EPR Paradox and the Physical Meaning of an Experiment in Quantum Mechanics

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Abstract

It is shown that there is one purely deterministic outcome when measurement is made on the state function chosen by EPR to describe the combined two-particle system – the distance between the two particles is preserved the same. Further, it is shown that, surprisingly, the Ψ-function designed according to QM leads to the following paradox – despite the fact that the two particles move in opposite directions, in time the distance between them becomes shorter and shorter.

As is known, Einstein, Podolsky and Rosen (EPR) [1] observe two particles which have interacted in the past (for $0 < t < \mathrm{T}$) but the interaction between them has ceased for times $t > \mathrm{T}$. Nevertheless, both particles, even at times $t > \mathrm{T}$ EPR consider to be described by one common state-function, namely $\Psi(x_1, x_2)$. The authors have chosen to explore one particular form of that state-function, namely

$$\Psi\left(x_1, x_2\right) = \int dp e^{\frac{i}{\hbar}(x_1 - x_2 - x_0)p}. \tag{1}$$

This function may be rewritten as

$$\Psi\left(x_1, x_2\right) = \int dp e^{-\frac{i}{\hbar}(x_2 + x_0)p} e^{\frac{i}{\hbar}x_1 p} \tag{2}$$

and then $e^{\frac{i}{\hbar}x_1 p}$ can be considered as the orthonormal basis of an observable $\mathcal{A}$ pertaining to particle 1 (position of particle 1) while the exponents

$e^{-\frac{i}{\hbar}(x_2+x_0)p}$ are the expansion coefficients. In other words, we may observe the above integral as the expansion of the Ψ-function along the continuous basis $e^{\frac{i}{\hbar}x_1 p}$.

When a measurement of the position of particle 1 at time t is carried out the Ψ-function collapses to the following function

$$f(x_1,x_2)=\delta(x-x_1)\delta(x-x_2-x_0) \tag{3}$$

and one immediately sees from eq.(3) that

$$x_1 = x_2 + x_0. \tag{4}$$

It can easily be seen also that if we carry out measurement of the momentum of the first particle at time t the Ψ-function collapses in such a way as to give eigenvalues p and –p of the first and the second particle, respectively.
Now, we may wish to explore what the reproducibility would be of the mentioned measurement (collapse of the same function $\Psi(x_1,x_2)$) at the same time t when carrying out repetitive measurements on it. Obviously, each time the operator $\mathcal{A}$ is applied on the function $\Psi(x_1,x_2)$ an eigenvalue of that operator will be produced. It should be noted, however, that, although the probability (expectation value) of obtainment of a given eigenvalue (from the set of the eigenvalues of the operator $\mathcal{A}$) is easily written as $< i\,|\mathcal{A}|\,j >$, the outcome from the process of a concrete measurement is completely unpredictable. In other words, as a result of a given measurement the obtainment of a given eigenvalue is completely random.

Interestingly, however, despite the fact that as a result of repetitious measurements (always done at time t) we will obtain different values of x_1 (resp. x_2), the separation between these two particles at each measurement will be one and the same – x_0. This will not be so noteworthy if there was not another fact concerning this system of two particles. As mentioned, if we happen to measure the momentum $\mathcal{A}$ of particle 1 we will each time (after each successive measurement) obtain a different value p of that momentum and, which is even more remarkable, as mentioned, this momentum will be exactly equal in size but opposite in sign to the momentum of particle 2. In other words, despite the fact that the two particles will tend to move, at time t, in opposite directions and after each measurement these equal but opposing velocities will be of different magnitude (absolute value of the velocity will differ at each measurement), the separation between the two particles will invariably be equal to a constant – x_0. Thus, although we cannot predict the outcome of the position or momentum measurement at time t we can with certainty predict that after each measurement the two particles will always be found at the same separation x_0.

Paradoxically, since the time t is arbitrary, the separation between the two particles will be one and the same, namely x_0, no matter at what point of time we make the experiment.

The above may appear like an interesting observation but it does not seem to lack physical meaning. Collapsed function is not a function of time and one may argue that nothing prevents it from having the above property.

We want now to explore this observation a little further. We want now to see what the time-propagation of the collapsed function

$$f\left(x_1,x_2\right)=\delta\left(x-x_1\right)\delta\left(x-x_2-x_0\right) \tag{5}$$

is. For this reason we may write the delta-functions explicitly

$$f\left(x_1,x_2\right)=\int dpe^{\frac{i}{\hbar}(x-x_1)p}\int dp'e^{\frac{i}{\hbar}(x-x_2-x_0)p'}=$$

$$\int dp\mathrm{A}\int dp'\mathrm{B}=\int dp\int dp'\mathrm{AB} \tag{6}$$

To see the development in time of above function we apply the propagator $e^{-iH_0\Delta t}$ where the Hamiltonian corresponds to a free particle (i.e. interaction between particles is zero, as required by EPR), i.e.

$$H_0 = \frac{p^2}{2m} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}. \tag{7}$$

Propagator including the positions of the two individual particles, instead of center-of-mass propagator yields the same result.

Let us now expand the propagator

$$e^{-iH_0\Delta t} \approx (1 - iH_0\Delta t) = \left(1 + i\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Delta t\right). \tag{8}$$

And now we are ready to apply the propagator (in its expanded form) on the function $f(x_1, x_2) = \int dp \int dp' \mathrm{AB}$ (notice the definition of A and B above):

$$(1 - iH_0\Delta t)\int dp \int dp' \mathrm{AB}. \tag{9}$$

It can easily be shown that it is equal to:

$$\int dp \int dp' e^{-i\frac{\hbar^2}{2m}\Delta t(p+p')^2} \mathrm{AB}.$$

(Details)

Above expression is the propagated function $f(x_1, x_2, t)$ which has the explicit form

$$\int dp'\left[\int dp e^{i(x-x_1-i\alpha p)p}\right]e^{i(x-x_2-x_0+i\alpha p')p'}. \tag{10}$$

(Details)

Last integral, however, immediately yields the following delta-functions

$$\delta\left(x-x_1-i\alpha p\right) \tag{11}$$

and

$$\delta\left(x-x_2-x_0+i\alpha p'\right). \tag{12}$$

From these delta-functions it follows that

$$x=x_1+\alpha p \tag{13}$$

and

$$x=x_2+x_0-\alpha p' \tag{14}$$

from where we have

$$x_1+\alpha p=x_2+x_0-\alpha p' \tag{15}$$

and, respectively, because $p'=-p$,

$$x_1=x_2+x_0. \tag{16}$$

As is seen from the above definition of α, namely this α is the parameter that is time-dependent – it turns out that the time-dependent parameter is eliminated when applying the propagator. This means that in time the collapsed function (at any moment of time) will be only coordinatedependent. In other words, the collapsed function (at any moment of time) will look exactly as the function which we were observing to collapse at a given time t – that is, even before applying the propagator.

Thus, it appears that in time the distance between the two particles will be always one and the same despite the fact that the two particles move in opposite directions.

One may argue that when collapsed state function is propagated the condition $p' = -p$ may not be valid anymore. Instead, a more general condition would apply, namely $p' = -np$, where $n = const$. In such a case we will have

$$x_1 + i\alpha p = x_2 + x_0 + i\alpha np \quad (17)$$

and therefore

$$x_1 - x_2 = x_0 + i\alpha(n-1)p \quad (18)$$

or

$$x_1 - x_2 = x_0 + ii\frac{\hbar^2}{2m}\Delta t(n-1)p \quad (19)$$

which is

$$x_1 - x_2 = x_0 - \frac{\hbar^2}{2m}\Delta t(n-1)p \quad (20)$$

Thus, it appears that as the time progresses and the particles move in opposite directions from one another, the distance between them gets shorter and shorter.

The state function chosen by EPR to describe the combined two-particle system is specially designed by them to show that the Ψ-functions used in

QM do not describe completely the state of the system. From the above argument it follows further that such specially designed function (and EPR have all the right to design any function they need to prove their point) appears to lead to conclusions that seem to even lack physical meaning.

One possible way of looking at it, in trying to resolve this probable problem, is to consider that after collapse the two particles fall on a sphere where their velocities can be of opposite directions while still maintaining the same distance. Further, in time this may be a shrinking sphere whereby these particles still can maintain somehow their opposite velocities while the distance between them gets closer and closer. This picture, however, does not seem to be supported by experimental evidence.

One last point – despite the generally probabilistic character of the outcome of measurement in QM, there appears to be an unsuspected deterministic outcome in this particular set up. No matter how uncertain, in general, the outcome regarding the state of the system may be after the experiment, there is something which we know for sure – the distance between particles will remain the same as a result of experiments of the type described. This seems to be an interesting deterministic feature of QM which probably is worth exploring further.

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References

1. Einstein A., Podolsky B., Rosen N., Phys. Rev., 47, 777-780 (1935).

I. As a helpful step, let us first of all observe what the second derivative of AB will look like:

$$\frac{\partial^2}{\partial x^2}\mathrm{AB} = \frac{\partial}{\partial x}\left(\frac{\partial \mathrm{A}}{\partial x}\mathrm{B} + \mathrm{A}\frac{\partial \mathrm{B}}{\partial x}\right) =$$

$$\frac{\partial^2 \mathrm{A}}{\partial x^2}\mathrm{B} + 2\frac{\partial \mathrm{A}}{\partial x}\frac{\partial \mathrm{B}}{\partial x} + \mathrm{A}\frac{\partial^2 \mathrm{B}}{\partial x^2}\,.$$

Now we are ready to do the derivation properly

$$\int dp \int dp'\big(-(p^2\mathrm{AB} + 2pp'\mathrm{AB} + p'^2\,\mathrm{AB}\big) =$$

$$\int dp \int dp'\big(-(p + p')^2\,\mathrm{AB}\big)\,.$$

Then, the whole expression will be

$$\int dp \int dp'\left(1 - \frac{\hbar^2}{2m}\Delta t\,(p + p')^2\right)\mathrm{AB}\,.$$

Now, once we have the whole expression we can restore the exponent

$$\int dp \int dp' e^{-i\frac{\hbar^2}{2m}\Delta t (p+p')^2}\,\mathrm{AB}\,.$$

II. Denote $i\dfrac{\hbar^2}{2m}\Delta t=\alpha(t)=\alpha$, then we have

$$\int dp\int dp'e^{-\alpha(p+p')^2}\mathrm{AB}$$

and substituting A and B we have

$$\int dp\int dp'e^{-\alpha(p+p')^2}e^{ip(x-x_1)}e^{ip'(x-x_2-x_0)}$$

which may also be written as

$$\int dp\int dp'e^{-\alpha p^2}e^{-\alpha p'^2}e^{-2\alpha pp'}e^{ip(x-x_1)}e^{ip'(x-x_2-x_0)}$$

or as

$$\int dp'\left[\int dpe^{-\alpha p^2}e^{-2\alpha pp'}e^{ip(x-x_1)}\right]e^{ip'(x-x_2-x_0)}e^{-\alpha p'^2}$$

respectively

$$\int dp'\left[\int dpe^{-\alpha p^2-2\alpha pp'+ip(x-x_1)}\right]e^{ip'(x-x_2-x_0)-\alpha p'^2}.$$

Now, because $p=-p'$ (and therefore $-\alpha p^2-2\alpha pp'=-\alpha p^2+2\alpha p^2=\alpha p^2$) we may write the above expression as

$$\int dp'\left[\int dpe^{ip\left(x-x_1\right)+\alpha p^2}\right]e^{ip'(x-x_2-x_0)-\alpha p'^2}$$

which is also

$$\int dp'\left[\int dpe^{ipx-ipx_1+\alpha p^2}\right]e^{ip'x-ip'x_2-ip'x_0-\alpha p'^2}=$$

$$\int dp'\left[\int dpe^{ipx-ipx_1-i^2\alpha p^2}\right]e^{ip'x-ip'x_2-ip'x_0+i^2\alpha p'^2}$$