Topological Phase Fields, Bäcklund Transformations, and Fine Structure

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Quantum coupling is defined by comparing the evolution of an input to an output phase, where the phase is evolving on a curved pseudospherical surface. The difference given by interference obeys a single-valuedness condition since the output phase is coupling back to the input phase. We arrive at Bäcklund transforms and corresponding sine-Gordon soliton equation. The idealized resonance or feedback condition corresponds to an oscillator potential that can be mapped by projective geometry to Coulomb coupling, where the effective coupling strength can be iteratively determined.

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Loop sub-loop coupling. In the previous work [1] spin-orbit coupling and precession induced by geometric phases was modelled by rolling cones or with a low-dimensional conic metric [2]. A phase can be a polarization or rotational property of a vector in Hilbert space. In the next step phase evolution was generalized to topological phase fields on pseudospherical surfaces allowing for solitonic signals [3]. In contrast to the previous work this paper does not start with conventional topological phases, it presents a simple model of non-linear topological phase field evolution given by the difference or interference between input and output of a nonlinear quantum impedance. With $\widetilde{\theta}$ as the input phase (the reference) and θ the output phase including geometric phase shift we have within the impedance the interference terms

- $(\theta+\widetilde{\theta})/2$: loop (common phase of subloops),
- $(\theta-\widetilde{\theta})/2$: subloop (differential or geometric phase),

where $\theta-\widetilde{\theta}$ is the sub-loop geometric phase evolution induced by "parallel transport". Since the output phase is coupling back to the input phase, single-valuedness in the quantum case requires that the phase difference given by interference on the closed loop must carry an integral number M of subloops. Consequently, the total system carries angular momenta distributed in M subloop units that carry vector properties (i.e. spins) that interfere with common loop vector properties (i.e. orbital spin). In a physical scenario modelling a quantum transformer of angular momentum with regular M-gonal symmetry, there are fields providing for coupling and momentum exchange. There are four types of coupling terms:

- $f(\theta+\widetilde{\theta})$: common loop momentum.
- $g(\theta+\widetilde{\theta})$: common loop field/force.
- $f(\theta-\widetilde{\theta})$: geometric ph./sub loop momentum.
- $g(\theta-\widetilde{\theta})$: geometric ph./sub loop field/force.

As a necessary condition for a closed–loop momentum exchange there are two orthogonal couplings ($x\perp y$):

$$(x)\quad f(\theta+\widetilde{\theta})=Mg(\theta-\widetilde{\theta}) \tag{1}$$

$$(y)\quad Mf(\theta-\widetilde{\theta})=g(\theta+\widetilde{\theta}) \tag{2}$$

M times the geometric phase induced (gauge) field strength balances the mean loop angular momentum, and M times the geometric phase induced (subloop) angular momentum balances the mean loop field strength. This means, that the geometric phase of the M subloop units is a generator of the spin-obit coupling field. Angular momentum is proportional to the temporal evolution of θ with $f(\xi)=\partial(\xi/2\pi)$. To ensure integrability, the field strength between sub-loop units in regular M-gonal symmetry will be proportional to distance with with $g(\xi)=M_g\sin(\xi/2)$. Now, the non-linear property of the "impedance" is given by the trigonometric field function, a consequence of curvature as can be directly shown: the resulting non-linear transformer can be exactly described by Bäcklund transformations (BT)

$$\begin{aligned}(\partial_x\widetilde{\theta}+\partial_x\theta)/M&=2\pi M_g\sin[(\widetilde{\theta}-\theta)/2],\\ (\partial_y\widetilde{\theta}-\partial_y\theta)M&=2\pi M_g\sin[(\widetilde{\theta}+\theta)/2],\end{aligned} \tag{3}$$

a manifestation of integrability [4, 5].

Sine-Gordon soliton. The BT eq.(3) are the generators of the sine-Gordon SG equation

$$\partial_x\partial_y\theta=-R\sin(\theta)/2, \tag{4}$$

where the second order equation eq.(4) arises as the integrability conditions of a pair of first order equations eq.(3), with i.e. $\partial_y(\partial_x\widetilde{\theta})=\partial_x(\partial_y\widetilde{\theta})$. For simplicity, $\widetilde{\theta}$ will serve as the special reference field of constant or stationary phase evolution given by the rather trivial case $\widetilde{\theta}=4\pi(\frac{1}{2}+n)$, with quantum gauge (or spin) dependent winding number $n=0,1,2,....$ This provides in eq.(3) for the simplification $\partial_x=M^2\partial_y$. Parametrizing the sub–loop with radius $r^2=x^2+y^2=(1+1/M^4)x^2=(M^4+1)y^2$ and $\partial_r^2=\partial_x^2+\partial_y^2$ we have

$$\mp\sqrt{M^2+1/M^2}\,\partial_r\theta=2\pi M_g\sin(\theta/2). \tag{5}$$

The nonlinear SG phase field evolves with a pseudospherical curvature constraint. This property is found with generalized Chebyshev coordinates on a plane $\mathcal{S}$ embedded in $\mathbb{R}^3$

$$ds^2=(dx)^2+(dy)^2+2\cos\theta dxdy \tag{6}$$

with scalar curvature $R = 2R_{1212}/\det(g_{ij})$ [5] of the generalized Chebyshev metric. There is a clear geometrical interpretation: the coordinate vector field is parallel transported along the signal/soliton vector field with respect to the Levi–Civita connection. A "privileged" surface of scalar curvature $R = -2$ is given i.e. by the Lobachevskian plane.

Topological solitons as solutions to eq.(4) and eq.(5) are spatially confined (localized), non-dispersive and non-singular solution of a non–linear field theory. In 2+1-dimensional gauge vortex scattering it follows from purely geometric considerations that the head-on scattering of M topological solitons (like monopoles, vortices, skyrmions, ...) distributed symmetrically around the point of scattering (relative angular separations $2\pi/M$) is by an angle π/M, independent of various details of the scattering [6]. In this case the initial configuration has the symmetry group of a regular M-gon, the "moduli space" of M vortices, $\mathcal{M}_M$[7].

Spin-orbit resonance condition. To approach the coupling resonance we can define potential relations. Applying the Euler-Lagrange equation to the SG soliton Lagrangian and correspondent Hamiltonian, the potential for a stationary solution is with $\partial_r^2\theta \propto \partial_\theta V$ and $V \propto (\partial_r\theta)^2 \propto f^2$ given by

$$\begin{aligned} 2V(\theta) &= (\partial_x\theta/M)^2 = (M\partial_y\theta)^2 \\ &= (M^2 + 1/M^2)(\partial_r\theta)^2 \\ &= 2\pi^2 M_g^2(1-\cos\theta), \end{aligned} \tag{7}$$

where $\cos\theta = 1 - 2\sin^2(\theta/2)$. Since the field strength $g(\theta)$ between sub-loop units in regular M-gonal symmetry is proportional to distance r, the coupling potential as the square of the field strength can be assigned to an oscillator potential

$$V_o(r) = \frac{1}{2}\left(\frac{r}{\rho}\right)^2 = \frac{1}{2}(\pi M_g r)^2 = -\frac{1}{4}Rr^2, \tag{8}$$

providing for resonant coupling and for the general condition

$$r = \mp 2\rho\sin(\theta/2). \tag{9}$$

Coupling and self-energy. From eq.(7) the self-energy term can be identified as a constant θ-independent Riemann curvature scalar $R = -2/\rho^2$, with eq.(4) $\pi M_g\rho = 1$. Therefore, it is plausible to decompose energy in eq.(7) into at least two terms: a self-energy term $\pi^2 M_g^2$ (of constant phase evolution) and a dynamic coupling term $\pi^2 M_g^2\cos\theta$ that accounts for the geometric field evolution based on the BT. Integrating eq.(12) provides for $\theta \propto r^2 + c$, where the integration constant c can be obtained by comparing the correspondent parts in eq.(7) and eq.(9). This provides for a dynamic coupling term $\mp\pi M_g\sqrt{M^2+1/M^2}\theta$ that can be combined with a self-energy term and integration constant to

$$V(\theta) = V_o(y) = \pi^2 M_g^2 \mp \pi M_g\theta\sqrt{M^2+1/M^2}. \tag{10}$$

With eq.(7) and eq.(10) we immediately obtain an iterative equation of phase shift in resonance

$$\theta\sqrt{M^2+1/M^2} = \pm\pi M_g\cos\theta, \tag{11}$$

where the coupling allows for two possible signs.

External coupling: Coulomb force. The stereographic projection of the stationary dynamics on the two-dimensional (pseudo)sphere within $PSL(2,\mathbb{R})$ connects angular variable and sub-loop radius r. Eq.(7) in combination with eq.(9) maps the local oscillator potential to the non–local Coulomb potential under stereographic projection given by the conventional Bohlin transformation $r_c \to r^2$ [8]. This relates the Coulomb system to the resonance oscillator potential and maps with eq.(5) and eq.(9) the square of the phase gradient to the potential

$$2\pi r_c = \rho^2\partial_r\theta \to 4\pi^2 r^2 = \rho^4(\partial_r\theta)^2. \tag{12}$$

The projective geometry with $PSL(2,\mathbb{R})$ has the typical parameterization

$$z = r_c e^{i\varphi} = \begin{cases} \rho\cot\frac{\theta}{2}e^{i\varphi} \text{ sphere;} \\ \rho\coth\frac{\theta}{2}e^{i\varphi} \text{ pseudosphere,} \end{cases} \tag{13}$$

where θ, φ are the (pseudo)spherical coordinates [8].

Berry's Topological Phase. The so-called complex SG theory (applied in nonlinear optics to the phenomenon of anomalously low energy loss in coherent optical pulse propagation) has coupled nonlinear partial differential equations in 1+1-dimensional spacetime given by the Maxwell and Bloch equations with two complex fields and one real field [9] and generalizes the sine-Gordon theory to the non–abelian cases on a coset $SU(N)/U(N-1)$ in terms of a deformed gauged Wess-Zumino-Witten action [9]. The optical Bloch equation can be used to describe dipole spin precession [10], where θ as the strength of the torque vector agrees with the conventional mechanical interpretation of the SG equation as a continuum limit of the infinite chain of coupled pendulum equations. This allows also to interpret the SG field as an precession angle field induced by the topological phase field. In the typical case of 'parallel transportation' the spin vector will come back after every loop with a directional change $\varphi_g(T)$ (a geometric phase) equal to the curvature enclosed by the path $\mathcal{C}$ [11]. This is the simple monochromatic case of a loop with conic precession on the sphere (rotation induced i.e. by a modulated magnetic field) with an additional geometric phase given by

$$\varphi_g(T) = 2\pi J(1-\cos\theta), \tag{14}$$

θ is the vertex cone semiangle. The total phase $\varphi(T) = 2\pi J$ can be decomposed into a dynamic coupling part $\varphi_d(T) = 2\pi J\cos\theta$ and a geometric part $\varphi_g(T)$, where $\cos\theta$ is a typical spin-orbit or spin-spin coupling term. A general spin-orbit coupling constant proportional to the evolution of the dynamical part can be defined as a combination of an M-dependent term and a $\cos(\theta)$-term

$$\alpha(M) = \frac{\triangle\varphi_d(T)}{\varphi(T)}, \quad \triangle\varphi_d(T) = \frac{\varphi_d(T)}{\sqrt{M^2 + 1/M^2}}. \quad (15)$$

Note, that α can also be defined with spin J proportionality (see the previous work [2], where $1/M$-term or y contribution eq.(2) is missing). Comparing eq.(14) with eq.(11) the central term is given by

$$\alpha(M) = \frac{\cos\theta}{\sqrt{M^2 + 1/M^2}} = \frac{\pm\theta}{\pi M_g}. \quad (16)$$

The coupling strength. Consequently, a mutual resonant coupling od two identical systems via Coulomb mapping [3] is proportional to θ and α. The iteration eq.(11) is invariant with respect to the inversion and duality $M \leftrightarrow 1/M$. Inversion is the central operation in linear fractional transformation between local and non–local holonomy. M as an integral quantum number describes the phase-locked and single-valued field [2] and provides for integrability. M-type inversion could also characterize the relations between the electric and magnetic monopole charge $(2ge)^2 = 1$ with $(2g/e)^2 = M^2$, and also between group and phase velocity of a wave packet in the ground state $v_g v_p = 1$ with $v_p/v_g = M^2$. The coupling constant and special θ-value or oscillation range is iteratively obtained in eq.(11), where $M = 137$ or $M = 1/137$ from the Dirac theory of magnetic monopoles [12] provides with $M_g = 1$ for $1/\alpha = 137.03600960$ that fits within some ppb's to the Sommerfeld fine structure constant obtained in neutron interferometry. The meaning of the number 137 remains unclear. Eq.(11) is an chaotic algorithm, bifurcation starts above a special values of M_g, one-dimensional conic coupling [2] has no $1/M^2$-term within the square root in eq.(16) and is for big M values slightly stronger [3].

[1] B. Binder,*Geometric Phase Locked in Fine Structure* (2002); PITT-PHIL-SCI00000782.

[2] B. Binder,*Iterative Interplay between Aharonov-Bohm Deficit Angle and Berry Phase* (2002); PITT-PHIL-SCI00000810.

[3] B. Binder,*Charge as the Stereographic Projection of Geometric Precession on Pseudospheres*; PITT-PHIL-SCI00000818.

[4] M. J. Ablowitz and H. Segur,*Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia, (1981).

[5] H. Belich, G. Cuba, R. Paunov,*Surfaces of Constant negative Scalar Curvature*, solv-int/9909018.

[6] R. MacKenzie, Phys.Lett. **B 352**, 96-98 (1995); hep-th/9503044.

[7] C. Taubes, Comm. Math. Phys. **72**, 277, 1980.

[8] A. Nersessian, G. Pogosyan, Phys.Rev. **D63**, 020103, (2001); quant-ph/0006118; math-ph/0010049.

[9] Q.-H. Park, H.J. Shin, "Nonabelian sine-Gordon theory and its application to nonlinear optics"; hep-th/9606094.

[10] F. Bloch, Phys. Rev. **70**, 460 (1946).

[11] M. V. Berry, Proc. Roy. Soc. Lond. A **392**, 45 (1984).

[12] P. A. M. Dirac, Proc. Roy. Soc. London A **133**, 60 (1931).