'' the asymptotic limits $\varphi' \circ \alpha$ and $\varphi'' \circ \alpha$ exist for $t \to \infty$ and are disjoint. Such state functionals are called *asymptotically disjoint.* It has been objected that processes with asymptotically disjoint final states require an infinite measurement time.[13] This is a misunderstanding:

[12] HEPP (1972), lemma 2, p.246.

[13] For example by BELL (1975), and again by LANDSMAN (1995), p.55.

every measurement in engineering physics is asymptotic. In the following we will show that disjoint quantum states lead to singular decision problems which engineers reject as unrealistic idealizations.

6 Robustness of statistical decision procedures

To illustrate the measurement process from an engineering point of view, we consider the simplest statistical decision test as commonly used in experimental science.[14] Decision procedures use attributes of empirical observations to achieve a *Boolean classification* of facts. Since empirical data are invariably contaminated with noise, all experimental observations have to be considered to be subject to random variations. For a formal development of a statistical theory of classification it is irrelevant what is the cause of these random variations. They may be due to measurement errors, external noises, imperfect experimental procedures, or residual non-Boolean quantum effects. In the framework of statistical classification theory an observation is considered as a sample value $x = X(\omega) \in \mathcal{X},\ \omega \in \Omega$, of a $\mathcal{X}$-valued random element X, defined on a Kolmogorov probability space (Ω, Σ, μ). The random element X may be a real-valued random variable ($\mathcal{X} = \mathbb{R}$), a random vector (in the case of n independent real-valued observed values we have $\mathcal{X} = \mathbb{R}^n$), or a stochastic process (for example $\mathcal{X} = \mathcal{S}'$).

For the following it is sufficient to consider the special case of a binary decision. Assume that an experimenter knows that an observation $x = X(\omega)$ comes either from a distribution with the probability measure μ' or from a distribution with the probability measure μ''. A statistical test is a method to assign the observation to one of the two populations. The error of a statistical test is defined as the probability that the decision is false. Independently of how we perform this test, the *minimal error probability* is given by[15]

$$e_{\min}(\mu', \mu'') \ := \inf_{\mathcal{B} \in \Omega} \{\mu'(\mathcal{B}) - \mu''(\Omega - \mathcal{B})\} \quad ,$$

The minimal error probability is not easy to evaluate, but it can be estimated with the aid of the Hellinger integral $H(\mu', \mu'')$,[16]

$$H(\mu', \mu'') \ := \int_{\Omega} \sqrt{f'(\omega) f''(\omega)}\, d\mu(\omega) \quad ,$$

$$f' := d\mu'/d\mu \quad , \quad f'' := d\mu''/d\mu \quad , \quad 0 \leq H(\mu', \mu'') \leq 1 \quad ,$$

[14] For a review of the application of statistical decision theory in engineering science compare for example chapters 18–23 in Middleton (1960).

[15] Rényi (1966), Rényi (1967).

[16] Kraft (1955), lemma 1, p.127. The so-called Hellinger integral has been introduced by Hellinger (1909) in his investigation of unitary invariants of selfadjoint operators. Compare also Hahn (1912). In a statistical context these integrals have been introduced by Bhattacharyya (1943). Later Kakutani (1948) stressed the fact that the Hellinger integral is an inner product. The Hellinger integral was again introduced by Matusita (1951) under the name "affinity".

$$1 - \sqrt{1 - H(\mu', \mu'')^2} \ \leq e_{\min}(\mu', \mu'') \ \leq H(\mu', \mu'') \quad .$$

Here μ is any dominating measure (for example $\mu = \frac{1}{2}\mu' + \frac{1}{2}\mu''$), so that both μ' and μ'' are absolutely continuous with respect to μ. Since the convergence of the Hellinger integral to zero is equivalent to the convergence of the minimal error probability to zero, we can use the Hellinger integral $H(\mu', \mu'')$ as a criterion for the "nearness" of the two probability measures μ' and μ''. The smaller the Hellinger integral, the easier is it to distinguish the two measures from each other. If the Hellinger integral vanishes, a perfect decision can be made with probability one. Such statistical tests are called *singular*. Singular binary decision problems are characterized by mutually singular probability measures. Two measures μ' and μ'' are called equivalent, $\mu' \sim \mu''$, if $f'(\omega)f''(\omega) > 0$ for μ-almost all $\omega \in \Omega$. Two measures μ' and μ'' are called mutually singular, $\mu' \perp \mu''$, if $f'(\omega)f''(\omega) = 0$ for μ-almost all $\omega \in \Omega$. It follows that

$$\begin{array}{lll} \mu' \sim \mu'' & \text{if and only if} & H(\mu', \mu'') = 1 \quad , \\ \mu' \perp \mu'' & \text{if and only if} & H(\mu', \mu'') = 0 \quad . \end{array}$$

Only in the unrealistic case of infinitely many measurements or an infinitely long data acquisition period one can possibly get an error-free decision with $H(\mu', \mu'') = 0$. In engineering science one requires that the performance of statistical procedures is insensitive to small deviations of the actual situation from the idealized theoretical model. Such tests are called *robust*. Singular tests deteriorate seriously for small deviations from the nominal model. Since one never knows the underlying probability distribution accurately, *singular tests are rejected by engineers as improperly posed.*[17]

7 Disjoint states and singular decision problems

The Hellinger integral can be used as a physically meaningful measure of the approximate disjointness of quantum states. For simplicity we restrict our discussion to boson-type environments which can be described by Weyl systems.

For finitely many degrees of freedom Weyl's canonical commutation relations are expressed in terms of unitary Weyl operators $\zeta \mapsto W(\zeta)$ over the phase space $\mathbb{C}^n$. They fulfill the commutation relations

$$W(\zeta)\, W(\zeta') = e^{i\sigma(\zeta|\zeta')/2}\, W(\zeta + \zeta') \ , \ W(\zeta)^* = W(-\zeta) \quad , \quad \zeta, \zeta' \in \mathbb{C}^n \quad .$$

The symplectic form is defined by $\sigma(\zeta|\zeta') := i\ (\zeta|\zeta') - i\ (\zeta'|\zeta)$, $(\zeta|\zeta') := \sum_k \zeta_k^*\, \zeta_k'$.

For *infinitely* many degrees of freedom, we have to proceed more carefully since the topology on the phase space $\mathbb{C}^\infty$ of Weyl's canonical commutation relations is not intrinsically defined. For the present setting a nuclear phase space is appropriate. We choose the Schwartz sequence space $\mathcal{S}$ and its topological dual $\mathcal{S}'$, defined by

$$\mathcal{S} := \{\, \zeta \,|\, \zeta_k \in \mathbb{C} \ , \ \lim_{k\to\infty} k^p |\zeta_k| = 0 \ \text{ for all positive integers } p \,\} \ ,$$

[17] Slepian (1958), Root (1963), Root (1964), Root (1968).

$$\mathcal{S}' := \{\, \zeta \,|\, \zeta_k \in \mathbb{C}\ ,\ \lim_{k\to\infty} (1+k^2)^{-p}\,|\zeta_k| = 0 \ \text{ for all sufficiently large integers } p \,\}\ .$$

With the canonical bilinear form $(\xi|\zeta) := \sum_{k=1}^{\infty} \xi_k^*\,\zeta_k$ ($\zeta \in \mathcal{S}$, $\xi \in \mathcal{S}'$) that links $\mathcal{S}$ and $\mathcal{S}'$, the symplectic form is again given by $\sigma(\zeta|\zeta') := i\ (\zeta|\zeta') - i\ (\zeta'|\zeta)$, $\zeta, \zeta' \in \mathcal{S}$. Weyl's canonical commutation relations over the symplectic group $\mathcal{S}\times\mathcal{S}$ are given by

$$W(\zeta)\,W(\zeta') = e^{i\sigma(\zeta|\zeta')/2}\,W(\zeta+\zeta')\ ,\ W(\zeta)^* = W(-\zeta) \quad , \quad \zeta,\zeta' \in \mathcal{S} \quad .$$

For infinite systems the phase space is no longer locally compact so that there exists no nontrivial translation-invariant measure. Yet, there are uncountably many inequivalent quasi-invariant measures. This fact implies that *there exist uncountably many physically inequivalent representations of Weyl's canonical commutation relations* so that the uniqueness theorem by Stone, von Neumann, Mackey and Loomis for systems with a locally compact phase space no longer holds.[18] The richness of inequivalent representations reflects the complexity of infinite systems which requires the use of the C*-algebraic approach with its distinction between intrinsic properties (described by the kinematical C*-algebra) and contextual properties (described by an appropriate W*-representation of the kinematical C*-algebra).

Every normal state functional φ on a W*-algebra $\mathfrak{M}_\rho$ generated by a W*-Weyl system $\zeta \mapsto W(\zeta)$ is uniquely characterized by its state-generating functional $\zeta \mapsto \mathcal{G}_\varphi(\zeta) := \varphi\,\{W_\varphi(\zeta)\}$. A straightforward generalization of a theorem by Cushen and Hudson[19] says that the product of two state-generating functionals is the characteristic functional of some probability measure. Let φ be a normal state functional and let ρ be a normal reference state functional with the state-generating functionals $\mathcal{G}_\varphi$ and $\mathcal{G}_\rho$, respectively. Then we call the probability measure μ_φ associated with the characteristic functional $\zeta \mapsto \mathcal{G}_\varphi(\zeta)\ \mathcal{G}_\rho(\zeta)$ the *Husimi measure* accompanying the state functional φ. It is defined on the measurable space $(\mathcal{S}', \Sigma_{\mathcal{S}'})$, where $\Sigma_{\mathcal{S}'}$ is the minimal σ-field containing all cylinder sets in the dual $\mathcal{S}'$ of the Schwartz sequence space $\mathcal{S}$. Minlos' theorem[20] allows a characterization of this probability measure via a symplectic Fourier transform

$$\int_{\mathcal{S}'} e^{i\sigma(\xi|\zeta)}\mu_\varphi(d\xi) = \mathcal{G}_\varphi(\zeta)\ \mathcal{G}_\rho(\zeta) \quad , \quad \varphi \in (\mathfrak{M}_\rho)_* \quad , \quad \zeta \in \mathcal{S} \quad .$$

In the traditional irreducible Hilbert-space representation and the special case of a single degree of freedom the Radon–Nikodým derivative h_φ of μ_φ with respect to the Lebesgue measure has been introduced by Husimi[21] as the expectation value of the density operator D_φ with respect to pure coherent states, $h_\varphi(z) = \langle z|D_\varphi|z\rangle$. Here, the state vector $|z\rangle$ is defined by

[18] Stone (1930), Neumann (1931). The generalization of this uniqueness theorem to locally compact groups is due to Mackey (1949) and Loomis (1952). Compare also the discussion by Hegerfeldt & Melsheimer (1969).

[19] Cushen & Hudson (1971), proposition 5, p.464.

[20] Minlos (1959). Compare also the review by Hida (1980), chapter 3.

[21] Husimi (1940), p.278.

$\langle z|W(\zeta)|z\rangle = \exp(-|\zeta|^2 + z^*\zeta - z\,\zeta^*)$, $z, \zeta \in \mathbb{C}$. In this case the reference state is the ground state of the harmonic oscillator with the state-generating function $\zeta \mapsto \exp(-|\zeta|^2)$. The Husimi function $z \mapsto h_\varphi(z)$ has the remarkable property that it is strictly positive, $h_\varphi(z) > 0$ for every $z \in \mathbb{C}$.[22]

The positivity properties of Husimi functions can be used to discuss the exact or approximate disjointness of normal state functionals. Consider two normal factor state functionals $\varphi, \phi \in (\mathfrak{M}_\rho)_*$ and their associated positive Husimi measures μ_φ and μ_ψ on the measurable space $(\mathcal{S}', \Sigma_{\mathcal{S}'})$. With respect to a dominating measure μ (for example $\mu = \frac{1}{2}\mu_\varphi + \frac{1}{2}\mu_\psi$) we can define positive Husimi functions as Radon–Nikodým derivatives $h_\varphi := d\mu_\varphi/d\mu$ and $h_\psi := d\mu_\psi/d\mu$. The corresponding Hellinger integral is defined by

$$H(\mu_\varphi, \mu_\psi) := \int_{\mathcal{S}'} \sqrt{h_\varphi\, h_\psi}\ \mu(d\xi) \quad .$$

For two factor state functionals φ and ψ we have:

$$\begin{aligned} \varphi \text{ and } \psi \text{ are equal, if and only if} \quad & H(\mu_\varphi, \mu_\psi) = 1 \quad , \\ \varphi \text{ and } \psi \text{ are classically equivalent, if and only if} \quad & H(\mu_\varphi, \mu_\psi) > 0 \quad , \\ \varphi \text{ and } \psi \text{ are disjoint, if and only if} \quad & H(\mu_\varphi, \mu_\psi) = 0 \quad . \end{aligned}$$

This result implies that the Hellinger integral can be used as a criterion for the approximate disjointness of two classically equivalent factor state functionals. The smaller the value of $H(\mu_\varphi, \mu_\psi)$, the more the state functionals φ and ψ behave like disjoint states.

Let φ and ψ be the state functionals of two *classically equivalent* initial factor states. If the dynamics is given by a one-parameter automorphism group $\{\alpha_t \,|\, t \in \mathbb{R}\}$, then the time-evolved state functionals are given by $\varphi_t := \varphi \circ \alpha_t$ and $\psi_t := \psi \circ \alpha_t$. Hepp's theorem implies that for $t \to \infty$ the time-evolved state functionals φ_t and ψ_t are still equivalent. If for $t \to \infty$ the asymptotic limits of the state functionals exist, and if the asymptotic final states are disjoint,

$$H(\mu_{\varphi_t}, \mu_{\psi_t}) > 0 \quad \text{for} \quad t < \infty \quad , \quad \lim_{t\to\infty} H(\mu_{\varphi_t}, \mu_{\psi_t}) = 0 \quad ,$$

then we call the two classically equivalent factor states φ_t ad ψ_t *asymptotically disjoint*. The convergence of the function $t \mapsto H(\mu_{\varphi_t}, \mu_{\psi_t})$ for $t \to \infty$ means that for every $\epsilon > 0$ there is a finite time $T < \infty$ such that $H(\mu_{\varphi_t}, \mu_{\psi_t}) < \epsilon$ for every $t \geq T$.

Every test for deciding whether the reduced state of the apparatus is given by the state functional φ_t or by the state functional ψ_t has to be made by *classical* measurements. Independently of how we carry them out, their error cannot be smaller than the minimal error probability $e_{\min}(\mu_{\varphi_t}, \mu_{\psi_t})$. If the state functionals φ_t and ψ_t are asymptotically disjoint the error probability can be made arbitrarily small, $e_{\min}(\mu_{\varphi_t}, \mu_{\psi_t}) \leq H(\mu_{\varphi_t}, \mu_{\psi_t}) \overset{t\to\infty}{\longrightarrow} 0$. For a given threshold level ϵ an effective measuring time T can be defined by $H(\mu_{\varphi_T}, \mu_{\psi_T}) =$

[22] Compare McKenna & Klauder (1964), Mehta & Sudarshan (1965), p.B277.

ϵ. It depends in an essential way on the interaction terms in the Hamiltonian of the joint system. To summarize: *Measurements with asymptotically disjoint final states do not require an infinite measuring time. In special cases, the effective measuring time T can be very short.*

8 A simple example

At this stage it is helpful to give a simple example. We consider a two-level quantum system which is linearly coupled to an environment consisting of many bosons. We use a Hamiltonian formulation and assume that the joint system "object system & environment" is a purely quantum-mechanical system, characterized by an algebra of observables with a trivial center. The object system is taken to be a two-level quantum system while the environment is modeled by many harmonic oscillators. The object system is assumed to be linearly coupled to the environment so that the Hamiltonian H of the joint system is given by

$$H/\hbar = \tfrac{1}{2}\,\omega\,\sigma_3 \otimes 1 + \sum\nolimits_k \omega_k\, 1 \otimes a_k^* a_k + \tfrac{1}{2}\,\sigma_3 \otimes \sum\nolimits_k \lambda_k\,(a_k^* + a_k) \quad ,$$

with the real-valued frequencies $\omega, \omega_k, \lambda_k$. The boson operators $a_1, a_2, a_3, \ldots$ are defined via the Weyl operator $W(\zeta) = \exp\{\sum_k (\zeta_k\, a_k^* - \zeta_l^* a_k)/2\}$, $\zeta \in \mathcal{S}$. The Pauli operator $\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \sigma_3)$ is characterized by $\boldsymbol{\sigma} = i\,\boldsymbol{\sigma} \times \boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^2 = \mathbf{1}$. This model is explicitly solvable for any number of the environmental degrees of freedom. Despite of its triviality, it is useful to demonstrate in a transparent way the emergence of classical observables and asymptotically disjoint quantum states.

The nature of the interaction between the object system and its environment is to a large extent determined by the so called *memory function* $t \mapsto K(t)$, defined by $K(t) := \sum_k (\lambda_k^2/\omega_k)\cos(\omega_k t)$, $t \in \mathbb{R}$. The memory function is real-valued and independent of the state of the system. We choose the parameters in the Hamiltonian in such a way that in the limit of infinitely many bosons we have $K(0) < \infty$, so that the Hamiltonian is bounded from below. In this case the function $t \mapsto K(t)$ is a characteristic function, hence the Fourier–Stieltjes transform $K(t) = \int_{-\infty}^{\infty} e^{i\lambda t} d\hat{K}(\lambda)$ of some distribution function $\lambda \mapsto \hat{K}(\lambda)$.

If the environment has only finitely many degrees of freedom, then the distribution function $\lambda \mapsto \hat{K}(\lambda)$ is necessarily discrete, so that the object system cannot exhibit a genuine relaxation behavior. To avoid unnecessary difficulties, we will choose a memory function without a discrete and without a singular part. In this case the distribution function is absolutely continuous and has a nonnegative derivative $\lambda \mapsto \hat{k}(\lambda) := d\hat{K}(\lambda)/d\lambda$ almost everywhere.

The emergence of asymptotically disjoint states and of induced superselection sectors depends in a crucial way on the low frequency behavior of the distribution density $\lambda \mapsto \hat{k}(\lambda)$. For the power law $\hat{k}(\lambda) \sim \lambda^\nu$ for $\lambda \to 0$ one distinguishes between ohmic, subohmic an superohmic interactions:

- if $\hat{k}(\lambda) \sim \lambda^\nu$ with $\nu > 0$, then the interaction is called *superohmic*,

- if $\hat{k}(\lambda) \sim \lambda^{\nu}$ with $\nu = 0$, then the interaction is called *ohmic*,
- if $\hat{k}(\lambda) \sim \lambda^{\nu}$ with $\nu < 0$, then the interaction is called *subohmic*.

For $\nu = 0$ the behavior of the system is called *ohmic* since in many models such an interaction leads to a dissipative term linearly proportional to a velocity.[23] Both ohmic and subohmic systems are distinguished by the so-called *infrared singularity*, characterized by $\sum_k(\lambda_k^2/\omega_k^2) = 2\int_0^\infty \hat{k}(\lambda)\lambda^{-1}d\lambda = \infty$. The infrared singularity is characteristic for electromagnetic interactions. It is due to the zero photon mass, which is responsible for their long-range and quasi-classical character. Since every quantum object is observed through its electromagnetic interactions, ohmic interactions establish a bridge between pure quantum systems and classical observational tools.

Since generality is not the point here, we restrict ourselves to an initial state which is given by a product state functional $\rho = \varphi \otimes \phi_\beta$ with an arbitrary state functional φ for the object system and a β-KMS functional ϕ_β for the environment with the thermal relaxation time $\tau_\beta = \hbar\beta/2$. On account of the simplicity of the model one proves without much difficulty that the expectation value $m_\beta^{\pm}(t) = \rho\{\sigma^{\pm}(t)\}$ of the Pauli operator in the Heisenberg picture $\sigma^{\pm}(t) := \{\sigma_1(t) \pm i\,\sigma_2(t)\}/2$ is given by

$$m_\beta^{\pm}(t) \;=\; m^{\pm}e^{\pm i\omega t}\exp\left\{-4\int_0^\infty \coth(\lambda\tau_\beta)\,\sin^2(\lambda t/2)\,\hat{k}(\lambda)\,\lambda^{-1}d\lambda\right\} \quad ,$$

with the initial value $m^{\pm} = \varphi(\sigma^{\pm})$. The state functional ρ_t of the joint system at time t can be written as

$$\rho_t = \tfrac{1}{2}\,\varphi(\sigma_3+1)\,\varphi^{+}\otimes\phi_\beta^{+}(t) \;+\; \tfrac{1}{2}\,\varphi(\sigma_3-1)\,\varphi^{+}\otimes\phi_\beta^{-}(t) \quad ,$$

where the object state functional $\varphi^{\pm}$ is characterized by $\varphi^{\pm}(\sigma_3 \pm 1) = 1$ and $\varphi^{\pm}(\sigma_3 \mp 1) = 0$. The approximate disjointness between the two factor state functionals $\phi_\beta^{+}(t)$ and $\phi_\beta^{-}(t)$ can be measured by the Hellinger integral

$$H\Big\{\mu_{\phi_\beta^{+}(t)}\,,\,\mu_{\phi_\beta^{-}(t)}\Big\} \;=\; \exp\left\{-2\int_0^\infty\{1-\exp(2\lambda\tau_\beta)\}\,\sin^2(\lambda t/2)\,\hat{k}(\lambda)\,\lambda^{-1}d\lambda\right\} \quad .$$

Using the well-known asymptotic expansion of Fourier integrals[24] one finds in an ontic description for a pure initial state with $\beta = \infty$,

$$\begin{aligned}
\lim_{t\to\infty} m_\infty^{\pm}(t)\,e^{\mp i\omega t} \;&>\; 0 \quad \text{for the superohmic case } \nu > 0 \quad ,\\
\lim_{t\to\infty} m_\infty^{\pm}(t)\,e^{\mp i\omega t} \;&=\; 0 \quad \text{for the ohmic and subohmic case } \nu \le 0 \quad ,\\
\lim_{t\to\infty} H\Big\{\mu_{\phi_\infty^{+}(t)}\,,\,\mu_{\phi_\infty^{-}(t)}\Big\} \;&>\; 0 \quad \text{for the superohmic case } \nu > 0 \quad ,\\
\lim_{t\to\infty} H\Big\{\mu_{\phi_\infty^{+}(t)}\,,\,\mu_{\phi_\infty^{-}(t)}\Big\} \;&=\; 0 \quad \text{for the ohmic and subohmic case } \nu \le 0 \quad .
\end{aligned}$$

[23] Compare Leggett et al. (1987), p.5

[24] Compare for example Lighthill (1958), chapter 4.

That is, *only if the interaction shows an infrared singularity (i.e. in the ohmic and subohmic case) the asymptotic final states are asymptotically disjoint.*

This behavior changes in an epistemic description. If the environmental initial state is given by a β-KMS state functional with $\beta < \infty$, then an ohmic interaction does no longer lead to asymptotically disjoint final states. For the sake of brevitiy we discuss only the high-temperature limit for which we get the following results:

$$\begin{aligned} \lim_{t\to\infty} m_\beta^\pm(t)\, e^{\mp i\omega t} &> 0 \quad \text{for } \nu > 1 \quad , \\ \lim_{t\to\infty} m_\beta^\pm(t)\, e^{\mp i\omega t} &= 0 \quad \text{for } \nu \leq 1 \quad , \\ \lim_{t\to\infty} H\Big\{\mu_{\phi_\beta^+(t)}\,,\,\mu_{\phi_\beta^-(t)}\Big\} &> 0 \quad \text{for every } \nu > -1 \quad . \end{aligned}$$

That is, in the high temperature limit the expectation values of $\sigma^+(t)$ and $\sigma^-(t)$ vanish asymptotically already for $\nu \leq 1$. Also, the off-diagonal elements of the reduced density operators (in the eigenbasis of σ_3) of the object system go to zero for $t \to \infty$ more rapidly than in the pure-state case. Yet, for any value of ν the Hellinger integral reaches asymptotically a nonvanishing constant so that the two epistemic states associated with the eigenstates of σ^+ and σ^- are never asymptotically disjoint. This example shows that *the asymptotic behavior of the reduced density operator of the object system says nothing about a possible decoherence in the sense of an exact or asymptotic disjointness.*

The qualitatively different behavior of the individual description (with a *pure* environmental initial state) and a statistical description (with a *nonpure* β-KMS initial state) can be interpreted as a signal-to-noise problem. The pure-state description corresponds to a detection problem of a noise-free signal while the epistemic description parallels the detection of a deterministic signal in the presence of noise. The condition $\beta > 0$ guarantees that the test is nonsingular, hence robust.

It is straightforward to prove that the epistemic quantum-mechanical expectation value $m_\beta^\pm(t)$ is the same as the stochastic mean value $\int_\Omega m_\infty^\pm(t|\omega)\, d\mu_\beta(\omega)$ of the single-system quantum-mechanical expectation value $m_\infty^\pm(t|\omega)$ of $\sigma^\pm(t|\omega)$ with respect to the pure ground-state functional φ. Here $t \mapsto \sigma^\pm(t|\omega)\}$ is the solution of Heisenberg's equation of motion with the time-dependent Hamiltonian $H(t|\omega) = H + \frac{1}{2}\hbar\omega\sigma_3 g_\beta(t|\omega)$, where the classical input force $t \mapsto g_\beta(t|\omega)$ is a tracjectory of a zero-mean Gaussian stochastic process with the covariance $2\tau_\beta^{-1}\int_0^\infty \cos(\lambda t)\,\hat{k}(\lambda)\, d\lambda$. This input force function corresponds to fractional noise with the low-frequency power spectrum $|\lambda|^\nu$. For ohmic interactions ($\nu = 0$) the input force g_β corresponds to a white noise process. For superohmic interaction ($\nu > 0$) the stochastic process g_β is a fractional derivative of white noise while for subohmic interactions ($\nu < 0$) g_β is a fractional integral of white noise.[25]

To summarize: *The epistemic description of the model by a β-KMS initial state of the environment can be interpreted as a statistical average of the*

[25] Compare Mandelbrot (1967), Mandelbrot & Ness (1968)

pure-state ontic description. The average is over an ensemble of external perturbations modeled as a classical stochastic processes with a power spectrum whose low-frequency part corresponds either to white noise or to fractional noise. The statistical averaging procedure deteriorates the signal-to-noise ratio of the decision test, but it also regularizes the asymptotic singular behavior of the ohmic and subohmic case. Although for $\beta < \infty$ there are no asymptotically disjoint states, for appropriate parameters in the Hamiltonian there can be approximately disjoint final states.

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