Mach's contribution to the origin of Inertia.

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1. Introduction.

In the middle of the nineteenth century the compact picture of Physics, that had been developing during the XVIII century, got into a crisis either for the appearance of new discoveries in the natural sciences or for the renewed scientist interest in the foundation of their subject. A decisive contribution for this process was supplied by Ernst Mach who showed the conceptual values and defects of Classical Mechanics with a learned historical analysis. In fact at the basis of the Machian work, besides the philosophical position that ties up with the traditional empiricism, there is the awareness that only the historical approach of physical principles clarifies their origin, meaning and accuracy. For example the concept of space and time were replaced by a formulation more pertaining to the experience after having showed the logical weakness and inconsistencies of the Newtonian definitions.

Mach's work produced a profitable debate among the german physicists and some of them tried to develop, deepen and transform in mathematical form his numerous critics and ideas. Among them the Einstein figure which got many suggestions from the reading of the "*Mechanic in its logical and historical development*" [1], evident either in the article of 1905 on the Special Relativity or in the laborious way that would carry him to the formulation of the General Relativity stands out best.

The key concepts of the Special Relativity are a group of sentences on space and time that Einstein proposes in the early pages of the article: for example he says that "*We have to take into account, that all our judgements in which time plays a part, are always judgement of simultaneous events. If for instance I say, 'the train arrived here at seven o'clock', I mean something like this: 'the pointing of the small hand of my watch to seven and the arrival of the train are simultaneous events.*".[2] The previous passage reminds us of the Machian philosophy [3] in two aspects at least: above all it derives from the observation of daily experiences like the train arrival or the rotation of the watch's hands and not from abstract considerations without direct correspondence; secondly it's like Mach's reflections on "absolute time". In the *Principia Mathematica* Newton had introduced the definition of "absolute time" like something that flows uniformly without any reference to external objects; Mach, on the contrary, thinks that when we point out the temporal changes of an object A - he had in mind the pendulum oscillations - we are simply saying that the states of A depend on, are in relations with those of another object B - for example the earth rotation (cfr Par. 3). If we compare the Einstein criticism to simultaneousness with Mach's considerations on absolute time, we note there is an analogy between the two argumentations; as the first one compares two events like the train arrival and the position of the watch's hands, the second confronts the states of A (pendulum oscillations) with those of B (the earth rotation).[3] More subtle, but equally effective was Mach's influence on the construction of General Relativity. One of the pillars of the new theory is the Equivalence Principle between gravitational and inertial mass, that Einstein reaches after a long research starting directly from the reading of some passages on inertia contained in the Mechanics. Mach's Principle, which Einstein formulates for the first time in 1912 [4] and that links the inertial properties of a body to the presence of the other masses in the universe, represents from one aspect an end point of Einstenian reflections on Mach's work, and, from another aspect, the stimulus to get through the question and reach a complete formulation of the Equivalence Principle [5].

If Einstein carried on the extreme consequences of Mach's ideas till the construction of a new theory, we must not forget that some second level physicists of the german area, like Benedict and Immanuel Friedländer and August Föppl, were the first to have developed them in a strictly mechanical framework. Their research was at first an experimental one and they propose a series of experiments to detect centrifugal forces and an inertial reference frame disconnected from the Newtonian concepts of absolute space and time.

All the experiments start from the reading of the Machian criticism to the concepts of absolute space and time, and its application to the experience of Newton's bucket. The English scientist had proposed the following experiment: we hang a bucket full of water to a rope and then we impart a continuous circular motion till the rope becomes stiff. Releasing the rope to its movement, the bucket begins a rotational motion, while the water inside stays at rest. The motion continues and the rotation passes from the vessel to the liquid up to the case with the bucket at rest and the free surface of water that becomes hollow. The rising of water along the sides of the vessel isn't due to the relative rotation to the borders, as we haven't the same effect when the bucket rotates and the water is at rest (verifying in any case a relative rotation), but represents the true motion of the liquid. The observed effect is explained by the presence of a *centrifugal force* that drives away the water particles from the rotational axis.

The concept of centrifugal force was developed by Huygens, deepened by Newton and accepted by the scientific community during Seventeenth century without any particular difficulties: the centrifugal force derived from the body's inertia and has been considered an intrinsic property to its state of motion. Newton used this property to prove the existence of the absolute motion: in fact the observation of centrifugal accelerations, that is something intrinsic to the state of motion, allows to distinguish between absolute and relative motion.

In the Mechanics, Mach tests the bucket experiment and reaches the conclusion that centrifugal forces are "*produced*" by the relative motion of water in respect to the earth mass and "*the other celestial bodies*", while its motion referred to "*the vessel sides*" doesn't produce any observable effect. His final result is an explanation of the experience in relative terms only, where concepts like absolute space, time and motion are absent. The centrifugal forces are described through the relative motion of the water according to the set of fixed stars and they don't represent the expression of a true rotation into an absolute space.

But Machian thought can be interpreted from another point of view. In fact he says that the centrifugal forces "*are produced by the relative motion*" to the mass of fixed stars, as if the stars had an active role in the determination of the forces. It would seem that Mach suggests a "*new physical mechanism*" which explains the origin of inertial forces in some causal interaction between the water in the bucket and the other universal masses [9].

Among the two interpretations the second one gained more success, as both Einstein and the german physicists directed their research to clarify with theoretical argumentations and practical experiments whether the second reading of the Machian passages could lead or not to new physical results. Admitting that the centrifugal accelerations observed in the bucket's experiment *are due to the relative motion* compared to the distant masses, we deduce that any rotational motion of sufficiently massive bodies will produce centrifugal forces directly observable. This conclusion is the starting point of the works of Friedländer , Föppl and Einstein.

Let's start to analyse briefly the works of these authors to show the remarkable importance of the Machian considerations on the vessel experiment.

In a monograph of 1896 Immanuel Friedländer, in collaboration with his brother Benedict, exposes a motion analysis taking suggestions either from Machian work or suggesting an original contribution [6].

Immanuel starts from the conviction – reached after long personal studies – that "*the appearance of centrifugal forces is only explicable in terms of relative motions of the physical system concerned, without resorting to any concept of absolute motion.*" This idea leads him to show two lines of research: the first reformulates the concept of moving mass's "vis viva" arriving at a new formulation of the Inertia Principle, the second shows through an experimental way that the present formulation of Mechanics is inadequate. The choice of Immanuel, and his brother, is the second one and they try (in particular Benedict) to find an experiment to determine centrifugal forces produced by relative motion. In fact the experience is constituted by a massive fly-wheel with a torsion balance in the center; the goal is to check if the relative motion of the wheel with reference to the torsion balance produces centrifugal forces, measured by the latter.

The german physicist Föppl published a monograph in February 1904, in which there's an experiment, like that of Focault's pendulum, to determine the correct position of an inertial reference frame. Föppl is interested in comparing two reference systems: that of fixed stars and the one determined by the pendulum experiment. Usually they're thought of as inertial and coincident, but according to our author's view they could defer by a little influence due to the earth rotation. The experiment result – of which the monograph was the report – was negative, but it's interesting to discover its conceptual foundations.

A reply came from the second article published in November 1904 [8], where Föppl quotes Mach as his source of inspiration. As in the case of the Friedländer brothers, Föppl makes the hypothesis that the relative rotational motions (the earth rotation) produce somehow inertial forces. He considers a set of n massive points divided in two other subsets G_1 and G_2; the first, moreover, is the greatest, while the second is in rotational motion. If we consider the two groups as two physical reference systems we can easily detect the G_1 set with the fixed stars and G_2 with the earth. We are now in the experimental conditions described in the communication of 1904. With the Machian considerations the subset G_2, in relative rotational movement with reference to G_1, will produce an influence on G_1 which appears in a Coriolis force directly observable.

At last we consider Einstein's article of 1912 [4]. He considers a massive spherical thin shell K, a massive point P at its centre and shows how the motion of K modifies the inertial properties of P ([1])

At the end of the article Einstein introduces Mach's Principle:"*. . . the presence of the inertial shell K increases the inertial mass of the material point P within it. This makes it plausible that the entire inertia of a mass point is the effect of the presence of all other masses,*

([1])the result reached by Einstein with a preliminary scalar theory of gravitation is the following:

1. the presence of the spherical shell of mass M and radius R increases the inertial mass m of the point at its centre of a quantity $k(\frac{mM}{Rc^2})$ where k is the universal gravitational constant.
2. An acceleration Γ of the shell leads to an acceleration $\gamma = \frac{3}{2}k(\frac{M}{Rc^2})\Gamma$ of the massive point m.

resulting from a kind of interaction with them. This is exactly the standpoint for which E. Mach has argued persuasively in his penetrating investigations of this matter" [4].

We have described shortly how a certain interpretation of the Machian passages can be considered the foundation of innovative physical researches; but it remains to verify its accuracy. In the following pages we'll try to explain this point and we'll arrive at the conclusion that the way followed by the Friedländer, Föppl and Einstein is a forced interpretation of the Machian arguments.

2. The relation between Mechanics and Physics and the discovery of the Universal Principle.

Mach's scientific and philosophical works are characterized by the analysis of some old philosophical questions which make sense because they involve the 800 science foundations. Questions as what is an observation, an experiment, a mass, the forces or a natural law are all subjects which Mach affronts during his intellectual track, employing his mathematical and physical skills along with his historical background. The last one in particular leads him to the analysis of the foundations of Physics and Mechanics and drives him through the research of their strict reciprocal relation. The link between Mechanics and Physics is typical of the philosophical reflection of XIX century and Mach takes part in the debate with an innovative and determinant contribution.

At the middle of the nineteenth century the dependence of the Physics from Mechanics is a result obtained first of all by d'Alembert's work, but it's Helmholtz who develops exhaustively the mechanicistic programme ("*On the force's conservation* – 1847 [10]). The research methodology of the Mechanicist is founded on the principle of reducing every natural transformation to a sufficient cause: that's why the natural phenomena must be split into others more simple through the research of their causes, till the last causes and so on to the elementary phenomena. The reductionism for the first time divides the research into two distinct activities: an experimental science and a theoretical one. The first has the task "*to search the laws, from which the individual processes of Nature can lead back to general rules*", for example Mariotte's or Gay Lussac's laws, even if they defect in not giving the causes of natural behaviour.

For example if we consider Boyle's law, we know how to join some properties of a gas like temperature and volume, but we remain doubtful on the origin of this behaviour. The research of the causes of the natural phenomenon goes beyond the simple description of the appearances and surpasses the experimental science arriving at the theoretical one: it has the precise target to "*find the unknown causes of the processes from their visible effects and try to understand them also through the causal law*" [10]. But how can we explain in Physics the unknown causes of the phenomena? The external objects are grouped under the abstract concept of matter, which has descriptive qualities, such as the position and the mass, along with efficient quality about which the experience informs us. In Physics the efficient factor is translated in the concept of force and so studying the last causes of the phenomena is equivalent to research the fundamental forces. Helmholtz goes on and specifies some properties of these forces. In fact if the fundamental elements to which each explanation leads back must maintain their characteristics, it derives that the only changes possible are those of position: so the forces are *Driving Forces*. In order to use the mechanical model of material points and central forces of attraction and repulsion, the natural systems must be split into smaller parts and the acting forces broken into their elementary components.

The task of Physics, and of Science in general, is now well detected: "*to reduce the na-*

tural phenomena to immutable forces, attractive or repulsive, whose intensity depends on the distance [10].

Helmholtz's reductionism was criticized by Mach in the work of 1872 [11] in which he studies the history and the origin of the Principle of labour conservation (now we'll talk of energy). The starting point of the Machian analysis is the concept of matter affronted from a sensistic position. The world that is presented to us by our sensibility has an extraordinary richness of qualities and considering the matter for its simple geometrical and mechanical properties, means to privilege without any reason the characteristics that don't present any ontological diversities in respect to the others. The reality we obtain through a sense isn't more real and doesn't have a different value from that furnished by the other senses.

Mach uses a particular word to indicate the data provided by sensibility: he considers them like a set of elements. Their use has the value to eliminate the intrinsic subjectivism contained in any form of anchorage of science to sensibility, purifying the language of the sensations from any possible personal interpretation. In this manner we say that the elements represent a common impersonal foundation over which we construct scientific knowledge [12] [13] [14].

Besides they allow us to surpass the mind-body distinction which is the foundation of the mechanical reduction: the fallacy of our sensibility, that forces us to go beyond the limits of experience, is admitted only by considering an effective separation between mind and body. However according to Mach the external objects are something that differ from us too, but we can perceive their behaviour only through mental structures that base themselves upon the elements; in other words "*the sensations are not representations of things, but on the contrary a thing is a mental representation of a set of relatively stable sensations*"[13]. The target of science is now reduced to the simple task of reproducing mentally the relations among sensible data. In few words a descriptive operation, but which must possess a qualitative aspect as any scientific knowledge. In Physics we find an example in the Principle that fix the convertibility of heat into mechanical labour, in which we detect a numerical connection between a mechanical quality and one that isn't. Then science has the instrument to link different processes to one another, and to connect them without transforming them qualitatively, free from suppressing the intuitive differences and without supposing that the senses deceive us giving only the appearance of things.

The base over which science is constructed is now determined: it's the elements. Each physical concept must reduce to them and when this doesn't happen the cause is in an imperfect correspondence with the experience. We are now in front of a typical error of metaphysical thought which conceives as real, fictitious entities.

But how can we detect the prejudices of our inclination to metaphysics? Our only possibility is the History of Science that enlightens certain mental processes and certain aptitudes of our mind. For example the Historical study of the Mechanics evolution shows us that the most well known science, that is used as paradigm, doesn't lay on stable and firm foundations. Then the attempt to reduce the knowable to this reveals to be fallacious, because we research something that goes beyond experience. Here is the appearance in scientific debate of imponderables or fluids to which we attribute existence, we debate on immeasurable properties, create problems without solutions. An example of this metaphysical attitude, for Mach, is the generalization made by Helmholtz of the *Principle of the absence of the Perpetuum mobile* which performs the reduction of each natural phenomenon to mechanical scheme.

We consider a system of n particles that evolves from an initial instant to a final one. The passage creates a labour quantity equivalent to the "*vis viva*" variation of the particles. Supposing we bring back the system to the initial state; can we make this passage without spending any labour? If this were possible, we would create a perpetuum mobile which produces labour from nothing, because we would cause energy from the initial to the final state but we would spend nothing to bring it back to the initial point. The experience ensures this behaviour is impossible. In its mathematical form the Principle of absence of Perpetuum mobile is translated in the *Principle of Vis viva conservation*, where the labour spent by the system in the passage from one state to the next one is equivalent to the difference of its Kinetic Energy. The law can be extended if we admit that the labour is produced by central forces (consideration which Helmholts accepts because the fundamental forces are central forces) obtaining in this manner the *Principle of the force conservation*. The labour that equates the Kinetic Energy of the system is now split into the sum of the products between the force (*Tension force*) interacting among two single particles and their reciprocal distance.

The mathematical form of the Principle becomes:

$$\frac{1}{2}\sum_n mv^2(t_2) - \frac{1}{2}\sum_n mv^2(t_1) = \int \sum_n (Xdx + Ydy + Zdz) \tag{1}$$

where the tension force has the components (X, Y, Z) and $v(t_1)$ and $v(t_2)$ are the velocities at the instants t_1 and t_2.

With the law (1) we have the instrument to perform the mechanical reduction of each natural phenomenon. In the work of 1847 Helmholtz applies the principle of force conservation not only to the fields already known, such as gravitation, the movement transmission through incompressible bodies, the movements of solid and liquid bodies perfectly elastic, but those which were studying for the first time in that period too (thermal, electrical and magnetic phenomena).

The principle of absence of the Perpetuum mobile leads to the law (1) but has a more general value. Besides it's the research foundation of each great past naturalist. For example Stevin considers it as an *instinctive experience*, whose contrary is absurd. He reaches the certainty that the chain put on a solid body of triangular form doesn't move and determines in this way the laws of inclined plane. Galileo conceives the same principle in the form of the law about which a body, by virtue of the falling velocity, rises at the same altitude; he uses it in the study of pendulum motion, and in the analysis of inertia law too. Even Huygens and the Bernoulli brothers use this principle in the works on the "vis viva" conservation. Although it has had a great influence on the evolution of Physics, its factual meaning remains uncertain. Mach noting this principle asserts something that is evident to everyone: that bodies won't rise on their own. They can't make this movement because they are linked to the earth by the gravitational attraction. In other words, a mass doesn't rise spontaneously because in this case it would break the dependence which links it to the other bodies. Mach expresses this conclusion with the following words: "*the principle of absence of perpetuum mobile is the principle of the natural universal relation*"; "*it depends on the knowledge of the universal natural dependence*". Therefore the principle expresses:

> *the experience result: the sensitive elements of the world α β γ ... appear as dependent on one another. The experience teaches us that the sensitive elements α β γ ... into which the world can be broken up are subject to variations and also that some of these elements are connected to others, as they appear and disappear together, or as the appearance of the elements of a type are linked to the disappearance of the elements of another.*

Therefore the notion of the reciprocal phenomena dependence is the starting point of each scientific knowledge and it will be the task of the different Science domains to define and measure the various relations. If to indicate this connection of facts, we prefer to use the term "causality", then we can identify the principle of absence of the perpetuum mobile with the principle of causality.

At the foundation of the causal concept there is the representation of a particular fact: the modification produced by our will in the environment in which we live. The connection between will and motion has taken the first relation of cause and effect, but it remains qualitative and forcefully indefinite. Only with the birth of Science there is the necessity to discover precise connections. The researcher proceeds circumscribing a set of relations from the totality of connections that the sensibility gives us; then he considers the relation which he has put in evidence as the fact in its totality and has the tendency to refer to the determinant fact a character of substantiality. Mach describes this methodology in the following manner:

> *When we speak about cause and effect, we put arbitrarily in evidence those circumstances, whose connection, in the representation of a thing, leads to an important result for us.*

Therefore there is a "modellization" at the foundation of the causality concept, a modellization which doesn't consider all the data given by the sensibility, but which makes emphasis only on some. On the contrary the scientist's activity must not confine itself to the creation of descriptive model of the experience, as already been taught by Helmholtz's philosophical position, but has the inclination to generalize the obtained principles and results (an example is the extension of the principle of "vis viva" conservation to the force conservation one) to create new models and experiences. What is Mach's position on this practise? He thinks theoretical activity is an important part of the scientist's work, but underlines the fact that the most scrupulous scientist shouldn't exchange this modellization with what is observed. His task is to reach a complete description of natural phenomena surpassing the hypothetical and temporary models. If therefore the experience is produced by theoretical models, as in Helmholts case, it must be integrated, in such a way that all the phenomenon is given, without omitting important aspects. In the Mechanics Mach remembers that the experience can't be "constructed" but it's necessary that it manifests itself. So the final goal remains that expressed by the universal principle: *the description of the dependence among the elements of sensibility in which the experience is broken.*

3. The motion representation in Mach and the analysis of some equivocal passages.

The highest point of the Machian historical-conceptual analysis is reached in some pages of the Mechanics, grouped in the paragraph "*Newton's ideas on time, space and movement*". In them Mach shows the conceptual weakness and the numerous logical inconsistencies of physical foundations such as the absolute space and time.

Close to this function of analysis and explanation, the same passages have represented a strong stimulus to the renewal of Physics, as giving evidence in some fundamental ideas of Einstein which have their roots in these pages (cfr. Introduction). Indeed the proposing effect leads us to consider Mach as a "forerunner" of intuitions which will have success in Physics; we think for example of the hypothesis of a causal link between the inertial properties of a body and the presence of the other masses of the universe (*Mach's Principle*).

On the standard of the Machian philosophy we analyse the indicated passages and we'll reach some conclusions that disavow in part this opinion. Each critique to the Mechanics foundations has as its starting point the principle of the universal dependence of the phenomena.

Let us consider the *time* concept: it's an instinctive notion that takes place from the existing narrow relation between the contents of our memory and the sensitive perception. For example when we look at the pendulum oscillations, we refer them to our conscience and have the impression that we describe something that flows independently from us. Here is then the opinion of the uselessness of the external objects and the consequent absoluteness of the concept which derives. In reality when we assert that an object A changes in time, we are comparing the states of A with those of an other object B whose states are considered as a unity of measurement. If we reconsider the Pendulum example, we note that its oscillations aren't a motion apart in an inert background constituted by the other objects, but represent a movement which is compared with the Earth's position during its daily revolution. In a similar manner, in the reporting of the time notion to the flow of our conscience we must not forget that we "*are part of nature with our thoughts*". Besides the notion of absolute space is represented by a uniform motion, but this isn't conceivable if we don't compare it to another movement. In fact we call uniform that motion in which equal increments of space correspond to equal space increases in a motion considered as a model. With a deeper insight we are in the condition similar to those of Thermometry: as in the studies of thermic phenomena we choose a volumetrical index which changes similarly to our sensation of heat and isn't exposed by the perturbations of our sensibility, so we'll use an arbitrary motion of parallel proceeding to our sensation of time to measure the time (for example the rotational motion of Earth). This operative definition of time eliminates each metaphysical obscurity and joins the notion to a comparison among movements: therefore time becomes the measure of relative positions among bodies in motion. If the time represents the more intimate expression of the reciprocal dependence of the natural phenomena - because our consciousness takes part - which is its most evident form? The answer is to research in the principles of Dynamics and in particular in the third.

We consider a simple example: examine a board L upon which rests a weight P. The third principle of Dynamics guarantees the body doesn't fall, because P exerts a pressure on the board to which it reacts with a contrary force. The two bodies L and P are in reciprocal dependence because they exchange pressure-forces to maintain the balance. But how does this interaction manifest itself? We know that pressure exerted by P is given by the product of its mass and the gravitational acceleration g. If we let the board fall with an acceleration g , we observe the absence of any pressure on L; if we force an acceleration a to the board, different from g, we obtain that the pressure of P on L is directly proportional to $(g + a)$. Then we conclude that the pressure of P on L is determined by the relative acceleration or, in other words, *their reciprocal interaction manifests with an exchange of accelerations.*

In a similar manner the mass concept is modified too. Let us consider two bodies A and B which are in mutual dependence; we deduce they communicate the accelerations $-a$ and $+a'$ respectively. Then we say B has $-a/a'$ times the mass of A. If we also choose the body A as a measure unit, we'll attribute the mass m to that body which impresses an acceleration equal to m times the acceleration that it receives by A.

It is a fact that the masses of the universe exchange accelerations one another's: Physics has the goal to describe this rapport without altering or surpassing the data given by the sensibility. It's important to note that this hypothesis joins different phenomena: in fact there is the gravitational interaction in which distant bodies exchange gravity accelerations, but there are also the rotational motions where centrifugal forces are present among the body in rotation and the masses to which we refer the motion. In this last case stating that masses exchange accelerations can be "deceptive" and leads us to describe the experimental data in a new way.

In the following pages we'll try to show how we could make this description and in the same time point out the correct interpretation of some equivocal passages that lead to consider Mach as a forerunner of innovative ideas.

Let's examine in the first place *Inertia Principle.*

Newtonian Mechanics considers the presence of an acceleration as the sign of a departure from a privileged state (Inertia) and explains this fact with the introduction of the force concept. The inertial motion is followed by the choice of a precise reference frame - the inertial one - which represents in mathematical form the concept of *absolute space and time.*

In formula the inertial motion is described by the equation:

$$\frac{d^2x}{dt^2} = \frac{d^2y}{dt^2} = \frac{d^2z}{dt^2} = 0 \tag{2}$$

where (x, y, z) are the absolute coordinates. Instead of linking motion to an inertial frame, therefore to an absolute space, it's convenient to report it to the other bodies of the universe which, on the other hand, represent the unique manner to determine reliably a reference frame. The experience teaches us that a body in uniform rectilinear motion has constant velocity and direction towards the fixed stars. Then we substitute the coordinates (x, y, z) of the old Newtonian attitude referring the body deportment to the other masses of the Universe. The Inertial Principle takes the following form: a mass n maintains constants module and direction of its velocity with reference to the other bodies of the universe m_1, m_2, m_3... placed at the distances r_1, r_2, r_3... if the relation is valid

$$\frac{d^2}{dt^2}\left[\frac{\sum_i m_i r_i}{\sum_i m_i}\right] = 0 \tag{3}$$

The (3) reduces to the (2) if we consider bodies sufficiently far and if we examine a sufficient number of them. We observe the following.

In the first place the (3) is a scalar equation and then it doesn't determine completely the motion of the body K; the velocity, being a three-dimensional vector, needs 3 scalar equations to find its deportment univocally.

Moreover (3) has an evident dynamical effect for the presence of masses m_i; if we indicate with μ the mass of body K, the equation (3) will be expressed more correctly by

$$\mu\left[\frac{d^2}{dt^2}\left(\frac{\sum_i m_i r_i}{\sum_i m_i}\right)\right] = 0 \tag{4}$$

an equation that derives from a Lagragian in which there are terms of interaction (μm_i). From this point of view (3) establishes an interaction between the masses of the universe and the body K, suggesting a link between the K's inertial deportment and the presence of the masses m_i [15].

We try to clarify this point.

The passage from (3) to (4) is correct, but it doesn't represent the Machian thought, rather an interpretation. We can consider Mach, writing (3), has in mind (4) or a Lagragian formulation of it and he doesn't reach a definite result for a lack of theoretical skills, but in the Machian writings it never presents (4) and even less an argumentation which supports it. Of course to support this interpretation of Inertial Principle we can use phrases and considerations of Mach, but a fulfilled reasoning as that of associating a Lagragian of interaction to (4) is never present.

The more equivocal passages which we can point out to support the "dynamical" interpretation of (3) (there are terms of interaction and a Lagragian too) are in the appendix of the writings of 1872. In it Mach analyses the inertial law and makes evident that the concept of absolute space, to which we refer the motion of a body, is a shortened description of the presence of the other masses and it's always associated with a precise reference frame: the sphere of fixed stars. Then he asks himself which form the law would take if the stars start a rotational motion. Would the inertial principle expressed by (3) still be valid?

Then he goes on:

> *... what share has every mass in the determination of direction and velocity in the law of inertia? No definite answer can be given to this by our experiences. We only know that the share of nearest masses vanishes in comparison with that of the farthest. We could, then, be able completely to make out the facts known to us if, for example, we were to make the simple supposition that all bodies act in the way of determination proportionately to their masses and independently of the distance, or proportionately to the distance, and so on. ... (5)*

Perhaps the (5) is the passage in which Mach seems to express precisely the idea of a certain influence exerted by the stellar masses in the determination of the body inertia; in fact he asks himself "*what share has every mass in the determination of direction and velocity in the law of inertia?*". Therefore could we consider Mach as a "forerunner" of the Einstenian positions? How could we interpret the term "share" and "determination" present in the passage? To clarify let's consider that the (5) is present in the work of 1872 and so it's the starting point of the Machian analysis on inertia. Is it possible to compare (5) with the final result expressed compactly by equation (3)? With a deeper insight we can make this comparison: for example in (5) as he asserts to express the contribution of each body, we could suppose it is proportional to its mass and distance, but in (3) the weighed sum at numerator signifies precisely this idea. Moreover in (3) we deduce that the masses near the test body give a lower contribution than the distant ones such as he prefigures in the writings of 1872 when he says "*We only know that the share of nearest masses vanishes in comparison with that of the farthest*". This aspect explains also why Mach in (3) has inserted the masses instead of limiting himself to give a formulation in which there were only the relative distances. In fact if Mach didn't intend to give an account of experimental data that the close masses have much less influence than the distant ones in the determination of the inertial deportment, we couldn't explain the massive terms present in (3). The introduction of the masses in the formula doesn't lead back to a proposal of an effective interaction with the body in examination, but as an attempt to describe an experimental fact.

Besides there is a further consideration to be made. If we admit the correct reasoning is the dynamical one, we can't continue in any case in this way due to some advice always present in the Machian work which is suggested in the same citation (5). In it Mach advises "*No definite answer can be given to this by our experiences*"; then a warning not to surpass the data produced by the experience. To demonstrate the fact that the Austrian scientist always maintains to this negative heuristic, we can report the considerations that Mach makes at the end of the discussion on inertia on the Mechanics. He reconsiders the validity of inertial law when the stars begin moving, but he only admits

> *It is impossible to say if the new expression* [the new definition of Inertia Principle] *could yet represent the condition of things if the stars begin to move one another. Such a general experience can't be constructed from a particular case we know. We must, on the contrary, wait until this experience presents itself. (6)*

Having surpassed the 'dynamic' interpretation of Inertia Principle, we now must specify the description of the reciprocal interaction among the universal masses. On this point Mach doesn't give a precise reasoning, but only some hints which represent – we hope – the path we should take to make a correct reformulation of mechanical foundations.

For example he borrows some results of thermodynamics or electromagnetism and uses them as analogies to clarify the fact that the accelerations take an important part in the relations among the masses. Mach remembers that in the natural phenomena the *differences* among certain quantities have a fundamental role: for example the differences of Potential give origin to processes which attempt to carry the system to a lower energetic level, or represent the departure of the system state from the average of the neighbouring states.([2]) We try to use this consideration in the case of acceleration too. If we considers bodies very distant from one another which move with constant velocity and direction with reference to fixed remote bodies, we note they vary their distances as a function of time. We can also say that very distant bodies change their distances in a manner so that they maintain their proportionality. If for example we consider the case of two distances r and ρ, we'll have $dr/d\rho = cost.$ Now we suppose the masses are in interaction, so that there is at least an acceleration $d^2r/dt^2 = a$ and consider that time at denominator is the measure of the distances of celestial bodies (or of the rotational angles). So the acceleration has the form $d^2r/d\rho^2 = a$ and represents a departure from the privileged state detected by $dr/d\rho = cost.$

The decisive point in this reasoning is to associate the presence of an acceleration with the comparison among the distances of celestial bodies, which is possible only because we consider time as a measure of relative positions. Then the acceleration is thought as a potential and reduced to the relative positions among celestial bodies.

That this is the way to follow is also confirmed by the considerations that Mach makes at the end of his analysis of absolute motion. He succeeds in cleaning the reference frame of every metaphysical absurdity, but it remains an abstraction which hasn't a direct verification on experience: he considers it an *external* reference. The scientist feels the need of *further insight* and the need to know *the immediate relations, the connections among the masses of the universe*: his programme is therefore the reduction of velocities and accelerations to relative positions among celestial bodies (the *immediate relations*) or in Machian terms "*There will hover before him as an ideal an insight into the principles of the whole matter, from which accelerated and inertial motions result in the same way.*".

Then Mach indicates the way the scientist will run through a complete formulation of Mechanics: express the interaction among bodies as a rapport of their relative positions.

It remains to meet the Machian arguments on the buck experiment, but now we have developed the instruments to interpret Mach's considerations correctly. In the Introduction we had briefly showed how in Newtonian mechanics the centrifugal forces represent a property of the motion state of a body and depend on its inertia. Now on the contrary, keeping in mind that masses are in reciprocal interaction according to Universal Principle, the centrifugal force of water isn't a *true motion* of liquid – as Newton thought – but represents solely the existent relation between the water mass and that of celestial bodies, their *interaction*.

Let's enter into some details. In the first place we note that some passages which are difficult to interpret are present in the analysis of vessel experiment: in fact we use some terms like *interaction*, *production*... which can have different meanings according to the philosophical background used. These passages are really the ones that allow us to consider Mach as a forerunner of inertia considerations as we said, but they have the correct meaning only in the light of Machian philosophy.

([2])usually we use expressions as $\frac{d^2U}{dx^2}$, $\frac{d^2U}{dy^2}$, $\frac{d^2U}{dz^2}$ where U is the Potential

For example when Mach asserts peremptorily "*Try to fix the newtonian vessel and rotate the sphere of fixed stars and then prove the absence of centrifugal forces*" it seems he wants to propose – if this be the case – that the relative rotation of stellar masses induces the presence of centrifugal forces. In reality he wants to underline that the vessel experiment can be described in two opposite manners: on the one hand we can consider the bucket in rotation and the stars motionless, on the other side we can imagine instead the vessel fixed and the set of stars in rotation around it. Mach doesn't distinguish between the two descriptions, rather he considers them equivalent because the unique motions he considers are the relative ones. In fact he underlines "*the world system is given only once, and the ptolemaic theory and the copernican one are only interpretations, and both equally valid*". Therefore the goal remains the description of what happens through relative concepts, removing metaphysical absurdities such as the absolute space and time. Then he affirms to support the two-fold description that "*the fundamental principles of mechanics can be formulated so that centrifugal forces are still present for relative rotational motions*".

If we examine the conclusions on the rotating vessel, we note the same ambiguity (cfr. Introduction) We can report two significant passages:

> *Newton's experiment with the rotating vessel of water simply informs us, that the relative rotation of water with respect to the sides of the vessel produce no noticeable centrifugal forces, but that such forces are produced by its relative rotation with respect to the mass of the earth and other celestial bodies (7)*

And in the following he goes on

> *No one is competent to say how the experiment would turn out if the sides of the vessel increased in thickness and mass till they were ultimately several leagues thick. (8)*

At a first reading it would seem the fixed stars have some influences on the presence of inertial forces and the latter are produced by the relative motion with respect to the other universal masses (reading of (7)). The idea is confirmed by the following consideration – the (8) – where, with a rhetorical argumentation of difficult interpretation, Mach seems to propose a new version of vessel experiment in which the sides become more massive and in this manner they produce centrifugal forces in the water.

To surpass this interpretation it's necessary to reread (7) and (8) in the light of Machian philosophy. Because in these passages a causal nexus seems to appear between universal masses and inertial forces, we must keep in mind the causal concept of Mach's thought (the principle of universal dependence of phenomena). Assuming that all the universal masses are in reciprocal relation and that the task of scientific study becomes the description of these relations, we deduce that the observation of centrifugal forces between the water and universal masses is the expression of their relation. When Mach asserts that inertial forces are produced with reference to the sphere of fixed stars, he isn't indicating some influence or some causal nexus between the latter and the liquid, rather he expresses in words the relation which is present among the elements of the physical system observed.

The *rigid legality among phenomena* – as he affirms in the writing of 1872 – is the economical description of relations among the elements of sensibility. In our case this description is reduced to express the connection between centrifugal forces in the bucket and the set of fixed stars.

Besides, if we must take into consideration what is shown by experience, it's senseless to immagine what would be the result of experiment if the sides of the vessel became more massive. In fact after (8) he says that "*the one experiment only lies before us, and our business is, to bring it into accord with the other facts known to us, and not with the arbitrary fictions of our imagination*". The (8) is an invitation not to surpass the limit of experience (what would happen if we made the hypothesis of some influence between the sides of the vessel and inertial forces) in order not to run the risk of imagining either new metaphysical entities, such as absolute space and time, or relations among elements of sensibility which aren't given and haven't any practical and conceptual utility.

Let's make some concluding remarks. The *Principle of universal dependence of natural phenomena*, which Mach declares in his work of 1872, is translated in the Mechanics in the assumption that masses exchange accelerations. How to describe this experimental data creates some interpretative problems. In fact there is a risk in considering the Machian passages as a proposal of some causal mechanism for the determination of inertia. In the light of Mach's philosophical positions this interpretation isn't correct for a two-fold reason:

1. he uses some terms that seem to indicate a causal nexus between universal masses and the inertial deportment (specially in vessel experiment) but his ideas on causality lead only to a description of what is observed. The exchange of accelerations among universal masses is reduced to a report on their relative positions.

2. The scientific research must maintain itself scrupulously to experience and not 'produce it' in any case. The risk that the scientist runs in abandoning this negative heuristic is the use of abstract concepts that aren't present in reality (absolute space and time). Mach never gives a 'dynamic scheme' which permits to obtain the inertial forces with respect to celestial masses, he limits himself to report the functional connections among natural phenomena. His argumentations on the vessel experiment are enclosed in this optic and represent a description of observed facts.

4. Conclusions.

Usually we attribute the consideration that celestial masses have a causal influence in the determination of inertia to Mach. This opinion is easily observable either in the writings of Friedländer, or in those of Föppl and it is present also in the work of Einstein. It originates from the reading of Mechanics and has the origin in a certain interpretation of the Principle of universal dependence of natural phenomena. We have noted in the historical study on Mechanics that the principle is transformed in the request that the masses exchange accelerations. It originates from a set of experimental evidences (such as the gravitational interaction, the proposed example of the weight on the board, the operational definition of mass, the Newton's vessel experiment...) and therefore is an expression of a fact common to a series of experiences. For every one of them the Physicist constructs a conceptual model whose task is to describe completely the data given by sensibility. The assumption that masses exchange reciprocal accelerations is then referred to a 'cluster' of models which explain a set of experiences. The causal nexus that each model carries within are reduced to describe certain dependencies among elements in which the experience has been divided or – in Machian terms – to report the observation that some elements of sensibility are linked to others, disappear together or the presence of elements of a type is linked with disappearance of other elements. So when Mach affirms that masses are in interaction exchanging accelerations, we don't need to ask what the cause of interaction is, searching something that surpasses experimental data, but our business is the description of functional connections extracted from experience. In the case of a mechanical phenomenon, for example, Mach indicates to us how to make a complete description by reducing the system deportment to the rapport among the relative position of its parts. In any case the experience is always the starting point of modellization and we must never reverse their rapport.

Our habit of theorizing induces us to search an origin, a cause in the sense of traditional Mechanicism, a *causal mechanism* which explains the observation of accelerations exchange among universal masses. The Principle of universal dependence frees itself in this manner from the experiences on which it is based and becomes a stimulus for theoretical activity. Therefore models of systems not yet observed and experimental consequences, that must be tested, are produced: so the modellization precedes the experience, rather "creates" it. It is on this line that the works of Friedländer, Föppl and Einstein, strongly interpreting the conclusions of vessel experiment, develop conceptual models and experiments to search the *origin* of inertial forces. What Mach's position is , regarding their works, is given by the author himself (at least for Friedläender and Föppl, but the consideration can be also extended to Einstein): at the end of analysis on absolute motion he asserts that "*. . . we must not underestimate even experimental ideas like those of Friedländer and Föppl, even if we do not yet see any immediate result from them*". However they are Hypothetical ideas that must be verified more precisely. On the contrary he thinks we must keep to what is immediately observable, but "*a glance from time to time into the depths of what is uninvestigated cannot hurt*" the natural scientist.

* * *

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