Higher-Dimensional Solitons Stabilized by Opposite Charge

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In this paper it is shown how higher-dimensional solitons can be stabilized by a topological phase gradient, a field-induced shift in effective dimensionality. As a prototype, two instable 2-dimensional radial symmetric Sine–Gordon extensions (pulsons) are coupled by a sink/source term such, that one becomes a stable 1d and the other a 3d wave equation. The corresponding physical process is identified as a polarization that fits perfectly to preliminary considerations regarding the nature of electric charge and background of 1/137. The coupling is iterative with convergence limit and bifurcation at high charge. It is driven by the topological phase gradient or non-local Gauge potential that can be mapped to a local oscillator potential under $PSL(2,\mathbb{R})$.

Introduction. Solitary waves were discovered in the first half of the nineteenth century by Rusell, the word soliton was invented by Kruskal, the sine–Gordon (SG) model by Skyrme [1]. Solitons retain their identity after collisions, can annihilate with anti–solitons, many–soliton solutions obey Pauli's exclusion principle. In 1+1–dim. space–time there are two non–trivial minimal quantum field theories which describe non–perturbative phenomena: the SG model and the massive Thirring model [2] (a self-coupled Dirac field, see the Lagrangians [3]), both are intimately related [4]. For nonlinear field theory models in 1+1–dim. space–time the equations of motion admit finite energy and finite width solutions called solitons [5]. In the previous paper fundamental three–dim. (3d) baryon particles have been assigned to soliton properties [6, 7]. But how can 3d soliton properties emerge, and how could a baryon-type 3d SG soliton subject to distortions be balanced and stabilized?

a. Time independent field equations. The low-dim. (bosonic) hermitian scalar field θ with Lagrangian density $\mathcal{L} = \frac{\mu}{2}\partial_\nu\theta\partial^\nu\theta \; - V(\theta)$ is a function of one space dimension and time (1+1–dim.). The time independent field equations reads $\mu\partial_r^2\theta = \partial_\theta V$ which can also be written as

$$\partial_t V = 0, \quad V(\theta) = \frac{\mu}{2}(\partial_r\theta)^2 \,, \tag{1}$$

where the sine–Gordon equation (SG)

$$\partial_{rr}\theta - V_0 \sin\theta = 0 \,, \tag{2}$$

has a potential given by

$$V(\theta) = V_0(1 - \cos\theta) \,. \tag{3}$$

In [7] it has been shown, that Rayleigh-type self–excited auto-parametric systems [8] can stimulate in a 3d-situation "whispering gallery modes" (that have been measured in [9, 10]) and model Coulomb interaction between sine-Gordon solitons. In [6] the same model has been be applied to determine the most likely Compton mass of the soliton. In this paper these results will be connected to the Skyrme baryon model and to the dissipative models including sink/source term representing the soliton charge.

b. Polarization and radiative coupling of pulsons. The isolated 1+1–dim. topology of a SG soliton is stable, integrable, and interaction-free. A single 2-dim. radial symmetric Sine–Gordon extension (pulsons [11]) is not stable. The dissipative property can be found by regarding the 1+2–dim. pulson solution with dissipative term $\partial_r\Theta/r$ and external coupling term $\pi M_g/M$

$$\partial_{rr}\Theta - V_0 \sin\Theta + \frac{\partial_r\Theta}{r} = \frac{\pi M_g}{M} \,. \tag{4}$$

Without coupling, the neutral or source free pulson ($M_g = 0$) is a breather-like slowly dying solution [11]. But if we choose the strength of the energy source/sink such, that $M_g \neq 0$ can compensate the dissipative term, the 2d - pulson can be either reduced to a pseudo–1d case $\partial_{rr}\Theta - V_0 \sin\Theta = 0$ by compensating the first–order term, or be promoted to a 3d radial wave equation $\partial_{rr}\Theta - V_0 \sin\Theta + 2\partial_r\Theta/r = 0$ depending on the sign in

$$\frac{\partial_r\Theta}{r} = \mp\frac{\pi M_g}{M} \,. \tag{5}$$

This means simply, that a dimensional shift (an additional phase gradient proportional to the radial distance) is induced by the Gauge potential of a sink or source. If there is a permanent external source of stochastic nature (thermal background radiation or fluctuations), it increases dimensionality and provides for a basic r–independent part in the potential $W(\Theta = 0) = V_0$. If two neutral pulsons become permanently polarized by adding a source term to one and a sink term with opposite sign to the other, there will be a radial coupling that can be assigned to a opposite signed "charge" $\pm M_g$ defined in eq.(5). The positive source $M_g > 0$ with positive charge will be assigned to the state that has increased dimensionality driven by a permanent external stochastic radiation source providing for $V_0 > 0$. In this case eq.(5) directly provides for

$$W(\Theta) = \frac{\mu}{2}(\partial_r\Theta)^2 + V_0 = \frac{\mu(1 + r^2)}{2}\left(\frac{\pi M_g}{M}\right)^2 \,. \tag{6}$$

As already shown in [12, 13, 14] eq.(5) is the projective condition necessary to adjust the topological phase fields on pseudospheres (constant negative curvature) that provides for the optimum feedback resonance, coupling strength, and corresponding fine structure of spin precession. Integrating eq.(5)

$$\Theta(r) - \Theta(0) = \int_0^r \partial_r \Theta dr' = \mp \frac{r^2}{2} \frac{\pi M_g}{M}\,, \tag{7}$$

relates the potential linearly to the scalar function Θ

$$W(\Theta) = \mp \Theta \mu \frac{\pi M_g}{M} + V_0, \quad \Theta(0) = 0\,, \tag{8}$$

that gives also the basic potential

$$W(0) = V_0 = \frac{\mu}{2} \left(\frac{\pi M_g}{M} \right)^2 . \tag{9}$$

The external source or thermal pool provides for the same scalar angle–dependent energy in both, the oscillator energy responsible for the dimensional change and polarization in eq.(8), and for the soliton coupling energy in eq.(3)

$$W(\theta_M) = V(\theta_M)\,. \tag{10}$$

Combining eq.(3) with eq.(8) via eq.(10) allows to determine the optimum phase shift of resonant coupling θ_M iteratively from

$$M\theta_M = \pm \pi M_g \cos\theta_M, \quad \theta_M = \pi\alpha_M, \tag{11}$$

see results in table I.

TABLE I:
Convergent fine structure (re)generation constants α_M for $M_g = 1$ and variable $M > 2$ [16], check also the simulation at [17].

M	$1/\alpha_M$
3	4.13669
4	4.96178
5	5.82662
6	6.72097
7	7.6371
8	8.56944
9	9.51399
10	10.46789
11	11.42906
12	12.39597
13	13.36747
137	137.03600941164

c. Topological and geometric phase interpretation. Two neutral pulsons can be mutually stabilized by assigning a positive "charge" ($M_g > 0$) to one and a "negative" charge ($M_g < 0$) to the other. This excites $M_g > 0$ to a pseudo 1+3–dim. soliton and reduces $M_g < 0$ to the pseudo 1+1–dim. topology of a SG–soliton. It is known that stereographic mapping can map a Coulomb potential to an oscillator potential and vice versa. With eq.(7), eq.(11), $\cos\theta = 1 - 2\sin^2(\theta/2)$, the projective relation is given by

$$r = 2\sin(\theta/2). \tag{12}$$

Interpretation: In the resonant case a topological phase scattering pattern is coupling back to the scatterer. The resulting non-linear scalar coupling field is a deficit angle field that obviously can be described by the sine-Gordon equation on pseudospheres with a local potential given by the square of the phase gradient. Eq.(12) is a projective scattering condition, where the isotropic radial coupling connects the pulsons by projective resonance. It is a projection under $PSL(2,\mathbb{R})$ that maps a local oscillator potential to a global Coulomb sink/source on the sphere and pseudosphere [15]. The mapping is directly controlled by the iterative solution eq.(11) providing for the generalized fine structure constants. The coupling or scattering condition eq.(12) maps the deficit angle to the proper spatial distance by relating the phase gradient potential to an oscillator potential while changing the dimension.

d. Coupling strength and energy. Compton scattering can model the quantum interaction of a linear wave and a particle. To define a coupling strength q between SG kink or antikink induced by the phase fluctuations generated by background radiation, it will be necessary to define some potential and energy relations in 1d and 3d. The 1d coupling energy can be defined by by a temporal average or mean unit energy E_{1d}

$$E_{1d} = q^2 \overline{(\partial_r \theta)^2} = 1\mu c^2 = 2q^2 \overline{V}\,, \tag{13}$$

where μ is a unit mass, c the light velocity, and $\overline{V} = V_0$ the mean background radiation energy. To compare our theoretical soliton coupling model to real existing couplings, mass/energy has to be quantified and geometrized. The mutual 1-d coupling to photons with amplitude/wavelenght fluctuation λ_μ can be regarded as a permanent Compton scattering process with mass–energy value related to λ_μ via Compton relation

$$E_{1d} = 2q^2 \overline{V} = \frac{hc}{\lambda_\mu}\,. \tag{14}$$

Let's connect pulsons at distance $2R$ by defining the 3d-potential in accordance with eq.(5)

$$\phi_{3d} = \frac{q}{4\pi R} = \frac{\partial_r \Theta}{4\pi R^2} = \mp \frac{M_g}{2MR}\,. \tag{15}$$

Generally, the Gauss relation can connect the 1-d coupling strength to a 3-d coupling strength with a spherical symmetric potential $\phi_{3d}(r)$ such, that the radial coupling energy is defined by

$$E_{3d}(r) = \frac{q}{\epsilon_0}\, \phi_{3d}(r)\,. \tag{16}$$

The coupling strength q is the charge defining the interaction

$$E_{3d}(r) = -\frac{1}{\epsilon_0}\int_\infty^r \phi_{3d}^2 4\pi dr' = \frac{q^2}{4\pi\epsilon_0 r}, \quad \phi_{3d} = \frac{q}{4\pi r}\,. \tag{17}$$

The fine structure constant is defined by

$$\alpha = \frac{q^2}{4\pi\epsilon_0\hbar c} = \frac{E_{3d}(\lambda_\mu)}{E_{1d}}\,, \tag{18}$$

where the relations at the special reference distance λ_μ given by dimensionless Planck units $h = c = \lambda_\mu = 1$ must obey the unit condition

$$E_{3d}(\alpha\lambda_\mu) = E_{1d} = \phi_{3d}(\alpha\lambda_\mu) = \phi_{1d} \equiv 1\,, \tag{19}$$

that provides for

$$\alpha = \frac{q}{4\pi}, \quad M = \left[\frac{1}{\alpha}\right]. \tag{20}$$

where [ ] means next higher integral value. Why integral? Because of single–valuedness the round-trip path fits integer numbers, similar to "whispering gallery modes" [7].

e. Concluding Remarks. It is an interesting question how stationary solitons (like breather) get their absolute mass/energy. To approach a 3d scattering we can compare to an ansatz for Skyrme fields in 3d [18] which uses rational maps between Riemann spheres under $PSL(2,\mathbb{R})$ and an $SU(2)$ valued Skyrme field. The lowest energy E_{1d} of Skyrmions which applies to the rational map ansatz, is more than a factor $12\pi^2$ lower than the energy given by the Lagrangian and called Fadeev-Bogomolny bound, see also [19, 20, 21]. In our case this coupling ratio is given by $q^{-2} = 12\pi^2$ in eq.(14). This could be found by treating the wave–soliton coupling as a Rayleigh–type auto–parametric system, see [6, 7]. Eq.(20) provides for $M = 137$ and for a plausible baryon energy limit: $\lambda_\mu \approx 1,31777... \cdot 10^{-15}$m or $E_{1d} = \mu c^2 = 940.86369...$ MeV extrapolated to Planck units $\lambda_\mu = c = \hbar = 1$ emerges as a system-invariant soliton mass scale that is 1.001382 times the neutron and 1.002762 times the proton scale [6]. The two oppositely charged 2-dim. radial symmetric pulsons stabilized and balanced by a topological phase gradient could be assigned to a proton–electron combination: one pulson is promoted to a positive charge with $M_g > 0$ as a pseudo 1+3–dim. soliton (the proton) while reducing the second pulson to a negative charge $M_g < 0$ and pseudo 1+1–dim. topology of a SG–soliton (the electron). The broken symmetry could be supported by the coupling field that could act as a flexible shield against external distortion or fluctuations in dimensionality. Generalizations to higher dimensions and source terms could be assigned to multiple charged nuclei, where the charge quantity $M_g q$ with $M_g \neq 0$ corresponds to a field–induced shift in effective dimensionality with $d - 2 = M_g$, stability and convergence criteria regarding eq.(11) for $1 \leq |M_g| \leq M$ can be found in [16]. To account for the (half) spin property of pulsons with cylindrical symmetry, it should be possible to characterize the polarized d=1/d=3 soliton system by two coupled two-spinors under $SU(2)$ including electromagnetic interactions defined by vector and scalar potentials responsible for the dimensional shift.

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