DEFINABILITY AND A NONLINEAR SIGMA MODEL

D.J. BENDANIEL
Cornell University
Ithaca NY, 14853, USA

Abstract. The concept of definability of physical fields is introduced and a set-theoretical foundation is proposed. In this foundation, we obtain a scale invariant nonlinear sigma model and then show that quantization of the model is necessary and sufficient for definability. We also obtain compactification of the spatial dimensions effectively and show its equivalence to quantization.

We look to provide a deep connection between physics and mathematics by requiring that physical fields be definable in a set-theoretical foundation. The well-known foundation of mathematics is the set theory called Zermelo-Fraenkel (ZF). In ZF, a set U of finite integers is definable if and only if there exists a formula $\Phi_U(n)$ from which we can unequivocally determine whether a given finite integer n is a member of U or not. That is, when a set of finite integers is not definable, then there will be at least one finite integer for which it is impossible to determine whether it is in the set or not. Other sets are definable in a theory if and only if they can be mirrored by a definable set of finite integers. Most sets of finite integers in ZF are not definable. Furthermore, the set of definable sets of finite integers is itself not definable in ZF. [1]

A physical field will be definable in a set-theoretical foundation if and only if the set of distributions of the field's energy among eigenstates (which is the set underlying thermodynamics) can be mirrored in the theory by a definable set of finite integers. This concept of definability is appropriate because, were there any physical field whose set of energy distributions among eigenstates was mirrored by an undefinable set of finite integers, that field would have at least one energy distribution whose presence or absence is impossible to determine, so the field could not be observable. Therefore, our task is to find a foundation in which it is possible to specify completely the definable sets of finite integers and to construct the fields mirrored by these sets.

The definable sets of finite integers cannot be specified completely in ZF because there are infinitely many infinite sets whose definability is undecidable. We must start with a sub-theory of ZF containing no infinite sets of finite integers. Then all sets of finite integers are *ipso facto* definable. This will mean, of course, that the set of all finite integers, called ω, cannot exist in that sub-theory. The set ω exists in ZF directly in consequence of two axioms: an axiom of infinity and an axiom schema of subsets. Thus, we must delete one or the other of these axioms. If we delete the axiom of infinity we will actually have no need for the axiom schema of subsets as well since all sets are finite. However that theory is too poor to obtain the functions of a real variable necessary for physical fields. So the task reduces to whether or not, starting by deleting the axiom schema of subsets from ZF yet retaining infinite sets, we can build a foundation in which we obtain just those physical fields mirrored by the definable sets of finite integers.

In the appendix we show eight axioms. The first seven are the axioms of ZF except that the axiom schema of replacement has been modified. The usual replacement axiom (AR) asserts that for any functional relation, if the domain is a set, then the range is a set. That axiom actually combines two independent axioms: the axiom schema of subsets, which we wish to delete, and an axiom schema of bijective replacement (ABR), which provides replacement only in one-to-one mappings. Accordingly, we can delete the axiom schema of subsets from ZF by substituting ABR for AR, forming the sub-theory ZF–AR+ABR.

We shall first discuss how ZF–AR+ABR differs from ZF. To do this, we look at the axiom of infinity. The axiom of infinity asserts the existence of at least one set ω* that contains, in general, infinite as well as finite ordinals. In ZF, we can obtain a set with just all the finite ordinals, the set ω, by using the axiom schema of subsets to establish the intersection of all the sets possible to create by the axiom of infinity. However, it has been shown that no model of ZF–AR+ABR contains ω [2].

This result confirms that the axiom schema of subsets is not contained in ZF–AR+ABR and requires the theory be uniformly dependent on ω*. A member of ω* is an "integer". An "infinite integer" is a member mapping one-to-one with ω*. A "finite integer" is a member that is not an infinite integer. In ZF–AR+ABR, any set of finite integers is finite. We denote finite integers by *i, j, k, ℓ, m* or *n*.

We now adjoin to ZF–AR+ABR an axiom asserting that all sets of integers are constructible. By constructible sets we mean sets that are generated sequentially by some process, one after the other, so that the process well-orders the sets. Goedel has shown that an axiom asserting that all sets are constructible can be added to ZF, giving a theory usually called ZFC^+.[3] It has also been shown that no more than countably many constructible sets of integers can be proven to exist in ZFC^+.[4] This result holds for the sub-theory ZFC^+–AR+ABR. Therefore we can adjoin to ZF–AR+ABR a new axiom asserting that all the subsets of ω* are constructible and that its power set is countable. We refer to these eight axioms as theory T.

As required for consistency, Cantor's proof cannot be carried out in T, so no uncountably infinite sets can be obtained with it.[5] All sets are countable, so the continuum hypothesis holds. However, as the axiom schema of subsets is not available, we cannot prove the induction theorem, so even many countable sets that exist in ZF do not exist in T. For example, we cannot sum infinite series, whereas in ZF infinite series play an important role in the development of mathematics. Nevertheless, using our axiom of constructibility we can obtain some functions of a real variable.

We first show that the theory T contains a real line, obtainable by a non-standard approach. Recall the definition of "rational numbers" as the set of ratios of any two members of the set ω, usually called **Q**. In T, we can likewise establish for ω* the set of ratios of any two of its integers, finite or infinite, as an "enlargement" of the rational numbers and we shall call this enlargement **Q***.

Two members of **Q*** are called "identical" if their ratio is 1. We now employ the symbol "≡" for "is identical to". An "infinitesimal" is a member of **Q*** "equal" to 0, i.e., letting y signify the member and employing the symbol "=" to signify equality, $y = 0 \leftrightarrow \forall k[y < 1/k]$. The reciprocal of an infinitesimal is "infinite". A member of **Q*** that is not an infinitesimal and not infinite is "finite", i.e., $[y \neq 0 \wedge 1/y \neq 0] \leftrightarrow \exists k[1/k < y < k]$. We apply this concept of equality to the interval between two finite members of **Q***; two finite members are either equal or the interval between them is finite. The constructibility axiom in T well-orders the power set of ω*, creating a non-standard real line **R*** consisting of binimals corresponding to subsets of ω*. A theory of e*quality-preserving* mappings between finite intervals of **R*** is uniformly dependent on ω* and gives functions of a real variable.

We first require equality-preserving homeomorphic mappings $\phi(x,u)$ between finite intervals X and U, where $x \in X$ and $u \in U$, such that $\forall x_1,x_2,u_1,u_2[\phi(x_1,u_1) \wedge \phi(x_2,u_2) \rightarrow [x_1 = x_2 \leftrightarrow u_1 = u_2]]$. These pieces are biunique and continuous. Functions of a real variable in T that are differentiable to all orders can now be built up from these biunique pieces. This build-up process is as follows: $\phi_1(x,u) \vee \phi_2(x,u)$ is a "function of a real variable" having the domain $X_1 \cup X_2$ and the range $U_1 \cup U_2$ if and only if $\forall x_1,x_2,u_1,u_2\forall n[\phi_1(x_1,u_1) \vee \phi_2(x_1,u_1) \wedge \phi_1(x_2,u_2) \vee \phi_2(x_2,u_2) \rightarrow [x_1 = x_2 \rightarrow u_1^{(n)} = u_2^{(n)}]]$ where $\phi_1(x,u)$ and $\phi_2(x,u)$ are biunique pieces and $u^{(n)}$ is the n^{th} derivative. All of these functions are of bounded variation. Any function of a real variable obeying these conditions is just a polynomial, since infinite series are unavailable in T. Constants are obtained directly from ABR. We shall focus on generating polynomials successively by an algorithm that is uniformly convergent to solutions of the Sturm-Liouville problem:

$$\int_a^b [p\left(\frac{du}{dx}\right)^2 - qu^2]dx = \lambda\int_a^b ru^2 dx \quad \text{where a} \neq \text{b, u}\frac{\text{du}}{\text{dx}} \equiv 0 \text{ at a and b} \qquad (1)$$

and p, q and r are functions of x.

The solutions to (1) are obtained by minimizing λ for $\int_a^b ru^2dx$ constant. This leads to an algorithm generating increasingly higher degree polynomials u_n, where n denotes the n[th] iteration, such that $\forall k \exists n \, [\int_a^b [p\left(\frac{du_n}{dx}\right)^2 - qu_n{}^2]dx - \lambda_n \int_a^b ru_n{}^2 dx < 1/k]$. We refer to polynomials of sufficiently high degree as an "eigenfunction". Every eigenfunction, since it is a polynomial, can be decomposed into "irreducible biunique eigenfunction pieces", as required in T.

We now show this theory is a foundation for fields governed by a nonlinear sigma model. Let us first consider two eigenfunctions, $u_1(x_1)$ and $u_2(x_2)$; for each let $p \equiv 1$, $q \equiv 0$ and $r \equiv 1$ and we shall call x_1 "space" and x_2 "time". It is well known that $\left(\frac{\partial u_1 u_2}{\partial x_1}\right)^2 - a\left(\frac{\partial u_1 u_2}{\partial x_2}\right)^2$ is the Lagrange density for a one-dimensional string and, by minimizing the integral of this function over all space and time, i.e., by Hamilton's principle, we can determine the field equations. We immediately extend this to separable bosonic strings in finitely many space-like (i) and time-like (j) dimensions. Extension to matrix fields is also possible. Since they are functions of real variables in T, the fields will be continuous, differentiable to all orders and of bounded variation, thus having no singularities.

Let $u_{\ell mi}(x_i)$ and $u_{\ell mj}(x_j)$ be eigenfunctions with non-negative eigenvalues $\lambda_{\ell mi}$ and $\lambda_{\ell mj}$, respectively. We define a "field" as a sum of eigenstates $\underline{\Psi}_m = \sum_\ell \Psi_{\ell m} \underline{i_\ell}$, $\Psi_{\ell m} = \prod_i u_{\ell mi} \prod_j u_{\ell mj}$, subject to the postulate that for every eigenstate m the value of the integral of the Lagrange density over $d\tau$, where $d\tau = \prod_i r_i dx_i \prod_j r_j dx_j$, is *identically* null:

$$\sum_\ell \int \left\{ \sum_i \frac{1}{r_i}\left[P_{\ell mi}\left(\frac{\partial \Psi_{\ell m}}{\partial x_i}\right)^2 - Q_{\ell mi}\Psi_{\ell m}^2\right] - \sum_j \frac{1}{r_j}\left[P_{\ell mj}\left(\frac{\partial \Psi_{\ell m}}{\partial x_j}\right)^2 - Q_{\ell mj}\Psi_{\ell m}^2\right]\right\} d\tau \equiv 0 \;\; \textit{for all } m. \quad (2)$$

In this integral equation the P and Q can be functions of any of the x_i and x_j, thus of any $\Psi_{\ell m}$ as well. This is a *nonlinear sigma model.* The $\Psi_{\ell m}$ are obtained by a generalized algorithm [6].

We can now show that the theory T implies a finite constant of action. Let us represent both

$$\sum_m \sum_\ell \int \left\{ \sum_i \frac{1}{r_i} \left[P_{\ell mi} \left(\frac{\partial \Psi_{\ell m}}{\partial x_i} \right)^2 - Q_{\ell mi} \Psi_{\ell m}^2 \right] \right\} d\tau \text{ and } \sum_m \sum_\ell \int \left\{ \sum_j \frac{1}{r_j} \left[P_{\ell mj} \left(\frac{\partial \Psi_{\ell m}}{\partial x_j} \right)^2 - Q_{\ell mj} \Psi_{\ell m}^2 \right] \right\} d\tau \text{ by } \alpha:$$

I. α is positive and must be closed to addition and to the absolute value of subtraction; thus α is an integer times a constant that is either infinitesimal or finite.

II. There is either no field (in which case $\alpha \equiv 0$) or otherwise in T α is non-infinitesimal, (in which case $\alpha \neq 0$); thus $\alpha = 0 \leftrightarrow \alpha \equiv 0$.

III. $\therefore \alpha \equiv \mathrm{n}\iota$, where n is an integer and ι is a finite constant such that $\alpha = 0 \leftrightarrow \mathrm{n} \equiv 0$.

Thus we have derived what is usually a postulate of quantum theory. For each eigenstate *m*:

$$\sum_\ell \int \left\{ \sum_i \frac{1}{r_i} \left[P_{\ell mi} \left(\frac{\partial \Psi_{\ell m}}{\partial x_i} \right)^2 - Q_{\ell mi} \Psi_{\ell m}^2 \right] \right\} d\tau \equiv \sum_\ell \int \left\{ \sum_j \frac{1}{r_j} \left[P_{\ell mj} \left(\frac{\partial \Psi_{\ell m}}{\partial x_j} \right)^2 - Q_{\ell mj} \Psi_{\ell m}^2 \right] \right\} d\tau \equiv \mathrm{n}\iota \qquad (3)$$

Finally, we note that the null postulate implies Hamilton's principle, as well:

$$\sum_\ell \int \left\{ \sum_i \frac{1}{r_i} \left[P_{\ell mi} \left(\frac{\partial \Psi_{\ell m}}{\partial x_i} \right)^2 - Q_{\ell mi} \Psi_{\ell m}^2 \right] - \sum_j \frac{1}{r_j} \left[P_{\ell mj} \left(\frac{\partial \Psi_{\ell m}}{\partial x_j} \right)^2 - Q_{\ell mj} \Psi_{\ell m}^2 \right] \right\} d\tau \equiv 0 \rightarrow$$

$$\delta \sum_\ell \int \left\{ \sum_i \frac{1}{r_i} \left[P_{\ell mi} \left(\frac{\partial \Psi_{\ell m}}{\partial x_i} \right)^2 - Q_{\ell mi} \Psi_{\ell m}^2 \right] - \sum_j \frac{1}{r_j} \left[P_{\ell mj} \left(\frac{\partial \Psi_{\ell m}}{\partial x_j} \right)^2 - Q_{\ell mj} \Psi_{\ell m}^2 \right] \right\} d\tau = 0 \quad \textit{for all m.}$$

While the model applies in finitely many space-like and time-like dimensions, we shall treat only the cases where the model has one time dimension. Then the constant ι is associated with each irreducible biunique time-eigenfunction piece and called "action". ι /(time-interval of the piece) is the concomitant "energy". Therefore *the model is intrinsically scale invariant.* If the eigenfunctions of the field are sinusoidal and n_m is the wave-number of the m^{th} eigenfunction in the time period ϑ, then the energy in the m^{th} eigenstate occurs in quanta of $4n_m\iota/\vartheta$ (so we obtain the usual "E = hν"). The "field energy" is the sum of energies in all of the eigenstates, $4M\iota/\vartheta$, where $M \equiv \sum j_m n_m$ and the integer j_m is the number of quanta in the m^{th} eigenstate. Thus $4M\iota \equiv \alpha$ for the field.

We have now set the stage for a discussion of the definability of the model in the theory T. Recall that every set (in T) of finite integers is finite and thus definable as that term has meaning in mathematical foundations. Thus a physical field will be definable in T if and only if the set of all distributions of field energy among eigenstates can be mirrored by some set (in T) of finite integers. We show easily that the quantized model with a single time dimension is definable in T:

For a given finite α, the ordered set of j_m corresponding to each distribution of energy among the eigenstates of the field can always be mapped to a different finite integer, e. g.,

$$\left\{ j_m \,|\, \sum_m^M j_m n_m \equiv M \right\} \Rightarrow \prod_m^M \left(P_m\right)^{j_m} \quad \text{where } P_m \text{ is the } m^{th} \text{ prime starting with 2.}$$

In T, the set of these finite integers for all distributions of energy exists and, moreover, every set of finite integers will mirror a distribution for some α. Thus quantization is sufficient for definability. Furthermore, we can show that quantization is also necessary for definability in T. Given a finite α, if ι were infinitesimal, then the integer M would have to be infinite. In that case, the set of all distributions of energy among eigenstates cannot be mirrored by any set (in T) of finite integers. Therefore, *quantization is necessary and sufficient for definability in T*. [7]

In conclusion, seeking a set-theoretical foundation for physical fields such that all fields are definable, we have invoked, instead of the usual foundation for mathematics ZF, a theory T, which is ZF−AR+ABR + Constructibility. In T, from a null hypothesis, we have obtained a scale invariant nonlinear sigma model and shown that quantization is necessary and sufficient for its definability. For finite energy, quantization is sufficient for definability of the model in ZF but is not necessary, i.e., is not implied by definability without postulates whose consistency with ZF has not been shown. Finally, while our development of the mathematics of T is far from rigorous or complete, the program seems clear and its implications for a theory of strings interesting enough to merit attention.

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1. Tarski, A., Mostowski, A. and Robinson, R.M., *Undecidable Theories*. North Holland, Amsterdam, 1953.

2. Holmes, R., Personal Communication, 1990.

3. Goedel, K., The consistency of the axiom of choice and of the generalized continuum hypothesis. *Annals of Math Studies*, 1940, **3**.

4. Cohen, P. J., *Set Theory and the Continuum Hypothesis*, New York, 1966.

5. The axiom schema of subsets is ∃u[[u = 0 ∨ ∃*xx*∈*u*] ∧∀*xx*∈u ↔ *x*∈*z* ∧ *X*(x)], where *z* is any set and ***X***(x) is any formula in which *x* is free and *u* is not free. The axiom enters ZF in AR but can also enter in the strong form of the axiom of regularity. (Note T has the weak form.) This axiom is essential to obtain the diagonal set for Cantor's proof, using x∉f(x) for ***X***(x), where f(x) is an assumed one-to-one mapping between ω* and P(ω*). The argument leads to the contradiction ∃c∈z***X***(c) ↔ ¬***X***(c), where f(c) is the diagonal set. In ZF, this denies the mapping exists. In T, the same argument instead denies the existence of the diagonal set, whose existence has been hypothesized while the mapping was asserted as an axiom. What if we tried another approach for Cantor's proof, by using ABR to get a characteristic function? Let ϕ(x,y) ↔ [***X***(x) ↔ y = (x,1) ∧ ¬***X***(x) ↔ y = (x,0)] and z = ω*. If c were a member of ω*, t = (c,1) and t = (c,0) both lead to a contradiction. But, since the existence of the diagonal set f(c) is denied and since a one-to-one mapping between ω* and P(ω*) is an axiom, as f(c) is not a member of P(ω*), so c cannot be a member of ω*. In T the characteristic function exists but has no member corresponding to a diagonal set.

6. The $u_{\ell mi}(x_i)$ and $u_{\ell mj}(x_j)$ are iterated using (1). The $p_{\ell mi}(x_i), q_{\ell mi}(x_i), p_{\ell mj}(x_j)$ and $q_{\ell mj}(x_j)$ will generally change at each iteration and are given by $p_{\ell mi} = \int \frac{P_{\ell mi}\Psi_{\ell m}^2 d\tau}{u_{\ell mi}^2 r_i dx_i} \Big/ \int \frac{\Psi_{\ell m}^2 d\tau}{u_{\ell mi}^2 r_i dx_i}$, etc.
Since the field is continuous, differentiable to all orders, of bounded variation and thus free of singularities, iterations for all $u_{\ell mi}(x_i)$ and $u_{\ell mj}(x_j)$ will converge jointly within a finite region.

7. Essentially the same reasoning can be applied to the spatial dimensions. The nonlinear sigma model is definable in T if and only if *M* is finite. In T, the range and domain of the irreducible biunique eigenfunction pieces in each of the spatial dimensions is finite (i.e., is not infinitesimal or infinite) and all functions are continuous. Thus, if any spatial dimension is infinite, α over all space-time, that is *M*, is infinite. If all spatial dimensions are finite, the model is quantized, hence definable in T and *M* is finite. The model is thus definable in T if and only if all the spatial dimensions are finite. We have obtained compactification effectively. Note that this is achieved without invoking boundary conditions. It follows that compactification of the spatial dimensions is equivalent to quantization.

Appendix

ZF - AR + ABR + Constructibility

Extensionality- Two sets with just the same members are equal.

$$\forall x \forall y [\forall z [z \in x \leftrightarrow z \in y] \rightarrow x = y]$$

Pairs- For every two sets, there is a set that contains just them.

$$\forall x \forall y \exists z [\forall w w \in z \leftrightarrow w = x \vee w = y]$$

Union- For every set of sets, there is a set with just all their members.

$$\forall x \exists y \forall z [z \in y \leftrightarrow \exists u [z \in u \wedge u \in x]]$$

Infinity- There is at least one set with members determined in infinite succession

$$\exists \omega^* [0 \in \omega^* \wedge \forall x [x \in \omega^* \rightarrow x \cup \{x\} \in \omega^*]]$$

Power Set- For every set, there is a set containing just all its subsets.

$$\forall x \exists P(x) \forall z [z \in P(x) \leftrightarrow z \subseteq x]$$

Regularity- Every non-empty set has a minimal member (i.e. "weak" regularity).

$$\forall x [\exists y y \in x \rightarrow \exists y [y \in x \wedge \forall z \neg [z \in x \wedge z \in y]]]$$

Replacement- Replacing members of a set one-for-one creates a set (i.e., "bijective" replacement).

Let $\phi(x,y)$ a formula in which x and y are free,

$$\forall z \forall x \in z \exists y [\phi(x,y) \wedge \forall u \in z \forall v [\phi(u,v) \rightarrow u = x \leftrightarrow y = v]] \rightarrow \exists r \forall t [t \in r \leftrightarrow \exists s \in z \phi(s,t)]$$

Constructibility- All the subsets of any ω* are constructible.

$$\forall \omega^* \exists S[(\omega^*,0) \in S \wedge \forall y E!z[y \neq 0 \wedge y \subseteq \omega^* \wedge (y,z) \in S \leftrightarrow (y \cup m_y - \{m_y\}, z \cup \{z\}) \in S]]$$

where m_y is the minimal member of y.