Time-reversal invariance and irreversibility in time-asymmetric quantum mechanics

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Abstract

The aim of this paper is to analyze the concepts of time-reversal invariance and irreversibility in the so-called 'time-asymmetric quantum mechanics'. We begin with pointing out the difference between these two concepts. On this basis, we show that irreversibility is not as tightly linked to the semigroup evolution laws of the theory -which lead to its non time-reversal invariance- as usually suggested. In turn, we argue that the irreversible evolutions described by the theory are coarse-grained processes.

1 Introduction

The problem of irreversibility owes its origin, in the nineteenth century, to the discussions of the founding fathers of statistical mechanics about the mechanical interpretation of the second law of thermodynamics. In those days, the problem consisted in seeking for an adequate account of the compatibility between the irreversible macroscopic evolutions described by thermodynamics and the reversible microscopic evolutions resulting from classical mechanics. In the beginning of the twentieth century, classical mechanics was replaced by quantum mechanics as the fundamental theory; however, this fact did not affect the core of the problem: independently of measurement, quantum evolutions turned out to be as reversible as the classical evolutions analyzed in the original formulation of the problem. During the first half of the twentieth century, the attempts to reconcile irreversible thermodynamic behavior with reversible quantum dynamics were confined to a background position in the face of the growing interest in the foundational problems of relativistic and quantum mechanics. It was just in the second half of the century, mainly since the 1960s, that the attention of the scientific community began to focus again on the problem of irreversibility. One of the main factors responsible for this change in the appreciation of the problem was the so-called 'time-asymmetric quantum mechanics' and its concept of intrinsic irreversibility, that is, the irreversibility due not to the interaction between a system and its environment, but to the dynamics of a closed system.

We shall subsume under the label 'time-asymmetric quantum mechanics school' ('TAQM-school', for short) the members and the works of two groups led by Arno Bohm at Austin and by Ilya Prigogine at Brussels. In spite of the differences between the two groups, it can be said that the main technical efforts of the TAQM-school has been directed to the formulation of a quantum mechanics capable of accounting for irreversible quantum phenomena, such as resonances, decaying processes, etc. Besides this general aim, the two groups agree on the use of rigged Hilbert spaces as the formal tool for addressing the issue of quantum irreversibility; according to their view, by means of this formalism it is possible to turn standard quantum mechanics into a 'time-asymmetric' theory where irreversible quantum descriptions can be obtained.

The aim of this paper is to analyze the main claims of the TAQM-school about irreversibility in the light of the distinction between the concepts of time-reversal invariance and reversibility. This task will allow us to argue that, to the extent that irreversibility and non time-reversal invariance are different concepts, the new quantum theory proposed by the TAQM-school has to include the formal resources necessary to account for each one of them. In fact, we shall identify two different elements in the formalism, each one of which is responsible for an independent and particular feature of the theory: its non time-reversal invariance and the irreversible character of its evolutions. As a consequence, we shall argue that, contrary to a common opinion, the theory's ability to account for irreversible quantum processes is independent of its 'time-asymmetry', that is, its non time-reversal invariance expressed by semigroup evolution laws. Finally, we shall show that, since Gamow vectors are functionals, the decaying Gamow vector decays only in a weak sense: this fact is the mathematical expression of the coarse-grained nature of the irreversibility described by the theory.

2 Disentangling concepts

The concepts of irreversibility and time-reversal invariance have been extensively discussed in the literature on philosophy of science. In this section our aim is not to address this old discussion; here we shall briefly consider the characterization of those concepts with the only purpose of supplying a conceptual basis for our analysis of TAQM-school's arguments.

> **Definition 1:** A dynamical equation (law) is *time-reversal invariant* if it is invariant under the application of the time-reversal operator **T.**

The time-reversal operator **T** performs the transformation $t \to -t$ and reverses certain magnitudes which depend on the particular theory considered. Nevertheless, the central idea is that **T** must reverse all the dynamical variables whose definitions in function of t are non-invariant under the transformation $t \to -t$. For instance, in classical particle mechanics, the action of **T** reverses the momenta but not the positions of the particles: $\mathbf{Tp} = -\mathbf{p}$ and $\mathbf{Tq} = \mathbf{q}$. In electromagnetism, in turn, **T** leaves the electric fields unchanged and reverses the velocities of the charges and also the magnetic fields to the extent that such fields change their direction in accordance with the velocities of the charges; then, $\mathbf{Tv} = -\mathbf{v}$, $\mathbf{TB} = -\mathbf{B}$ and $\mathbf{TE} = \mathbf{E}$ (for details, *cf.* Earman 2002). As a result, given a time-reversal invariant equation L, if $e(t)$

is a solution of L, then $\mathbf{T}e(t)$ is also a solution. If we call $\mathbf{R}$ the operator that reverses the proper magnitudes but not the time variable, the time-reversal state corresponding to $e(t)$ is:

$$\mathbf{T}e(t) = \mathbf{R}\,e(-t) \tag{1}$$

Although the concept of irreversibility has received many different characterizations in the literature, here we shall not discuss each one of them in detail. Since the irreversible evolutions treated by TAQM are usually decaying processes in closed and spatially bounded systems, we shall adopt a characterization based on the concept of attractor[1]. As it is well known, an attractor is defined as a subset of the phase space toward which a set of evolutions tends for $t \to \pm\infty$. We can extend this definition by considering a generalized concept of attractor as a subset of the set of the possible states of a system toward which a set of evolutions tends for $t \to \pm\infty$; this concept can be applied not only to phase spaces, but also to any kind of sets of states. Examples of generalized attractors are the attractors of classical dynamical systems (fixed point, limit cycle, fractal, etc.) and any classical or quantum equilibrium state. With this characterization, the concept of reversibility can be defined as follows:

> **Definition 2:** A solution (evolution) $e(t)$ of a dynamical equation is *reversible* if it has no generalized attractors, for any representation of $e(t)$.

When the time dependent state $e(t)$ can be represented as an n-uple of dynamical variables in phase space, $e(t) = (v_1(t), ..., v_n(t))$, reversibility requires that, for any dynamical variable $v_i(t)$, the limit $\lim_{t\to\pm\infty} v_i(t)$ does not exist. In this case it can be said that the evolution $e(t)$ is reversible if it has no attractors in phase space.

[1] In fact, the irreversible processes studied by TAQM are decaying processes, as the decay of excited states of molecules and nuclei, the weak decay of elementary particles or certain resonances such as those of the neutral Kaon system. In these cases, the time evolution tends to a final equilibrium state from which the system cannot escape: the irreversibility of the process is due precisely to the fact that the evolution leaving the equilibrium state is not possible. Since the aim of the TAQM is to find the adequate dynamical equations to describe this kind of irreversible behavior, in this context the concept of irreversibility can be elucidated in terms of the notion of attractor with no loss of generality.

Without entering into a discussion on the details of these two definitions (for further discussions, *cf.* Albert 2000, Arntzenius 2004), it is quite clear that the concepts of time-reversal invariance and irreversibility are different to the extent that they apply to different mathematical (physical) entities:[2] whereas time-reversal invariance is a property of dynamical equations and, *a fortiori*, of the sets of its solutions, reversibility is a property of a single solution of a dynamical equation. Furthermore, as previously characterized, both properties are not even correlated; in fact, they can be combined with each other in the four possible cases. For instance, in classical particle mechanics the application of the time-reversal operator $\mathbf{T}$ results:

$$\mathbf{T}q = q \qquad \mathbf{T}\dot{q} = -\dot{q} \tag{2}$$

$$\mathbf{T}p = -p \qquad \mathbf{T}\dot{p} = \dot{p} \tag{3}$$

Nevertheless, depending on the particular form of the Hamiltonian we can obtain:

- **Time-reversal invariance and reversibility**: Let us consider the harmonic oscillator with Hamiltonian:

  $$H = \frac{1}{2m}p^2 + \frac{1}{2}k^2 q^2 \tag{4}$$

  The dynamical equations are time-reversal invariant as can be proved by introducing eqs.(2) and (3) in:

  $$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad \dot{p} = -\frac{\partial H}{\partial q} = -k^2 q \tag{5}$$

  As a result, the set of trajectories in phase space is symmetric with respect to the q-axis. On the other hand, the solutions $q(t)$ and $p(t)$ have the following form:

  $$q(t) = C\cos(\omega t + \alpha) \qquad p(t) = Cm\omega\sin(\omega t + \alpha) \qquad \text{with } \omega = \frac{k^2}{m} \tag{6}$$

  and, therefore, they have no limit for $t \to \pm\infty$. In other words, each trajectory is reversible since it is a closed curve in phase space.

[2] For simplicity, from here we will not distinguish between mathematical entities (equations and solutions) and physical entities (laws and evolutions), using both indistinctly.

- **Time-reversal invariance and irreversibility.** Let us consider the pendulum with Hamiltonian:

$$H = \frac{1}{2m}\, p_\theta^2 - \frac{k^2}{2}\, \cos\theta \tag{7}$$

Again the dynamical equations are time-reversal invariant since $\mathbf{T}\theta = \theta$:

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m} \qquad\qquad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -\frac{k^2}{2}\sin\theta \tag{8}$$

Therefore, the set of trajectories in phase space is symmetric with respect to the θ-axis. However, not all the solutions are reversible. In fact, when $H = \frac{k^2}{2}$, the evolution is irreversible since it tends to $\theta = \pi$, $p_\theta = 0$ ($\theta = -\pi$, $p_\theta = 0$) when $t \to \infty$ ($t \longrightarrow -\infty$) (*cf.* Tabor 1989). For $H < \frac{k^2}{2}$ (oscillating pendulum) and $H > \frac{k^2}{2}$ (rotating pendulum), the evolutions are reversible.

- **Non time-reversal invariance and reversibility.** Let us now consider the modified oscillator with Hamiltonian:

$$H = \frac{1}{2m}\, p^2 + \frac{1}{2}\, K(p)^2\, q^2 \tag{9}$$

where $K(p) = K_+$ when $p \geq 0$, $K(p) = K_-$ when $p < 0$, and K_+ and K_- are constants. This means that $\mathbf{T}K_+ = K_-$. As a consequence, if $K_+ \neq K_-$, the dynamical equations are non time-reversal invariant since, for $p \geq 0$:

$$\dot{p} = -K_+^2 q \qquad \mathbf{T}\,\dot{p} = -\mathbf{T}K_+^2\,\mathbf{T}q = -K_-^2\, q \neq -K_+^2 q \tag{10}$$

and for $p < 0$:

$$\dot{p} = -K_-^2 q \qquad\quad \mathbf{T}\,\dot{p} = -\mathbf{T}K_-^2\,\mathbf{T}q = -K_+^2\, q \neq -K_-^2 q \tag{11}$$

Nevertheless, the solutions $q(t)$ and $p(t)$ are, for $p \geq 0$:

$$q(t) = C_1 \cos(\omega_+ t + \alpha_{+n}) \quad p(t) = C_1 m\omega_+ \sin(\omega_+ t + \alpha_{+n}) \tag{12}$$

and for $p < 0$:

$$q(t) = C_2 \cos(\omega_- t + \alpha_{-n}) \quad p(t) = C_2 m\omega_- \sin(\omega_- t + \alpha_{-n}) \tag{13}$$

where $\omega_\pm = \frac{K_\pm^2}{m}$ and the constants $\alpha_{\pm n}$ change from one cycle n to the next cycle $n+1$ in such a way that the solutions turn out to be continuous. It is clear that these solutions have no limit for $t \to \pm\infty$: each trajectory is reversible since it is a closed curve in phase space.

- **Non time-reversal invariance and irreversibility.** Let us consider a damped oscillator represented by the following dynamical equation:

$$\ddot{q} + A^2 \dot{q} + k^2 q = 0 \tag{14}$$

Given that $\mathbf{T}\, \ddot{q} = \ddot{q}$, the equation is non time-reversal invariant since, under the application of $\mathbf{T}$, it becomes:

$$\ddot{q} - A^2 \dot{q} + k^2 q = 0 \tag{15}$$

On the other hand, the solutions $q(t)$ have the following form:

$$q(t) = \mathrm{Re}\left[q_0\, e^{-i\alpha t}\right] = q_0 \cos\omega t\, e^{-\gamma t} \qquad \text{with } \alpha = \omega - i\gamma \tag{16}$$

Here $\cos\omega t$ is the oscillating factor and $e^{-\gamma t}$ is the damping factor. As a consequence, the evolutions are irreversible since they tend to zero for $t \to \infty$.

Up to this point, we have considered general definitions of time-reversal invariance and irreversibility. But since here we are interested in quantum mechanics, we shall restrict our attention on the following kind of evolutions:

$$e_t = U_t\, e_0 \tag{17}$$

where e_0 and e_t are vector states. The evolution operator is a unitary operator $U_t = e^{-iGt}$, where G is the self-adjoint generator of the evolution.[3] This kind of evolutions are solutions of dynamical equations of the form:

$$i\frac{de_t}{dt} = G\, e_t \tag{18}$$

[3] This analysis can also be applied to classical mechanics in the Koopman (1931) formulation.

Such a dynamical equation is *time-reversal invariant* since the time-reversal operator $\mathbf{T}$ acts on i and G as follows: $\mathbf{T}i = \mathbf{R}i = -i$ (what makes $\mathbf{T}$ and $\mathbf{R}$ antilinear[4]), $\mathbf{T}G\mathbf{T}^{-1} = \mathbf{R}G\mathbf{R}^{-1} = G$ (or, equivalently, $\mathbf{T}^{-1}G\mathbf{T} = \mathbf{R}^{-1}G\mathbf{R} = G$). In fact, under the application of $\mathbf{T}$, eq.(18) preserves its form:

$$\mathbf{T}\left(i\frac{de_t}{dt}\right) = -i\frac{d\mathbf{T}e_t}{d(-t)} = i\frac{d\mathbf{T}e_t}{dt} = \mathbf{T}\left(G\, e_t\right) = \mathbf{T}G\mathbf{T}^{^{-1}}\ \mathbf{T}e_t = G\ \mathbf{T}e_t \tag{19}$$

It is easy to prove that, if such an equation is time-reversal invariant, the set $\{U_t : t \in \mathbb{R}\}$ of evolution operators forms a group. In particular, there exists an operator

$$U_{-t} = \mathbf{R}^{-1}U_t\mathbf{R} \quad \text{such that} \quad U_tU_{-t} = I \tag{20}$$

where I is the identity operator. In fact,

$$U_{-t} = \mathbf{R}^{-1}U_t\mathbf{R} = e^{\mathbf{R}^{-1}[-iGt]\mathbf{R}} = e^{i\mathbf{R}^{-1}G\mathbf{R}t} = e^{iGt} \tag{21}$$

and, therefore, $U_tU_{-t} = e^{-iGt}\ e^{iGt} = I$. It is also easy to show how Loschmidt's paradox arises in this case. Let us consider an initial state e_0 that evolves under U_t to $e_t = U_te_0$. If now the state e_t is reversed, $\mathbf{R}e_t = \mathbf{R}U_te_0$, the further evolution under U_t, $U_t\mathbf{R}e_t = U_t\mathbf{R}U_te_0$, will lead to the reversed original state $\mathbf{R}e_0$; in fact, since $\mathbf{R}^{-1}U_t\mathbf{R}U_t = U_{-t}U_t = I$:

$$\mathbf{R}e_0 = U_t\mathbf{R}U_te_0 \quad \Rightarrow \quad e_0 = \mathbf{R}^{-1}U_t\mathbf{R}U_te_0 = U_{-t}U_te_0 = e_0 \tag{22}$$

On the other hand, the evolutions of the form $e_t = U_t\ e_0$ are *always reversible:* they have no limit for $t \to \pm\infty$ because the unitary operator U_t does not change the angle of separation (the inner product) or the distance (the square modulus of the difference) between vectors representing two different states. However, as it is well known, irreversible and, therefore, non-unitary evolutions can be obtained from the original reversible unitary dynamics by the introduction of some sort of coarse-graining. In its traditional form, a coarse-grained description arises from a partition of a phase space into discrete an disjoint cells: this mathematical procedure defines a projector (*cf.* Mackey 1989) whose action is to eliminate some components of the state

[4]If $\mathbf{T}$ were linear, $\mathbf{T}H\mathbf{T}^{-1} = -H$ (where H is the Hamiltonian), with the consequence that, for any state of energy E, there would be another state of energy $-E$. The antilinearity of $\mathbf{T}$ avoids this 'anomalous' situation (for a detailed discussion on this point, *cf.* Castagnino and Lombardi 2004b).

vector corresponding to the original description. If this idea is generalized, coarse-graining can be conceived as a projection that reduces the number of components of a vector e_t representing a state; the new coarse-grained state e_{cgt} then results:

$$e_{cgt} = \Pi \, e_t \tag{23}$$

where Π is a projector, that is, $\Pi^2 = \Pi$. The evolution represented by e_{cgt} may now be irreversible; this is the case when there exists the limit

$$\lim_{t \to \pm\infty} e_{cgt} = \lim_{t \to \pm\infty} \Pi \, e_t \tag{24}$$

This situation is usual in the description of classical systems where, under conditions of high instability, irreversible (non-unitary) coarse-grained evolutions can be obtained from the underlying reversible (unitary) dynamics. But it is worth stressing that e_t and e_{cgt} correspond to *different descriptive levels*: e_t is a reversible unitary evolution, and there is no way of extracting irreversibility in this level; e_{cgt} is an irreversible non-unitary evolution, but it can be defined only in a coarse-grained level of description.

On the basis of this elucidation of the concepts of time-reversal invariance and reversibility, *the problem of irreversibility* can be stated in a simple way: *how to explain irreversible evolutions in terms of time-reversal invariant laws.* With this characterization, there is no conceptual puzzle in the problem of irreversibility: in principle, nothing prevents a time-reversal invariant equation from having irreversible solutions. However, difficulties arise when we are dealing with dynamical equations having unitary solutions: as we have seen, since unitary evolutions are always reversible, it is necessary to go to a different level of description in order to obtain irreversibility. This point will be relevant in the discussions about irreversibility as obtained by the TAQM-school.

At this point it is worth while to emphasize that we have not talked at all about the arrow of time, a problem addressed by Bohm and Prigogine in their works. In Prigogine's version of TAQM, the second law of thermodynamics is introduced at the microscopic level as the criterion for retaining the future directed decaying evolutions and discarding the past directed ones (*cf.* Prigogine and George 1983, Antoniou and Prigogine 1993). In turn, Bohm appeals to the 'preparation-registration arrow of time' (Bohm *et al.* 1994, Bohm and Wickramasekara 2002, Bohm *et al.* 2003a), rooted in an idea that can be traced back to the works of Günther Ludwig (1983-1985): according to Bohm, this arrow expresses the asymmetry of the boundary conditions introduced by macroscopic preparation and registration devices, which are not described by quantum theory (*cf.* Bohm *et al.* 1997). We shall not address these points in the present paper since, as it has been noted (*cf.* Sklar 1974 and, more recently, Castagnino *et al.* 2003, Castagnino and Lombardi 2005), the problems of irreversibility and of time's arrow, even if related to each other, are conceptually different: whereas the problem of irreversibility asks for the explanation of irreversible evolutions in terms of time-reversal invariant laws, the problem of the arrow of time is concerned with the possibility of establishing a non-conventional and theoretically founded difference between the two directions of time. The discussion of the problem of the arrow of time in TAQM will be the subject of a future paper.

3 The rigged Hilbert space formalism

For the TAQM-school, the main reason to work with rigged Hilbert spaces is their ability to model irreversible physical phenomena, such as decaying processes, resonances and approach to equilibrium. In this section we shall describe the central features of this formalism; this will be necessary to identify, in the following sections, the key formal elements responsible for the non time-reversal invariance of the theory and for the irreversibility of its evolutions.

A rigged Hilbert space (RHS) or Gel'fand triplet (Gel'fand and Vilenkin 1964) is a triplet of spaces:

$$\mathbf{\Phi} \subset \mathcal{H} \subset \mathbf{\Phi}^{\times} \tag{25}$$

where:

- i.) The intermediate space $\mathcal{H}$ is an infinite-dimensional separable Hilbert space.

- ii.) The space $\mathbf{\Phi}$ is a topological vector space, which is dense in $\mathcal{H}$. This means that, for any $\psi \in \mathcal{H}$ and for any positive number $\varepsilon > 0$, there is another vector $\phi \in \mathbf{\Phi}$ such that $||\psi - \phi|| < \varepsilon$. The space $\mathbf{\Phi}$ has its own topology which is stronger than the topology that $\mathbf{\Phi}$ possesses as a subspace of $\mathcal{H}$.[5] The topology in $\mathbf{\Phi}$ is not given by a norm, but in the cases of physical interest, by a countable infinite family of norms; in these cases, $\mathbf{\Phi}$ has the structure of a metric space.

- iii.) The space $\mathbf{\Phi}^\times$ is the antidual space of $\mathbf{\Phi}$, and the vectors $F \in \mathbf{\Phi}^\times$, $F : \mathbf{\Phi} \to \mathbb{C}$, are functionals. The action of $F \in \mathbf{\Phi}^\times$ on $\phi \in \mathbf{\Phi}$ is usually expressed as $F(\phi)$ or, in Dirac's notation, $\langle \phi | F \rangle$. The functionals F fulfill the following conditions:

  - a.) *Antilinearity*: For any $\phi, \varphi \in \mathbf{\Phi}$ and $\alpha, \beta \in \mathbb{C}$:

  $$F(\alpha\,\phi + \beta\,\varphi) = \alpha^*\,F(\phi) + \beta^*\,F(\varphi) \tag{26}$$

    where the star denotes complex conjugation.

  - b.) *Continuity:* If $\phi_n \longmapsto \phi$ in $\mathbf{\Phi}$, then $F(\phi_n) \longmapsto F(\phi)$ in $\mathbb{C}$.[6]

In a RHS, $\mathcal{H} \subset \mathbf{\Phi}^\times$ means that any vector belonging to $\mathcal{H}$ can be viewed as a functional on $\mathbf{\Phi}$. If $\psi \in \mathcal{H}$, then the functional F_ψ belonging to $\mathbf{\Phi}^\times$ is uniquely defined by ψ as:

$$\langle \varphi | F_\psi \rangle := \langle \varphi | \psi \rangle \tag{27}$$

where $\langle \varphi | \psi \rangle$ is the usual scalar product in $\mathcal{H}$.

RHSs have an important property that is essential to understand the definition of the Gamow vectors. Under general assumptions, an operator A on $\mathcal{H}$ can be extended into the antidual $\mathbf{\Phi}^\times$ as $A^\times$ by the *duality formula*:

$$\langle A^\dagger \phi | F \rangle = \langle \phi | A^\times\, F \rangle\,, \qquad \forall\, \phi \in \mathbf{\Phi}\ ,\ \forall\, F \in \mathbf{\Phi}^\times \tag{28}$$

where $A^\dagger$ is the adjoint of A and $A^\times$ is a linear and continuous operator on $\mathbf{\Phi}^\times$ (*cf.* Schäffer 1970).[7] This property also applies when A is self-adjoint;

[5] This can be explained by noticing that the topology in $\mathbf{\Phi}$ has more open sets, and, consequently, more neighborhoods and less convergent sequences than $\mathcal{H}$.

[6] This is sufficient to define continuity in the case that $\mathbf{\Phi}$ is a metric space. Otherwise, it is necessary to generalize this condition to more general structures called 'nets' and 'filters' (*cf.* Schäffer 1970).

[7] The assumptions are: (i) the domain $\mathcal{D}(A^\dagger)$ of $A^\dagger$ includes the space $\mathbf{\Phi}$: $\mathbf{\Phi} \subset \mathcal{D}(A^\dagger)$, (ii) for each $\phi \in \mathbf{\Phi}$, $A^\dagger\phi \in \mathbf{\Phi}$; in this case we say that $\mathbf{\Phi}$ reduces $A^\dagger$, and(iii) the operator $A^\dagger$ is continuous on $\mathbf{\Phi}$ in the own topology of $\mathbf{\Phi}$.

in this case, $A^\dagger = A$, and the duality formula becomes:

$$\langle A\phi|F\rangle = \langle \phi|A^\times F\rangle \,, \qquad \forall\, \phi \in \mathbf{\Phi} \,,\ \forall\, F \in \mathbf{\Phi}^\times \tag{29}$$

Different realizations of RHSs have been used in the physical literature for distinct purposes. For instance, the Schwartz space[8] S is included and dense in the Hilbert space of complex square integrable functions in the real line, $L^2(\mathbb{R})$ (*cf.* Reed and Simon 1972). The RHS:

$$S \subset L^2(\mathbb{R}) \subset S^\times \tag{30}$$

has been used to give a rigorous mathematical foundation to Dirac's formalism. This realization is time-reversal invariant and irreversibility is not discussed in this case.

In the case of time-asymmetric quantum mechanics, Hardy functions play a central role. A function $f(x)$ is a *Hardy function on the upper (lower) half-plane* $\mathrm{Im}\, z > 0$ ($\mathrm{Im}\, z < 0$) of the complex plane, $f(x) \in \mathcal{H}^2_+$ ($f(x) \in \mathcal{H}^2_-$), iff:

- i.) $f(x)$ is a complex function of real variable, $f : \mathbb{R} \to \mathbb{C}$.
- ii.) $f(x)$ represents the boundary values of an analytic function $f(z)$ on the upper (lower) half plane $\mathrm{Im}\, z > 0$ ($\mathrm{Im}\, z < 0$) of the complex plane. This means that, for any $y_0 > 0$, $y_0 \in \mathbb{R}$, the complex function of complex variable $f(z) = f(x + iy_0)$ ($f(z) = f(x - iy_0)$) is analytic in the upper (lower) half-plane. In this case it is said that $f(z)$ is the analytical continuation of the function $f(x)$ in the upper (lower) half-plane of the complex plane.
- iii.) The following inequality holds:

$$\sup_{y_0>0} \int_{-\infty}^{\infty} |f(x \pm iy_0)|^2 \, dx \le K \qquad \text{with } K > 0 \tag{31}$$

[8]A function $f(x)$ is a Schwartz function, $f(x) \in S$, iff: (i) $f(x)$ is a complex function of real variable: $f : \mathbb{R} \to \mathbb{C}$, (ii) $f(x)$ is continuous and derivable to any order, and (iii) $f(x)$ tends to 0 for $|x| \to \infty$ faster than the inverse of any polinomial:

$$\lim_{|x|\to\infty} x^n f(x) = 0 \qquad \forall n = 0, 1, 2, ...$$

The same property is valid for the derivatives of $f(x)$.

where the sign $+$ $(-)$ corresponds to functions defined on the upper (lower) half plane, and the constant K depends on $f(z)$.

As a consequence of this definition, any Hardy function on the upper (lower) half-plane, $f_+(x) \in \mathcal{H}^2_+$ ($f_-(x) \in \mathcal{H}^2_-$), is a limit of a complex function $f_+(z)$ ($f_-(z)$) that is analytic in the upper (lower) half plane (*cf.* Koosis 1980):

$$\lim_{y_0 \to 0} f_\pm(z) = \lim_{y_0 \to 0} f_\pm(x \pm iy_0) = f_\pm(x) \qquad a.e. \tag{32}$$

where the almost everywhere restriction is referred to the Lebesgue measure on $\mathbb{R}$, and $f_+(x) \in L^2(\mathbb{R})$ ($f_+(x) \in L^2(\mathbb{R})$). Therefore, any $f(z)$ analytic in the upper (lower) half-plane fulfilling condition iii.) uniquely determines its boundary values on the real line, given by a Hardy function on the upper (lower) half-plane. After Titchmarsh's theorem (Titchmarsh 1937), the reciprocal is also true: any Hardy function on the upper (lower) half-plane, uniquely determines a complex function $f_+(z)$ ($f_-(z)$) that is analytic in the upper (lower) half-plane:

$$f_\pm(z) = f_\pm(x \pm iy_0) = \pm \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f_\pm(x)}{x - z}\, dx \tag{33}$$

and that fulfills condition iii.). Consequently, the functions on a half plane with this property can be identified with their boundary values on the real axis and, therefore, it is usual to assign the name 'Hardy function' also to them. Furthermore, it can be proved that the Hardy spaces $\mathcal{H}^2_+$ and $\mathcal{H}^2_-$ have the following important properties:

- a.) $\mathcal{H}^2_+$ and $\mathcal{H}^2_-$ have no other function in common than the zero function:

$$\mathcal{H}^2_+ \cap \mathcal{H}^2_- = \{0\} \tag{34}$$

- b.) $\mathcal{H}^2_+$ and $\mathcal{H}^2_-$ are closed subspaces of $L^2(\mathbb{R})$ and, therefore, Hilbert subspaces of $L^2(\mathbb{R})$:

$$\mathcal{H}^2_+ \subset L^2(\mathbb{R}) \qquad \mathcal{H}^2_- \subset L^2(\mathbb{R}) \tag{35}$$

- c.) The direct sum of $\mathcal{H}^2_+$ and $\mathcal{H}^2_-$ coincides with $L^2(\mathbb{R})$:

$$\mathcal{H}^2_+ \oplus \mathcal{H}^2_- = L^2(\mathbb{R}) \tag{36}$$

A Hardy function on the upper (lower) half-plane is called 'smooth' if it is infinitely differentiable and fast decreasing. Therefore, the space of smooth Hardy functions on the upper (lower) half-plane is the intersection between the Hardy space $\mathcal{H}^2_+$ ($\mathcal{H}^2_-$) and the Schwartz space S: $S \cap \mathcal{H}^2_+$ ($S \cap \mathcal{H}^2_-$). As a consequence of a theorem by Paley and Wiener (1934), the intersections $S \cap \mathcal{H}^2_\pm$ are not only non-trivial but also dense in $\mathcal{H}^2_\pm$. If we endow $S \cap \mathcal{H}^2_\pm$ with the metric topology inherited from S, it can be shown that (*cf.* Bohm and Gadella 1989):

$$S \cap \mathcal{H}^2_\pm \subset \mathcal{H}^2_\pm \subset (S \cap \mathcal{H}^2_\pm)^\times \tag{37}$$

are RHSs. However, these are not yet the RHSs used in time-asymmetric quantum mechanics. As a consequence of a result due to van Winter (van Winter 1974), any Hardy function on the upper (lower) half plane, $f_+(z)$ ($f_-(z)$), is uniquely determined by its boundary values on the positive real semiaxis $\mathbb{R} = [0, \infty)$. Therefore, instead of working with $S \cap \mathcal{H}^2_\pm$, we can work with $S \cap \mathcal{H}^2_\pm\Big|_{\mathbb{R}^+}$, that is, the restriction of the functions of $S \cap \mathcal{H}^2_\pm$ to $\mathbb{R}^+$. Since it can be proved that both $S \cap \mathcal{H}^2_\pm\Big|_{\mathbb{R}^+}$ are dense in $L^2(\mathbb{R}^+)$ (*cf.* Bohm and Gadella 1989), then:

$$S \cap \mathcal{H}^2_\pm\Big|_{\mathbb{R}^+} \subset L^2(\mathbb{R}^+) \subset \left(S \cap \mathcal{H}^2_\pm\Big|_{\mathbb{R}^+}\right)^\times \tag{38}$$

are also RHSs: these are the particular realizations used by the TAQM-school in its time-asymmetric quantum mechanics.

4 Rigged Hilbert spaces in time-asymmetric quantum mechanics

The TAQM-school works with vector states $\phi^\pm \in \mathbf{\Phi}_\pm$, whose corresponding wave functions in the energy representation, $\phi^\pm(\omega)$, belong to the space $S \cap \mathcal{H}^2_\pm\Big|_{\mathbb{R}^+}$:

$$\phi^\pm \in \mathbf{\Phi}_\pm \qquad\qquad \phi^\pm(\omega) = \langle \omega | \phi^\pm \rangle \in S \cap \mathcal{H}^2_\pm\Big|_{\mathbb{R}^+} \tag{39}$$

As it is well known, in standard quantum mechanics the time evolution of a state vector belonging to the Hilbert space $\mathcal{H}$ is given by the unitary evolution obtained from the Schrödinger equation:

$$\forall \varphi \in \mathcal{H} \qquad \varphi(t) = U_t\, \varphi = e^{-iHt} \varphi \qquad \text{for } -\infty < t < \infty \tag{40}$$

where H is the Hamiltonian operator and $U_t = e^{-iHt}$ is a well defined unitary operator on $\mathcal{H}$. Let us now consider the action of the evolution operator $U_{-t} = e^{iHt}$ on the vectors $\phi^{\pm} \in \mathbf{\Phi}_{\pm}$, where U_{-t} is the adjoint (inverse) of U_t. This strategy is justified by the fact that we want to obtain the evolution operator $U_t^{\times}$ for the vectors belonging to $\mathbf{\Phi}_{\pm}^{\times}$ and, therefore, we have to begin with the adjoint of U_t: $U_t^{\dagger} = U_{-t}$ (*cf.* the duality formula 28):

$$\langle U_t^{\dagger} \phi | F \rangle = \langle \phi | U_t^{\times} F \rangle \,, \qquad \forall\, \phi \in \mathbf{\Phi} \;,\; \forall\, F \in \mathbf{\Phi}^{\times} \tag{41}$$

The operator U_{-t} is well defined on $\mathbf{\Phi}_{\pm}$; however, its behavior on $\mathbf{\Phi}_{+}$ and $\mathbf{\Phi}_{-}$ is very different for different values of t. In fact, it is desired that the action of e^{iHt} on a function $\phi^{+}(\omega)$ lead to a new function $\varphi^{+}(\omega)$ such that:

$$\text{If } \phi^{+}(\omega) \in S \cap \mathcal{H}^2_{+}\Big|_{\mathbb{R}^+}, \text{ then } \varphi^{+}(\omega) = e^{i\omega t} \phi^{+}(\omega) \in S \cap \mathcal{H}^2_{+}\Big|_{\mathbb{R}^+} \tag{42}$$

In other words, e^{iHt} should turn smooth Hardy functions on the upper half-plane into smooth Hardy functions on the upper half-plane. But this requirement is not fulfilled for all values of t since the third property in the definition of the Hardy functions (*cf.* eq.(31)) does not hold for $t < 0$. Precisely, only for $t \geq 0$:

$$\sup_{y_0 > 0} \int_{-\infty}^{\infty} |\phi^{+}(\omega + iy_0)\, e^{i(\omega + iy_0)t}|^2\, dx = \sup_{y_0 > 0} \int_{-\infty}^{\infty} |\phi^{+}(\omega + iy_0)|^2\; e^{-2y_0 t}\, dx \leq K \tag{43}$$

Furthermore, one can show that, for each $t_0 < 0$, there exists a function $\phi^{+}(\omega) \in S \cap \mathcal{H}^2_{+}\Big|_{\mathbb{R}^+}$ such that $e^{-i\omega t_0} \phi^{+}(\omega) \notin S \cap \mathcal{H}^2_{+}\Big|_{\mathbb{R}^+}$. Therefore, condition (42) only holds for $t \geq 0$. An analogous argument can be applied to functions $\phi^{-}(\omega) \in S \cap \mathcal{H}^2_{-}\Big|_{\mathbb{R}^+}$: the evolution operator $e^{i\omega t}$ turns smooth Hardy functions on the lower half-plane into smooth Hardy functions on the lower half-plane only for $t \leq 0$ (*cf.* Bohm and Gadella 1989).

The above results can be summarized as follows:

$$\text{If } \phi^{+} \;\in\; \mathbf{\Phi}_{+}, \text{ then } \varphi^{+} = e^{iH_{+}t} \phi^{+} \in \mathbf{\Phi}_{+} \quad \text{for } t \geq 0 \tag{44}$$

$$\text{If } \phi^{-} \;\in\; \mathbf{\Phi}_{-}, \text{ then } \varphi^{-} = e^{iH_{-}t} \phi^{-} \in \mathbf{\Phi}_{-} \quad \text{for } t \leq 0 \tag{45}$$

where the semigroup generators $H_\pm$ are the restrictions of the self-adjoint operator H to the subspaces $\mathbf{\Phi}_\pm$. In turn, the evolution operators $U_t^+ := e^{-iH_+t}$ and $U_t^- := e^{-iH_-t}$ can be extended into the antiduals $\mathbf{\Phi}_\pm^\times$ by means of the duality formula (28):

$$
\begin{aligned}
\langle U^+_{-t}\phi^+|F^+\rangle &= \langle\phi^+|U_t^{+\times}F^+\rangle \;\forall\,\phi^+\in\mathbf{\Phi}_+\,,\;\forall\,F^+\!\in\mathbf{\Phi}_+^\times,\,\forall t\geq 0 \quad (46)\\
\langle U^-_{-t}\phi^-|F^-\rangle &= \langle\phi^-|U_t^{-\times}F^-\rangle \;\forall\,\phi^-\in\mathbf{\Phi}_-\,,\;\forall\,F^-\!\in\mathbf{\Phi}_-^\times,\,\forall t\leq 0 \quad (47)
\end{aligned}
$$

where:

$$
U_t^{+\times} = e^{-iH_+^\times t} \qquad\qquad U_t^{-\times} = e^{-iH_-^\times t} \tag{48}
$$

and $H_\pm^\times$ are the extensions of the self-adjoint operator H to the subspaces $\mathbf{\Phi}_\pm^\times$ (*cf.* Bohm and Scurek 2000, Bohm *et al.* 2003a). Observe that $U_t^{+\times}$ and $U_t^{-\times}$ are operators defined on $\mathbf{\Phi}_+^\times$ and $\mathbf{\Phi}_-^\times$ respectively. Equations (46) and (47) can, therefore, be written as follows:

$$
\begin{aligned}
\langle e^{iH_+t}\phi^+|F^+\rangle &= \langle\phi^+|e^{-iH_+^\times t}F^+\rangle \;\forall\,\phi^+\in\mathbf{\Phi}_+\,,\;\forall\,F^+\!\in\mathbf{\Phi}_+^\times,\,\forall t\geq 0 \quad (49)\\
\langle e^{iH_-t}\phi^-|F^-\rangle &= \langle\phi^-|e^{-iH_-^\times t}F^-\rangle \;\forall\,\phi^-\in\mathbf{\Phi}_-\,,\;\forall\,F^-\!\in\mathbf{\Phi}_-^\times,\,\forall t\leq 0 \quad (50)
\end{aligned}
$$

Summing up, the choice of Hardy functions for this particular realization of the RHS is what allows the TAQM-school to obtain evolution operators that form semigroups, instead of the group evolution operators of the traditional Hilbert space formulation of quantum mechanics.

In addition to the states $\phi^\pm\in\mathbf{\Phi}_\pm$ with smooth wave functions $\phi^\pm(\omega)$, this realization of the RHS formalism introduces new generalized vectors, that is, functionals on the spaces $\mathbf{\Phi}_\pm$. Loosely speaking, in a RHS, the smaller the space $\mathbf{\Phi}$ is, the bigger the space $\mathbf{\Phi}^\times$ is. In this particular realization, the spaces $\mathbf{\Phi}_\pm$ are restricted enough to permit their antiduals $\mathbf{\Phi}_\pm^\times$ to contain not only the Dirac kets, but also more general kets. In fact, besides to eigenkets with real eigenvalues, the spaces $\mathbf{\Phi}_\pm^\times$ may contain also eigenvectors of the Hamiltonian having complex eigenvalues. For instance, there may exist a vector $\Psi^D\in\mathbf{\Phi}_+^\times$, called '*decaying Gamow vector*', and a vector $\Psi^G\in\mathbf{\Phi}_-^\times$, called '*growing Gamow vector*', such that they are eigenvectors of $H_+^\times$ and $H_-^\times$ with complex eigenvalues $z_R=\omega_R-i\frac{\Gamma}{2}$ and $z_R^*=\omega_R+i\frac{\Gamma}{2}$ respectively, with $\Gamma>0$ (*cf.* Bohm and Gadella 1989, Bohm *et al.* 2003b):

$$
H_+^\times\,\Psi^D \;=\; z_R\,\Psi^D=(\omega_R-i\frac{\Gamma}{2})\,\Psi^D \tag{51}
$$

$$H_-^\times\, \Psi^G \;\; = \;\; z_R^*\, \Psi^G = (\omega_R + i\frac{\Gamma}{2})\, \Psi^G \tag{52}$$

The Gamow vectors are related with resonances, which are usually described by means of the analytical continuation of the scattering operator S in the energy representation, $S(\omega)$:[9] the analytical continuation of $S(\omega)$ in the upper and the lower half-planes of the complex energy plane possesses at least a pair of complex conjugate poles at the points z_R and z_R^*, which turn out to be the complex eigenvalues of the Hamiltonian (*cf.* Gadella 1997).[10] The imaginary part of these eigenvalues is precisely what breaks down the unitary character of the time evolution and makes it possible to obtain exponentially growing and decaying states. In fact, since the Gamow vectors belong to the antidual spaces $\mathbf{\Phi}_\pm^\times$, their time evolution has to be computed by means of the duality formulas (49) and (50):

$$\langle e^{iH_+ t}\phi^+ | \Psi^D \rangle \;\; = \;\; \langle \phi^+ | e^{-iH_+^\times t} \Psi^D \rangle \qquad \forall\, \phi^+ \in \mathbf{\Phi}_+ \,,\, \forall t \geq 0 \tag{53}$$

$$\langle e^{iH_- t}\phi^- | \Psi^G \rangle \;\; = \;\; \langle \phi^- | e^{-iH_-^\times t} \Psi^G \rangle \qquad \forall\, \phi^- \in \mathbf{\Phi}_- \,,\, \forall t \leq 0 \tag{54}$$

Therefore, for $\forall t \geq 0$:

$$\langle \phi^+ | e^{-iH_+^\times t} \Psi^D \rangle \,=\, \langle \phi^+ | \Psi^D \rangle \, e^{-i(\omega_R - i\frac{\Gamma}{2})t} \,=\, \langle \phi^+ | \Psi^D \rangle \, e^{-i\omega_R t} \, e^{-\frac{\Gamma}{2}t} \tag{55}$$

This expression represents an exponentially decaying process with lifetime $\tau = \frac{2}{\Gamma}$, whose limit when t goes to infinity results:

$$\lim_{t\to\infty} \langle \phi^+ | e^{-iH_+^\times t} \Psi^D \rangle \,=\, \lim_{t\to\infty} \langle \phi^+ | \Psi^D \rangle \, e^{-i\omega_R t} \, e^{-\frac{\Gamma}{2}t} = 0 \tag{56}$$

This means that, for $t \to \infty$, the decaying Gamow vector Ψ^D exponentially decays in a *weak sense*. Analogously, the growing Gamow vector Ψ^G exponentially decays in a *weak sense* for $t \to -\infty$.

At this point we have all the conceptual and formal elements for assessing the claims of the TAQM-school about irreversibility.

---

[9] As poles of the resolvent, Gamow vectors were first introduced by Grossmann (1964), independently of RHSs. Later, they were unexpectedly discovered in the RHS formalism as generalized eigenvectors of self-adjoint operators with complex eigenvalues (Lindblad and Nagel 1970). The association between the poles of the S-matrix with the vectors in the RHS was established in the 1980s (Bohm 1981, Gadella 1983, 1984).

[10] Second and higher order poles of the scattering operator S are treated in Bohm *et al.* 1997 and in Antoniou *et al.* 1998.

5 Time-reversal invariance and irreversibility in TAQM-school

One of the main purposes of the TAQM-school is to obtain a formulation of quantum mechanics capable of explaining irreversible quantum phenomena; according to their view, the use of RHSs is what turns standard quantum mechanics into a time-asymmetric theory where irreversible quantum descriptions are possible. But we have seen that non time-reversal invariance and irreversibility are different concepts; therefore, it is worth while to ask how and by means of which formal resources the new formalism accounts for these two different features.

In its many works, the TAQM-school seems to suggest that, in the RHS formalism, the fact that evolutions are described by means of semigroups rather than groups is what permits irreversibility to be modeled in a natural way. For instance, according to Antoniou and Prigogine (1993), semigroups are the formal elements that describe the intrinsic irreversibility of large Poincaré systems where the number of degrees of freedom tend to infinite and 'continuous sets of resonances' arise (for a detailed discussion, *cf.* Bishop 2004). In turn, according to Bohm, in the new formulation of quantum mechanics, "*the semigroup arrow is interpreted as microphysical irreversibility*" (Bohm *et al.* 1994, p.2593) and "*the semigroup* $e^{-iH_+^\times t}$*,* $t \geq 0$*, expresses intrinsic irreversibility on the microphysical level*" (Bohm and Harshman 1998, p.233; for a similar claim, *cf.* p.189). In particular, Bohm asserts that: "*It was realized by Antoniou that the RHS of Hardy class functions provided the suitable mathematical framework for describing irreversibility at the microscopic level*" (Bohm *et al.*, 1997, p.491). These claims show that the TAQM-school seems to establish a close link between the non time-reversal invariance of the theory, expressed by semigroup evolution laws, and the irreversibility of the processes described by it, as if the irreversible character of the particular evolutions depended on the fact that they are described by semigroups. In fact, the school's proposal has been interpreted in this sense. For instance, Bishop says that "*one of the important features of the RHS is that evolution operators are often elements of semigroups rather than groups, so that irreversible behavior can be modeled naturally*" (Bishop 2004b, p.17; for similar claims, *cf.* Bishop 2003), and that "*compared to the standard HS framework, the RHS framework provides a significant advantage in the de-*

scription of irreversible processes in that semigroup evolutions arise naturally in the latter" (Bishop 2004a, p.1685), since "*semigroups of operators are the appropriate operators for the evolution of intrinsically irreversible processes*" (Bishop 2004a, p.1679). The same idea reappears when the author discusses, in particular, the works of Prigogine's and Bohm's groups: "*the intrinsic irreversibility of LPS [large Poincaré systems] must be described by semigroups*" (Bishop 2004b, p.18); "*these semigroups fall out of the analysis quite naturally in the RHS framework providing a rigorous description of irreversible behavior in a scattering experiment*" (Bishop 2004a, 1680). We shall argue that this way of conceiving the contributions of the TAQM-school is misguided: when the RHS's realization used by the TAQM school is analyzed from a mathematical viewpoint, the supposed link between the semigroup evolution laws ant the irreversibility of the processes described by the theory is not as close as those claims seem to suggest.

As we have seen, when time evolutions are governed by a unitary operator U_t, the time-reversal invariance of the evolution law implies that the evolution operators U_t ($t \in \mathbb{R}$) form a group, in particular, that there exists an operator U_{-t} such that $U_t U_{-t} = I$. But the time-evolutions in TAQM's formalism are described by the operators $U_t^{+\times}$ and $U_t^{-\times}$, which are defined only for $t \geq 0$ and $t \leq 0$, respectively; this means that $U_t^{+\times}$ with $t < 0$ and $U_t^{-\times}$ with $t > 0$ do not exist and, therefore, the sets $\{U_t^{+\times} : 0 \leq t \in \mathbb{R}\}$ and $\{U_t^{-\times} : 0 \geq t \in \mathbb{R}\}$ of evolution operators form two semigroups. This fact is what breaks down the time-reversal invariance of the original theory: now we have two semigroup evolution laws, each one of which is non time-reversal invariant. In turn, semigroups arise as a result of using Hardy functions in this particular realization of the RHS formalism. As it was explained, the impossibility of defining an evolution operator for $-\infty < t < \infty$ depends on *the third property in the definition of the Hardy functions*, which implies that, for any Hardy function $\phi^{\pm}(\omega)$ and for any $y_0 > 0$, the functions $\phi^{\pm}(\omega \pm i y_0)\, e^{i(\omega \pm i y_0)t}$ must be square integrable and all the integrals must be bounded by the same constant $K > 0$. This means that the non time-reversal invariance of the theory is a consequence of working with a particular realization of the RHS formalism, based on Hardy functions.

On the other hand, an evolution is irreversible if it has a limit for $t \rightarrow \pm\infty$. In the TAQM' formalism, irreversibility is introduced by the fact that processes that exponentially decay (grow) as $e^{-\frac{\Gamma}{2}t}$ ($e^{\frac{\Gamma}{2}t}$) can be obtained: they have a well defined limit for $t \rightarrow \infty$ ($t \rightarrow -\infty$). And this, in turn, depends on the existence of the decaying (growing) Gamow vector Ψ^D (Ψ^G), which is an eigenvector of the Hamiltonian with complex eigenvalue $z_R = \omega_R - i\frac{\Gamma}{2}$ ($z_R^* = \omega_R + i\frac{\Gamma}{2}$). The possibility of defining Gamow vectors is a result of using functions that can be analytically continued in the lower and in the upper half-planes of the complex energy plane: each pair of Gamow vectors correspond to the resonance determined by the pair of poles z_R and z_R^* of those analytical continuations. But the property of having analytical continuation in the half-planes of the complex plane is weaker than the property of being a Hardy function since it is only *the second property in the definition of the Hardy functions*. This means that the existence of Gamow vectors depends neither on the use of Hardy functions, nor on the semigroup description of the evolution law. For instance, Gamow vectors can be defined as functionals on spaces of the form $\mathcal{F}(\mathcal{D}(\mathbb{R}))$, where $\mathcal{F}$ stands for the Fourier transformation and $\mathcal{D}(\mathbb{R})$ is the space of infinitely differentiable complex functions with compact support on the real line $\mathbb{R}$. Each function of $\mathcal{F}(\mathcal{D}(\mathbb{R}))$ is entirely analytical and, considered as a complex function of the real variable $\mathbb{R}$, a Schwartz function. It can be proved that the space $\mathcal{F}(\mathcal{D}(\mathbb{R}))\Big|_{\mathbb{R}^+}$ of the restrictions to the positive semiaxis $\mathbb{R}^+$ of the functions belonging to $\mathcal{F}(\mathcal{D}(\mathbb{R}))$ is dense in $L^2(\mathbb{R}^+)$. Therefore, the triplet:

$$\mathcal{F}(\mathcal{D}(\mathbb{R}))\Big|_{\mathbb{R}^+} \subset L^2(\mathbb{R}^+) \subset \left(\mathcal{F}(\mathcal{D}(\mathbb{R}))\Big|_{\mathbb{R}^+}\right)^\times \tag{57}$$

is a realization of the RHS $\mathbf{\Phi} \subset \mathcal{H} \subset \mathbf{\Phi}^\times$. In this RHS we can define two functionals $\Psi^D$and Ψ^G belonging to $\mathbf{\Phi}^\times$ such that they are eigenvectors of the extension $H^\times$ of the total Hamiltonian H to $\mathbf{\Phi}^\times$, with complex eigenvalues $z_R = \omega_R - i\frac{\Gamma}{2}$ and $z_R^* = \omega_R + i\frac{\Gamma}{2}$ respectively (as in eqs. (51) and (52)). Here z_R and z_R^* are also the pair of complex conjugate poles of the analytical continuation of the scattering operator S in the energy representation. However, in this case the space $\mathbf{\Phi}$ is invariant under the action of the whole group $U_t = e^{-iHt}$ and, therefore, this group can be extended to the antidual $\mathbf{\Phi}^\times$ as $U_t^\times$. As a consequence, the Gamow vectors $\Psi^D$and Ψ^G evolve, for all $\phi \in \mathbf{\Phi}$ and *for all values of* t, as:

$$\langle\phi|U_t^\times\Psi^D\rangle = \langle\phi|\Psi^D\rangle\, e^{-i\omega_R t}\, e^{-\frac{\Gamma}{2}t} \tag{58}$$

$$\langle\phi|U_t^\times\Psi^G\rangle = \langle\phi|\Psi^G\rangle\, e^{-i\omega_R t}\, e^{\frac{\Gamma}{2}t} \tag{59}$$

This clearly shows that the very use of a RHS where the Gamow vectors can be defined does not lead by itself to a semigroup description of the evolution law; this is the consequence of the construction of the spaces $\mathbf{\Phi}_{\pm}$ via Hardy functions.

Summing up, the non time-reversal invariance of the theory proposed by the TAQM-school is a consequence of the use of Hardy functions in the realization of the RHSs; the irreversibility of certain evolutions is obtained by means of Gamow vectors, which depend on working with functions that can be analytically continued in the two half-planes of the complex energy plane and, in particular, on the existence of poles in those continuations. But the existence of Gamow vectors does not depend on the use of Hardy functions. Therefore, the theory's ability to describe irreversible evolutions does not depend on its non time-reversal invariance. In fact, irreversible evolutions can also arise in a time-reversal invariant theory based on an adequate RHS. For instance, Gamow vectors can be obtained in a realization of the RHS formalism in terms of functions that have analytical continuations but are not Hardy functions (*cf.* Castagnino and Laura 1997, Castagnino *et al.* 2002). In this case, one may eventually define a pair of structures to describe resonances, one for positive and the other for negative values of time (*cf.* Castagnino *et al.* 2001). Here resonances are also related with the poles of the corresponding function of complex variable; however, since the constraint imposed by the Hardy functions does not exist, the time evolutions are governed by group evolution laws and, as a consequence, the theory remains as time-reversal invariant as standard quantum mechanics in separable Hilbert space. So, the assumption that the RHS of Hardy class functions provides the suitable mathematical framework for describing irreversibility is, at least, misleading. Conversely, in particular situations, the non time-reversal invariant theory of the TAQM-school may describe only reversible evolutions: this is the case of periodic systems, in whose descriptions Gamow vectors do not take part. This fact is not surprising to the extent that the new formalism must be capable of accounting also for the traditional reversible evolutions described by standard quantum mechanics. These arguments clearly show that the links between non time-reversal invariance and irreversibility are not as strong as it is usually suggested in the literature on TAQM.

6 The coarse-grained nature of irreversibility

The main scientific concern of Prigogine has been the status of the second law of thermodynamics and, in general, of irreversible processes. According to him, if the second law has to be considered as a fundamental law, it cannot be the result of coarse-graining, since coarse-grained descriptions are unavoidably subjective, only due to our calculation techniques and measurement limitations. On this basis, through all his works Prigogine was a bitter enemy of any sort of coarse-graining: for him, it is absurd to conceive irreversible processes, such as the combustion in a furnace or the burning of a candle, as dependent on the observer and his experimental capacities (*cf.* Prigogine and Stengers 1979). For this reason, Prigogine directed his efforts to obtain an account of the objective and, from his viewpoint, necessarily non coarse-grained irreversibility. Here our aim is neither to evaluate Prigogine's motivations nor to discuss the alleged relationship between coarse-graining and subjectivity. Our only purpose is to analyze the way in which irreversibility arises in TAQM-school's proposal in order to determine if the RHS formalism avoids coarse-graining as Prigogine assumes.

In time-asymmetric quantum mechanics, the elements of the antidual spaces $\mathbf{\Phi}_{\pm}^{\times}$ represent '*generalized states*' (*cf.* Bohm *et al.* 2003b), which evolve under the action of the operators $U_t^{\pm\times}$ as follows:

$$\varphi^+(t) = U_t^{+\times}\varphi^+ = e^{-iH_+^\times t}\varphi^+ \qquad \forall\, \varphi^+ \in \mathbf{\Phi}_+^\times, \forall t \geq 0 \tag{60}$$

$$\varphi^-(t) = U_t^{-\times}\varphi^+ = e^{-iH_-^\times t}\varphi^+ \qquad \forall\, \varphi^- \in \mathbf{\Phi}_-^\times, \forall t \leq 0 \tag{61}$$

In the case in which the generalized state φ^+ is the decaying Gamow vector Ψ^D, Bohm claims that the evolution is given by[11] (*cf.* Bohm *et al.* 2003a, p.129, eq.(42)):

$$\Psi^D(t) = e^{-iH_+^\times t}\,\Psi^D = e^{-i\omega_R t}\,e^{-\frac{\Gamma}{2}t}\,\Psi^D \qquad \forall t \geq 0 \tag{62}$$

Therefore:

$$\lim_{t\to\infty}\Psi^D(t) = \lim_{t\to\infty} e^{-i\omega_R t}\,e^{-\frac{\Gamma}{2}t}\,\Psi^D = 0 \tag{63}$$

These two last equations suggest that the Gamow vectors represent states in the same sense as the vectors belonging to the spaces $\mathbf{\Phi}_\pm$ do, and that they

[11]Note that our sign convention does not coincide with Bohm's, since he uses $\phi^\pm \in \mathbf{\Phi}_\mp$.

decay "by themselves": "*the Gamow vector for the Friedrichs model decay exponentially in the future,* $t > 0$" (Antoniou *et al.* 2001). In other words, it seems that the RHS formalism permits quantum states to evolve in an irreversible way: quantum irreversible phenomena would be defined in the same level of description as the reversible processes described by standard quantum mechanics. However, this interpretation can be questioned when the mathematical nature of the Gamow vectors in the context of the RHS formalism is taken into account.

Let us remember that the vectors belonging to $\mathbf{\Phi}^{\times}$ are *functionals* acting on the elements of $\mathbf{\Phi}$. As a consequence of their very mathematical nature, they have meaning not as isolated elements, but only in their relationship with the vectors belonging to $\mathbf{\Phi}$:

$$\phi \in \mathbf{\Phi}\ ,\ F \in \mathbf{\Phi}^{\times} \qquad F(\phi) = \langle \phi | \, F \rangle \tag{64}$$

Since the Gamow vectors are particular functionals belonging to $\mathbf{\Phi}^{\times}_{\pm}$, they only have mathematical meaning in expressions of the form:

$$\begin{aligned} \phi^{+} &\in \mathbf{\Phi}_{+} \qquad \langle \phi^{+} | \, \Psi^{D} \rangle & (65) \\ \phi^{-} &\in \mathbf{\Phi}_{-} \qquad \langle \phi^{-} | \, \Psi^{G} \rangle & (66) \end{aligned}$$

Therefore, in mathematical precise terms, their time evolution has to be computed as:

$$\begin{aligned} \langle \phi^{+} | e^{-iH_{+}^{\times} t} \Psi^{D} \rangle &= \langle \phi^{+} | \Psi^{D} \rangle \, e^{-iz_{R} t} \qquad \forall t \geq 0 & (67) \\ \langle \phi^{-} | e^{-iH_{-}^{\times} t} \Psi^{G} \rangle &= \langle \phi^{-} | \Psi^{G} \rangle \, e^{-iz_{R}^{*} t} \qquad \forall t \leq 0 & (68) \end{aligned}$$

with $z_R = \omega_R - i\frac{\Gamma}{2}$. This means that the equations (62) and (63) are not mathematically correct: what decays as t goes to infinity is not $\Psi^{D}(t)$ but $\langle \phi^{+} | \Psi^{D}(t) \rangle$. As a consequence, if we want to conceive $\Psi^{D}(t)$ as a generalized state, we only can strictly say that the expectation value of the observable $A = | \, \phi^{+} \rangle \langle \phi^{+} |$ in the state $\Psi^{D}(t)$ decays exponentially:

$$\langle A \rangle_{\Psi^{D}(t)} = |\langle \phi^{+} | \Psi^{D}(t) \rangle|^{2} \; = |\langle \phi^{+} | \Psi^{D} \rangle|^{2} \, e^{-\Gamma t} \tag{69}$$

In other words, whereas $\langle A \rangle_{\Psi^{D}(t)}$ exponentially decays as t goes to infinity:

$$\lim_{t \to \infty} \langle A \rangle_{\Psi^{D}(t)} = 0 \tag{70}$$

the generalized state $\Psi^D(t)$ has only a *weak limit*:[12]

$$w - \lim_{t\to\infty} \Psi^D(t) = 0 \tag{71}$$

This weak limit means that the generalized state $\Psi^D(t)$ decays in a coarse-grained sense *from an observational point of view*, that is from the perspective given by the observable $A = |\,\phi^+\rangle\langle\phi^+|$, for any $\phi^+ \in \mathbf{\Phi}_+$.[13] The basis for this claim is that $\langle A\rangle_{\Psi^D}$ results from a projection of the vector $\Psi^D(t)$ onto a subspace defined by the operator A. But, as we have seen in Section 2, a projection amounts to a coarse-graining that reduces the number of components of the vector representing the state. Therefore, $\langle A\rangle_{\Psi^D(t)}$ is the result of a coarse-graining introduced by the observable A onto the evolving state $\Psi^D(t)$. In fact, since $A^2 = A$, the observable A can be conceived as a projector Π:

$$A = |\,\phi^+\rangle\langle\phi^+| = \Pi \tag{72}$$

Then, we can define a coarse-grained state Ψ^D_{cg} as:

$$\Psi^D_{cg} = \Pi\,\Psi^D = |\,\phi^+\rangle\langle\phi^+|\Psi^D\rangle \tag{73}$$

With this definition:

$$\begin{aligned} |\Psi^D_{cg}\rangle\langle\Psi^D_{cg}| &= |\,\phi^+\rangle\langle\phi^+|\Psi^D\rangle\langle\Psi^D|\phi^+\rangle\langle\phi^+| = \\ &= |\langle\phi^+|\Psi^D\rangle|^2\,|\phi^+\rangle\langle\phi^+| = \langle A\rangle_{\Psi^D}\,|\phi^+\rangle\langle\phi^+| \end{aligned} \tag{74}$$

In other words, the expectation value of the observable $A = |\,\phi^+\rangle\langle\phi^+|$ in the state Ψ^D can be viewed as the result of the action of the projector $\Pi = A$ on the vector Ψ^D. On this basis we can understand why $\langle A\rangle_{\Psi^D(t)}$ is a coarse-grained magnitude: strictly speaking, this coarse-grained magnitude is what

[12] Let us consider a vector space V endowed with an inner product $(.|.)$. We say that the sequence $\{a_n\}, a_n \in V$ has a weak limit $a \in V$, that is:

$$w - \lim_{n\to\infty} a_n = a$$

iff

$$\lim_{n\to\infty} (a_n|b) = (a|b)$$

for any vector $b \in B$, where B is a subspace of V.

[13] An analogous argument directed to show the coarse-grained nature of decoherence can be found in Castagnino and Lombardi 2004a.

decays for $t \to \infty$, and not the generalized state $\Psi^D(t)$ as Bohm's equations (62) and (63) seem to suggest.

The conclusion of this argument is that, in spite of the efforts directed to extracting irreversibility from quantum mechanics, there is no magic in physics or in mathematics. If the evolutions of the quantum states of a closed system are governed by unitary evolution operators, they have no generalized attractors. Non-unitary evolutions can only be obtained by going to a level of description different than the descriptive level with unitary evolutions. In the case of the evolution described by Gamow vectors in RHS, the coarse-grained magnitude that decays as t goes to infinity is the expectation value of the observable $A = |\,\phi^+\rangle\langle\phi^+|$ in the generalized state Ψ^D, for any $\phi^+ \in \mathbf{\Phi}_+$, and there is no quantum law that prevents it from having this kind of behavior.

We do not want to finish this section without mentioning another argument which seems to be in conflict with the conception of a Gamow vector as representing a truly quantum state: there is no unique, universally accepted way of defining averages on Gamow vectors, in particular, mean values of the energy (*cf.* Civitarese *et al.* 1999, Civitarese and Gadella 2004). Furthermore, it is possible to construct an algebraic formulation for quantum states and observables that includes singular states (*cf.* Antoniou *et al.* 1997; Antoniou and Suchanecki 1997) and functionals having the properties of the Gamow vectors (*cf.* Castagnino *et al.* 2001); these functionals have well defined averages of the Hamiltonian, but all of them vanish. The consequence of these results is that either we do not have a clear definition of energy on Gamow vectors or this energy always vanish identically.

7 Conclusions

The questions related with time are one of the areas of inquiry where physics and philosophy are more strongly intertwined. In particular, the phenomenon of irreversibility has engaged the attention of many authors since the birth of thermodynamics. This is precisely the case of the TAQM-school with its proposal of a new quantum mechanics based on the use of RHSs: this formalism would turn quantum mechanics into a non time-reversal invariant (time-asymmetric) theory which permits irreversible behavior to be modeled in a precise way. In this paper we have assessed this claim on the basis of a detailed analysis of the formalism proposed by the school.

As we have seen, although in principle nothing prevents a time-reversal invariant theory from describing irreversible evolutions, in the case of dynamical equations with unitary solutions, time-reversal invariance and reversibility seem to go hand-in-hand. Nevertheless, even in this case both properties are different to the extent that they are related with distinct features of the formalism: whereas time-reversal invariance implies the *group structure* of the evolution operators, reversibility is a consequence of the *unitary* character of such operators. Therefore, even if the time-reversal invariance of a theory is broken down by means of semigroup evolution laws, this fact does not affect the reversible character of the evolutions if they are still described by unitary operators. The only way to extract irreversibility from unitary processes is by means of some mathematical procedure that leads to a level of description different from the original dynamical level, where non-unitary evolutions can be obtained.

These general considerations are directly applicable to TAQM. In this case:

- the non time-reversal invariance of the theory is due to the semigroup structure of the evolution laws which, in turn, is a consequence of the use of a particular realization of the RHS based on Hardy functions.

- the irreversibility of the evolutions is due to the existence of Gamow vectors in RHS, which depends on the use, not of Hardy functions, but of functions having analytical continuations in the half-planes $\operatorname{Im} z > 0$ and $\operatorname{Im} z < 0$ of the complex plane.

Since irreversible evolutions given by Gamow vectors can be obtained in time-reversal invariant versions of the theory where evolutions are described by groups, irreversibility does not depend on the semigroup evolution laws as the literature on the subject usually seems to suggest. On the other hand, since Gamow vectors in RHS are functionals, the decaying Gamow vector does not strictly decay in the infinite time limit, as it could be wrongly interpreted from TAQM school's presentation; the decaying Gamow vector only decays in a weak sense. In spite of Prigogine's aversion to coarse-graining, what strictly decays as time goes to infinity is a coarse-grained magnitude which involves a generalized projection of the decaying Gamow vector.

In spite of these considerations, it is worth stressing that the arguments presented here only concern the interpretation of the conceptual conclusions that can be drawn from the physical work of the TAQM-school. As a consequence, they do not diminish the scientific value of the school's contributions: time-asymmetric quantum mechanics is a powerful theory for the description of intrinsic irreversibility. In this sense, many irreversible phenomena have been adequately modeled with the new formalism; for instance, it has been successfully applied to nuclear physics and to the Lee-Oehme-Yang theory for the neutral Kaon. In this paper, our aim has been to contribute to the conceptual understanding and the interpretation of a theory whose fruitful scientific results have been widely received by the physical community.

8 References

- Albert, D. (2000), *Time and Chance*, Harvard University Press, Cambridge.
- Antoniou, I., Gadella, M. and Pronko, G. (1998), "Gamow vectors for degenerate scattering resonances", *Journal of Mathematical Physics*, **39**, 2459-2475.
- Antoniou, I., Gadella, M., Karpov, E., Prigogine, I. and Pronko, G. (2001), "Gamov algebras", *Chaos, Solitons and Fractals*, **12**, 2757-2775.
- Antoniou, I. and Prigogine, I. (1993), "Intrinsic irreversibility and integrability of dynamics", *Physica A*, **192**, 443-464.
- Antoniou, I. and Suchanecki, Z. (1997), "Quantum systems with diagonal singularity", *Advances in Chemical Physics*, **99**, 299-332.
- Antoniou, I., Suchanecki, Z., Laura, R. and Tasaki, S. (1997), "Intrinsic irreversibility of quantum systems with diagonal singularity", *Physica A*, **241**, 737-772.
- Arntzenius, F. (2004), "Time reversal operations, representations of the Lorentz group, and the direction of time", *Studies in History and Philosophy of Modern Physics*, **35**, 31-44.
- Bishop, R. (2003), "A tale of two arrows", *International Journal of Theoretical Physics*, **42**, 2371-2377.

- Bishop, R. (2004a), "Arrow of time in rigged Hilbert space quantum mechanics", *International Journal of Theoretical Physics*, **43**, 1675-1687.
- Bishop, R. (2004b), "Nonequilibrium statistical mechanics Brussels-Austin style", *Studies in History and Philosophy of Modern Physics*, **35**, 1-30.
- Bohm, A. (1981), "Resonance poles and Gamow vectors in the rigged Hilbert space formulation of quantum mechanics", *Journal of Mathematical Physics*, **22**, 2813-2823.
- Bohm, A., Antoniou, I. and Kielanowski, P. (1994), "A quantum mechanical arrow of time and the semigroup time evolution of Gamow vectors", *Journal of Mathematical Physics*, **36**, 2593-2604.
- Bohm, A. and Gadella, M. (1989), *Dirac Kets, Gamow Vectors and Gelfand Triplets*, Springer Lecture Notes in Physics, Vol **348**, Springer Verlag, New York and Berlin.
- Bohm, A., Gadella, M. and Mithaiwala, M. (2003a), "Time asymmetric quantum theory: foundations and applications", in *The Physics of Communication, Proceedings of the XXII Solvay Conference on Physics*, World Scientific, Singapore, 117-138.
- Bohm, A. and Harshman, N. L. (1998), "Quantum theory in the rigged Hilbert space. Irreversibility from causality", in A. Bohm, H. D. Doebner and P. Kielanowski (eds.), *Irreversibility and Causality: Semigroups and Rigged Hilbert Spaces. Lectures Notes in Physics 504*, Springer-Verlag, Berlin.
- Bohm, A., Loewe, M. and Van de Ven, B. (2003b), "Time asymmetic quantum theory. modifying and axiom of quantum physics", *Fortschritte der Physik,* **51**, 551-568.
- Bohm, A., Maxson, S., Loewe, M. and Gadella, M. (1997), "Quantum mechanical irreversibility", *Physica A*, **236**, 485-549.
- Bohm, A. and Scurek, R. (2000), "The phenomenological preparation-registration arrow of time and its semigroup representation in the RHS

quantum theory", in H. D. Doebner, S. T. Ali, M. Keyl and R. F. Werner (eds.), *Trends in Quantum Mechanics*, World Scientific, Singapore-London.

- Bohm, A. and Wickramasekara, S. (2002), "A new topology for an axiom of quantum mechanics", *Irreversible Quantum Dynamics, Lectures Notes in Physics 622*, Springer-Verlag, Berlin.
- Castagnino, M., Gadella, M., Id Betan, R. and Laura, R. (2001), "G-amow functionals on operator algebras", *Journal of Physics A*, **34**, 10067-10084.
- Castagnino, M., Id Betan, R., Laura, R. and Liotta, R. (2002), "Quantum decay processes and Gamov states", *Journal of Physics A*, **35**, 6055-6074.
- Castagnino, M. and Laura, R. (1997), "Minimal irreversible quantum mechanics: Pure-state formalism", *Physical Review A*, **56**, 108-119.
- Castagnino, M. and Lombardi, O. (2004a), "Self-induced decoherence: a new approach", *Studies in History and Philosophy of Modern Physics*, **35**, 73-107.
- Castagnino, M. and Lombardi, O. (2004b), "The generic nature of the global and non-entropic arrow of time and the double role of the energy-momentum tensor", *Journal of Physics A*, **37**, 4445-4463.
- Castagnino, M. and Lombardi, O. (2005), "The global non-entropic arrow of time: from global geometrical asymmetry to local energy flow", *Synthese*, forthcoming.
- Castagnino, M., Lombardi, O. and Lara, L. (2003), "The global arrow of time as a geometrical property of the universe", *Foundations of Physics*, **33**, 877-912.
- Civitarese, O. and Gadella, M. (2004), "Physical and mathematical aspects of Gamow states", *Physics Reports*, **396**, 41-113.
- Civitarese, O., Gadella, M. and Id Betan, R. (1999), "On the mean value of the energy for resonant states", *Nuclear Physics A*, **660**, 255-266.

- Earman, J. (2002), "What time reversal invariance is and why it matters", *International Studies in the Philosophy of Science*, **16**, 245-264.

- Gadella, M. (1983), "A description of virtual scattering states in the rigged Hilbert space formulation of quantum mechanics", *Journal of Mathematical Physics*, **24**, 2142-2145.

- Gadella, M. (1984), "On the RHS description of resonances and virtual states", *Journal of Mathematical Physics*, **25**, 2481-2485.

- Gadella, M. (1997), "Derivation of Gamow vectors for resonances in cut-off potentials", *Letters in Mathematical Physics*, **41**, 279-290.

- Gel'fand, I. and Vilenkin, N. (1964), *Generalized Functions*, *Vol 3: Theory of Differential Equations*, Academic Press, New York.

- Grossmann, A. (1964), "Nested Hilbert spaces in quantum mechanics I", *Journal of Mathematical Physics*, **5**, 1025-1037.

- Koopman, B. (1931), "Hamiltonian systems and transformations in Hilbert space", *Proceedings of the National Academy of Sciences*, **17**, 315-318.

- Koosis, P. (1980), *Introduction to H^p Spaces*, Cambridge University Press, Cambridge.

- Lindblad, G. and Nagel, B. (1970), "Continuous bases for unitary irreducibe representations of SU(1,1)", *Annales de l'Institut Henri Poincaré A*, **13**, 27-56.

- Ludwig, G. (1983-1985), *Foundations of Quantum Mechanics*, Vol.I and Vol.II, Springer, Berlin.

- Mackey, M. C. (1989), "The dynamic origin of increasing entropy", *Review of Modern Physics*, **61**, 981-1015.

- Paley, R. and Wiener, N. (1934), *Fourier Transforms in the Complex Domain*, American Mathematical Society, New York.

- Prigogine, I. and George, C. (1983), "The second law as a selection principle: the microscopic theory of dissipative processes in quantum systems", *Proceedings of the National Academy of the Sciencies of the United States of America*, **80**, 4590-4594.

- Prigogine, I. and Stengers, I. (1979), *La Nouvelle Alliance*, Gallimard, Paris.

- Reed, M. and Simon, B. (1972) *Functional Analysis*, Academic Press, New York.

- Schäffer, H. H. (1970), *Topological Vector Spaces*, Springer, New York-Heidelberg-Berlin.

- Sklar, L. (1974), *Space, Time and Spacetime*, University of California Press, Berkeley.

- Tabor, M. (1989), *Chaos and Integrability in Nonlinear Dynamics*, John Wiley and Sons, New York.

- Titchmarsh, E. C. (1937), *Introduction to the Theory of Fourier Integrals*, Clarendon Press, Oxford.

- van Winter, C. (1974), "Complex dynamical variables for multiparticle systems with analytic interactions. I", *Journal of Mathematical Analysis and Applications*, **47**, 633-670.