On the Lorentz Invariance of Maxwell's Equations

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Abstract
It is shown that, contrary to existing opinion, Maxwell's equations are not invariant in form under Lorentz transformations.

The invariance in form of Maxwell's equations*) under Lorentz transformations (Lorentz covariance of Maxwell's equations) is considered a major achievement of the Special Theory of Relativity (STR) [1]. The invariance in question is often cited as an example of the "beauty" of the theory.**)

However, a closer inspection of the Lorentz-transformed Maxwell's equations reveals that they differ in form from the Maxwell equations before the transformation, i.e. in fact, Maxwell's equations are not invariant in form (covariant) under Lorentz transformations.

Indeed, the Lorentz-transformed components of Maxwell's equations (shown on p.52 of [1]) contain the following expressions – $\beta\left(\mathrm{Y} - \frac{\mathrm{v}}{\mathrm{c}}\mathrm{N}\right)$ and $\beta\left(\mathrm{N} - \frac{\mathrm{v}}{\mathrm{c}}\mathrm{Y}\right)$ etc. (**cf. Addendum 1)**. However, as shown in [2], by neglecting the terms of <u>third</u> and higher order of $\frac{\mathrm{v}}{\mathrm{c}}$ we may write the first of the above expressions, $\beta\left(\mathrm{Y} - \frac{\mathrm{v}}{\mathrm{c}}\mathrm{N}\right)$, explicitly as (**cf. Addendum 2**)

$$\mathrm{Y} - \frac{\mathrm{v}}{\mathrm{c}}\mathrm{N} + \mathrm{Y}\frac{1}{2}\frac{\mathrm{v}^2}{\mathrm{c}^2}$$

As was discussed in the mentioned paper [2] the terms containing the <u>second</u> power of $\frac{\mathrm{v}}{\mathrm{c}}$ in the last expression can be of comparable magnitude and therefore if the third term is neglected, the second term must also be neglected. This is because the quantity N in the term $\frac{\mathrm{v}}{\mathrm{c}}\mathrm{N}$ implicitly contains a velocity, u, over c term (**cf. Addendum 3)**. And because nothing restricts velocity, u, from being comparable to velocity v, the second term can also be of the order of second power in $\frac{v}{c}$. Thus, the second and third term can be of comparable magnitude and one may write

$$\beta\left(Y - \frac{v}{c}N\right) = Y - \cancel{\frac{v}{c}N} + \cancel{Y\frac{1}{2}\frac{v^2}{c^2}} = Y \qquad (1)$$

On the other hand, the second of the above expressions, $\beta\left(N - \frac{v}{c}Y\right)$, can be represented as

$$N - \frac{v}{c}Y + \frac{1}{2}\frac{v^2}{c^2}N$$

or, neglecting the term containing <u>third</u> power of $\frac{v}{c}$ – namely, $\frac{1}{2}\frac{v^2}{c^2}N$ (recall that component N contains, in general, a quantity of the order of $\frac{v}{c}$ (**cf. Addendum 3))** the expression becomes:

$$\beta\left(N - \frac{v}{c}Y\right) = \left(N - \frac{v}{c}Y\right) \qquad (2)$$

So far we applied approximations to the leading order of $\frac{v}{c}$, as required in [1].***)

We can now substitute equations (1) and (2) in the expressions of the transformed Maxwell's equations (cf. middle of p.52 of [1]) (**cf. Addendum 1**) and obtain the following forms Maxwell's equations (for free space) in component form as follows

$$\frac{1}{c}\frac{\partial X}{\partial \tau}=\frac{\partial}{\partial \eta}\left(N-\frac{v}{c}Y\right)-\frac{\partial}{\partial \zeta}\left(M+\frac{v}{c}Z\right) \quad \frac{1}{c}\frac{\partial L}{\partial \tau}=\frac{\partial}{\partial \zeta}\left(Y-\frac{v}{c}N\right)-\frac{\partial}{\partial \eta}\left(Z+\frac{v}{c}M\right)$$

$$\frac{1}{c}\frac{\partial Y}{\partial \tau} = \frac{\partial L}{\partial \zeta} - \frac{\partial}{\partial \xi}\left(N - \frac{v}{c}Y\right) \qquad \frac{1}{c}\frac{\partial}{\partial \tau}\left(M + \frac{v}{c}Z\right) = \frac{\partial Z}{\partial \xi} - \frac{\partial X}{\partial \zeta}$$

$$\frac{1}{c}\frac{\partial Z}{\partial \tau} = \frac{\partial}{\partial \xi}\left(M + \frac{v}{c}Z\right) - \frac{\partial L}{\partial \eta} \qquad \frac{1}{c}\frac{\partial}{\partial \tau}\left(N - \frac{v}{c}Y\right) = \frac{\partial X}{\partial \eta} - \frac{\partial Y}{\partial \xi} \tag{3}$$

From equation (3) the vector form of Maxwell's equations (for free space) after the Lorentz transformations is obtained:

$$\frac{1}{c}\frac{\partial \mathbf{E}}{\partial \tau}=\nabla\times\mathbf{B'} \text{ and } -\frac{1}{c}\frac{\partial \mathbf{B'}}{\partial \tau}=\nabla\times\mathbf{E} \tag{4}$$

This, however, is in disagreement with the first postulate – "The Principle of Relativity." Recall that the form of Maxwell's equations under discussion, after the application of the first postulate, should be

$$\frac{1}{c}\frac{\partial \mathbf{E'}}{\partial \tau}=\nabla\times\mathbf{B'} \text{ and } -\frac{1}{c}\frac{\partial \mathbf{B'}}{\partial \tau}=\nabla\times\mathbf{E'} \tag{5}$$

In other words, the forms of Maxwell's equations after the Lorentz transformations, namely

$$\frac{1}{c}\frac{\partial \mathbf{E}}{\partial \tau}=\nabla\times\left(\mathbf{B}+\frac{1}{c}(v\times\mathbf{E})\right) \text{ and } -\frac{1}{c}\frac{\partial\left(\mathbf{B}+\frac{1}{c}(v\times\mathbf{E})\right)}{\partial \tau}=\nabla\times\mathbf{E} \tag{6}$$

are obviously not the same as the forms of Maxwell's equations (for free space) before the transformations:

$$\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} \quad \text{and} \quad -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E} \tag{7}$$

This demonstrates, contrary to the widespread belief, that Maxwell's equations are not Lorentz invariant in form (are not Lorentz covariant).

*) In [1] equations observed here are called Maxwell-Hertz equations for empty space.
**) Regarding the "beauty" of a theory one may encounter extreme views expressed by some. In [3] Dirac writes: "… it is more important to have beauty in one's equations than to have them fit experiment."
***) Note, however, that lack of Lorentz invariance of the Maxwell equations can be demonstrated also when no approximation has been made (**cf. Addendum 2**).

References

1. Einstein A., Pages 37-65 From "*The Principle of Relativity*", Dover, 37-65 (1905) – English translation of the original Einstein A., Annalen der Physik, **17**, 891-921 (1905).

2. Noninski V.C., *Special Theory of Relativity and the Lorentz Force*, philsci-archive.pitt.edu, document: PITT-PHIL-SCI00001006 (2003).

3. Dirac P. A. M., *The Evolution of the Physicist's Picture of Nature*, Sci. Am., **208**, 45-53 (1963).

Addendum 1

$$\frac{1}{c}\frac{\partial X}{\partial \tau}=\frac{\partial}{\partial \eta}\left\{\beta\left(N-\frac{v}{c}Y\right)\right\}-\frac{\partial}{\partial \zeta}\left\{\beta\left(M+\frac{v}{c}Z\right)\right\} \quad \frac{1}{c}\frac{\partial L}{\partial \tau}=\frac{\partial}{\partial \zeta}\left\{\beta\left(Y-\frac{v}{c}N\right)\right\}-\frac{\partial}{\partial \eta}\left\{\beta\left(Z+\frac{v}{c}M\right)\right\}$$

$$\frac{1}{c}\frac{\partial}{\partial \tau}\left\{\beta\left(Y-\frac{v}{c}N\right)\right\}=\frac{\partial L}{\partial \zeta}-\frac{\partial}{\partial \xi}\left\{\beta\left(N-\frac{v}{c}Y\right)\right\} \quad \frac{1}{c}\frac{\partial}{\partial \tau}\left\{\beta\left(M+\frac{v}{c}Z\right)\right\}=\frac{\partial Z}{\partial \xi}\left\{\beta\left(Z+\frac{v}{c}M\right)\right\}-\frac{\partial X}{\partial \zeta}$$

$$\frac{1}{c}\frac{\partial}{\partial \tau}\left\{\beta\left(Z+\frac{v}{c}M\right)\right\}=\frac{\partial}{\partial \xi}\left\{\beta\left(M+\frac{v}{c}Z\right)\right\}-\frac{\partial L}{\partial \eta} \quad \frac{1}{c}\frac{\partial}{\partial \tau}\left\{\beta\left(N-\frac{v}{c}Y\right)\right\}=\frac{\partial X}{\partial \eta}-\frac{\partial}{\partial \xi}\left\{\beta\left(Y-\frac{v}{c}N\right)\right\}$$

where

$$\beta=\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

Addendum 2

Indeed, observe one of the y'-axis components of the electric field which is in fact an equation from [1] (page 54) – (same conclusions apply also to y'-axis component of the magnetic field and z'-components of the electric and the magnetic field):

$$Y' = \beta\left(Y-\frac{v}{c}N\right)=\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}\left(Y-\frac{v}{c}N\right)$$

and apply the expansion

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)a^{n-2}}{2}b^2 + \ldots$$

to obtain

$$\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{v^2}{c^2}\right)^2}{2} + \ldots$$

For the sake of this discussion, consider only the first two terms of the expansion and obtain

$$\beta = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2}$$

Thus, the expression $Y' = \beta\left(Y - \frac{v}{c}N\right)$ may be rewritten in the following way:

$$Y' = \beta\left(Y - \frac{v}{c}N\right) =$$

$$\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}\left(Y - \frac{v}{c}N\right) \approx$$

$$\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right)\left(Y - \frac{v}{c}N\right) =$$

$$Y - \frac{v}{c}N + Y\frac{1}{2}\frac{v^2}{c^2} - \frac{1}{2}\frac{v^3}{c^3}N$$

When contemplating the above it is necessary to remember that the components of the electric field X, Y, Z and magnetic field L, M and N refer to the source creating these fields and not to the test charge (test charge in this case is of magnitude $q = 1$). Velocity, v, however, refers to the test charge. A good explanation of the discussed physical situation is given in French A.P., *Special Relativity*, W.W.Norton & Co., New York, 1968, p.234. In this respect, specially note that the $\frac{v^2}{c^2}$ term in the factor β has nothing to do with the source of electric fields and is connected only with the test charge. Therefore, nothing prohibits the above-shown expansion of $\beta = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$ for reasonable values of v.

Thus, in the Lorentz force formula $\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)$ the quantity q is the magnitude of the test charge and $\mathbf{v}$ is the velocity of the test charge while the fields $\mathbf{E}$ and $\mathbf{B}$ are created by the moving source charge.

Now, because the fields $\mathbf{E}$ and $\mathbf{B}$, respectively their components Y and N, are created by the source charge, in order to obtain certain required values of Y and N, we may choose freely the magnitude of the source charge and its velocity **u** (**cf. Addendum 3**).

Thus, suppose, for instance, that we choose the magnitude of the source charge and its velocity **u** such that components Y and N yield $\frac{\mathrm{N}}{\mathrm{Y}} = \frac{1}{2}\frac{\mathrm{v}}{\mathrm{c}}$. It is remarkable, that under the latter condition, after the Lorentz transformations, it will not matter whether we choose to approximate to the leading order of $\frac{\mathrm{v}}{\mathrm{c}}$ or we choose not to make any approximation (of course, retaining terms containing third and higher powers in $\frac{\mathrm{v}}{\mathrm{c}}$ will be unreasonable), the outcome will be that the terms containing second order in $\frac{\mathrm{v}}{\mathrm{c}}$ will vanish. This is important since even before any Lorentz transformations are carried out terms containing second order in $\frac{\mathrm{v}}{\mathrm{c}}$ exist in the formulas of classical electrodynamics and are not usually ignored. As an example one may consider the above-mentioned expression for the Lorentz force – components of Lorentz force contain terms $\frac{\mathrm{v}}{\mathrm{c}}\mathrm{N}$ and $\frac{\mathrm{v}}{\mathrm{c}}\mathrm{M}$ which are, as seen above, second order in $\frac{\mathrm{v}}{\mathrm{c}}$ and yet they are not ordinarily ignored in classical electrodynamics.

The example above, concerning the magnitudes of Y and N, is sufficient to help one conclude that the Lorentz transformations do not give proper form of the Maxwell equations, that is, a form which would be in concordance with the first postulate of STR.

Of course, one may determine for oneself how much the situation will change in cases when the velocity **u** of the source is greater or less than **v**. One can also examine for oneself what the effect of changing q would be and convince oneself that conditions such as $\frac{\mathrm{N}}{\mathrm{Y}} = \frac{1}{2}\frac{\mathrm{v}}{\mathrm{c}}$ can be fulfilled.

Addendum 3

From Biot-Savart Law [cf. Slater J.C., *Electromagnetism*, Dover, New York, p.54, 1969 or Eyges L., *The Classical Electromagnetic Field*, Dover, New York, 1972, p.117] the component N of the magnetic field is (in Gaussian units)

$$N = B_z = \frac{q}{c}\frac{(\mathbf{u}\times\mathbf{r})_z}{r^3} = \frac{q}{c}\frac{(u_x y - u_y x)}{r^3}$$

where **u** is the velocity of the source charge of the magnetic field (notice that the magnitude of **u** differs from the velocity of the unit electric point charge **v**; nothing prevents, however, the magnitudes of **u** and **v** to be made comparable, therefore, the second term and the third term are of the same order of magnitude with respect to $\frac{v}{c}$ and neglecting one term as opposed to the other is inconsistent), q is the charge of the source of the electromagnetic field, **r** is the radius-vector from the source charge q to the unit electric point charge (test charge), c is the velocity of light.

Also,

$$L = B_x = \frac{q}{c}\frac{(\mathbf{u}\times\mathbf{r})_x}{r^3} = \frac{q}{c}\frac{(u_y z - u_z y)}{r^3}$$

and

$$M = B_y = \frac{q}{c}\frac{(\mathbf{u}\times\mathbf{r})_y}{r^3} = \frac{q}{c}\frac{(u_z x - u_x z)}{r^3}$$

We should note again that when contemplating the above it seems necessary not to forget to take into account the fact that the components of the electric field X, Y, Z and magnetic field L, M and N refer to the source creating these fields and not to the test charge. A good explanation of the discussed physical situation is given in French A.P., *Special Relativity*, W.W.Norton & Co., New York, 1968 p.234. Thus, in the Lorentz force formula $\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)$ the quantity q is the magnitude of the test charge and $\mathbf{v}$ is the velocity of the test charge while the fields $\mathbf{E}$ and $\mathbf{B}$ are created by the moving source charge.

Addendum 4

Let us observe as an example the explicit form of the second line of the transformed equations on page 52 of [1]:

$$\frac{1}{c}\frac{\partial}{\partial \tau}\left\{\beta\left(Y - \frac{v}{c}N\right)\right\} = \frac{\partial L}{\partial \zeta} - \frac{\partial}{\partial \xi}\left\{\beta\left(N - \frac{v}{c}Y\right)\right\}.$$

Since, as already discussed, $\beta\left(Y - \frac{v}{c}N\right) = Y$ and $\beta\left(N - \frac{v}{c}Y\right) = \left(N - \frac{v}{c}Y\right)$, to the leading order of approximation, we have

$$\frac{1}{c}\frac{\partial Y}{\partial \tau} = \frac{\partial L}{\partial \zeta} - \frac{\partial}{\partial \xi}\left(N - \frac{v}{c}Y\right). \tag{8}$$

Although eq.(8) is not exactly of the form claimed on p.52 (second transformed equation therein), it still can be analyzed following the author of [1]. We thus obtain:

$$Y' = Y$$

and

$$N' = \left(N - \frac{v}{c}Y\right)$$

Obviously, the obtained form of Y' is not what that form is claimed to be on p.53 of [1].
As another example, observe the <u>sixth</u> equation on p.53 of [1]

$$\frac{1}{c}\frac{\partial}{\partial \tau}\left\{\beta\left(N - \frac{v}{c}Y\right)\right\} = \frac{\partial X}{\partial \eta} - \frac{\partial}{\partial \xi}\left\{\beta\left(Y - \frac{v}{c}N\right)\right\}$$

$$\frac{1}{c}\frac{\partial}{\partial \tau}\left(N - \frac{v}{c}Y + \cancel{\frac{1}{2}\frac{v^2}{c^2}}N - \cancel{\frac{1}{2}\frac{v^3}{c^3}}Y\right) = \frac{\partial X}{\partial \eta} - \frac{\partial}{\partial \xi}\left(Y - \frac{v}{c}N + Y\frac{1}{2}\frac{v^2}{c^2}\right)$$

note that $\frac{1}{2}\frac{v^2}{c^2}N$ is of the order of third power in $\frac{v}{c}$

$$\frac{1}{c}\frac{\partial}{\partial\tau}\left(N - \frac{v}{c}Y\right) = \frac{\partial X}{\partial\eta} - \frac{\partial}{\partial\xi}\left(Y - \frac{v}{c}N + Y\frac{1}{2}\frac{v^2}{c^2}\right)$$

We can now neglect the second and the third term in the parentheses on the right-hand side because both terms contain $\frac{v}{c}$ of second order. We obtain

$$\frac{1}{c}\frac{\partial}{\partial\tau}\left(N - \frac{v}{c}Y\right) = \frac{\partial X}{\partial\eta} - \frac{\partial Y}{\partial\xi}$$

and now the comparison with

$$\frac{1}{c}\frac{\partial N'}{\partial\tau} = \frac{\partial X'}{\partial\eta} - \frac{\partial Y'}{\partial\xi}$$

yields

$$Y' = Y$$

and

$$N' = \left(N - \frac{v}{c}Y\right)$$

If we apply the same arguments with regard to the rest of the equations on p.52 of [1], after the Lorentz transformation we obtain, in contrast with the equations on p.53 of [1], the following set of equations:

$$\begin{aligned} X' &= X & L' &= L \\ Y' &= Y & M' &= \left(M + \frac{v}{c}Z\right) \\ Z' &= Z & N' &= \left(N - \frac{v}{c}Y\right) \end{aligned} \qquad (9)$$