Reconstruction of quantum theory

Alexei Grinbaum

LPHS – Archives Henri Poincaré (CNRS UMR 7117),
23 bd Albert Ier, 54015 Nancy, France
Email alexei.grinbaum@polytechnique.edu

Abstract

What belongs to quantum theory is no more than what is needed for its derivation. Keeping to this maxim, we record a paradigmatic shift in the foundations of quantum mechanics, where the focus has recently from interpreting to reconstructing quantum theory. Several historic and contemporary reconstructions are analyzed, including the ones due to Hardy, Rovelli, and Clifton, Bub and Halvorson. We conclude by discussing the importance of a novel concept of intentionally incomplete reconstruction.

1 What is wrong with interpreting quantum mechanics?

Ever since the first days of quantum mechanics physicists as well as philosophers tried to interpret it, understanding this task as a problem of giving to the new physical theory a clear meaning. One of the principal reasons why one has always felt the need for interpretations has to do with the puzzling aspect of the formalism of quantum mechanics, usually referred to as the measurement problem. Reversible unitary evolution of the wave function, according to standard quantum mechanics, at the moment of measurement is replaced by an irreversible transformation known as wavefunction collapse. First and foremost, interpretations of quantum mechanics aimed at making sense of this surprising change in the theory's dynamics, sometimes taking the collapse at face value and claiming its fundamental irreducible role, or sometimes going to another extreme and denying the collapse altogether. However, looking globally, the enterprize of interpreting quantum mechanics failed: today we still have no consensus on what *the* meaning of quantum theory is. None of the proposed answers has won overall acceptance. Perhaps the most remarkable manifestation of the failure to interpret quantum mechanics is the attitude taught to most young physicists in lecture rooms and research laboratories in the last half century, “Shut up and calculate!” [37]

Why did attempts at a univocal interpretation fail? Many answers are possible, and among them we favor two, both showing that there is an intrinsic deficiency in the idea of interpreting a physical theory with the help of philosophical instruments only.

The first answer is that to a physical theory one would naturally like to give a *physical* meaning in the Greek sense of $\varphi\acute{\upsilon}\sigma\iota\varsigma$, i.e. we—as part of the physicists' audience—expect to be told a true story about nature. This is because we casually tend to apply physical theory to the phenomenal world to learn something about the latter, and not the world to physical theory in order to invent a meaning of the theory. Physical theory is above all a tool for predicting the yet unobserved phenomena; so employing existing knowledge and experience of the world to interpret physics runs counter to its basic function as a scientific theory. However, notwithstanding such an against-the-grain direction in which a philosophical interpretation operates, the former does not necessarily lead to formal contradiction that would invalidate the interpretation program logically; more modestly but perhaps no less irritatingly, at the end one is often left with a feeling of being excluded from mainstream research. Further, as the physics of today is inseparable from mathematics, a meaning cannot be physical and thus satisfactory if it is merely heaped over and above the mathematical formalism of quantum mechanics, instead of coming all the way along with the formalism as it rises in a derivation of the theory.

The second answer is that we live in a situation where objective truth has been appropriated by science, and to pass public ratification every increase in knowledge must confront experimental setups. In this world an interpretation can only be considered satisfactory when it becomes an integral part of science. This is not unprecedented in the history of ideas: indeed, many philosophical questions with the advent of empirical science ceased to be perceived as philosophical and are now treated as scientific. Even such a pronounced critic of the idea that physics can have implications for philosophy as Steven Weinberg admits that, exceptionally, physics and philosophy can be connected in that "discoveries in physics sometimes reveal that topics that had been thought to be proper subjects for philosophical argument actually belong in the province of ordinary science" [57]. To be able to convince Weinberg and many of those scientists who remain sceptical about the philosophical debate over interpretations of quantum mechanics, the problem of interpreting quantum mechanics must be moved into the area of science; only then will the puzzling discord disappear.

2 Reconstruction of physical theory

2.1 Schema

We call *reconstruction* the following schema adopted to the needs of quantum theory and different from the notion of rational reconstruction introduced by Carnap [9]. Theorems and major results of physical theory are formally derived from simpler mathematical assumptions. These assumptions or axioms, in turn, appear as a representation in the formal language, of a set of physical principles. Thus, reconstruction consists of three stages: first give a set of physical principles, then formulate their mathematical representation, and finally derive from here the formalism of the theory.

Contrary to the case of interpretation, the three-stage structure of reconstruction permits the latter to acquire supplementary persuasive power which arises from the use of mathematical derivation. Established as valid mathematical results, theorems and equations of the theory become unquestionable and free of suspicion. 'Why is it so?'—'Because we derived it.' The question of meaning, previously asked with regard to the formalism, is removed and now bears, if at all, only on the selection of the principles. No room for mystery remains in what concerns the meaning of the theory's mathematical apparatus. As one of the consequences, in the reconstruction program the measurement problem loses the central role it has occupied for the success of an interpretation of quantum mechanics. In reconstruction, one makes sense of all of the formalism on the basis of first principles, and whatever mathematical element is contained in the formalism that is used in a particular reconstruction, this element becomes meaningful thanks to the first principles.

Explanatory power of the reconstruction is a power of explanation of where the structure of the theory comes from; not necessarily a power of explanation by the theory, of the real world. Traditionally, interpretations focused on the latter task and gave less attention to the former one. Reconstruction shifts this focus area: its added value for better understanding quantum theory originates in the new insights into the structure of the theory, made possible thanks to the use of mathematical derivation.

2.2 Selection of the first principles

Anyone who wishes to attempt a reconstruction of physical theory must formulate the foundational principles which he or she believes plausible and translate them into mathematical axioms. Then the rest of the theory will be constructed 'mechanically,' by means of a formal derivation. The choice of axioms must be the only allowed freedom in the whole construction. It is commonplace to say that it is not easy to exhibit an axiomatic system that would stand to such requirements, especially in the case of quantum theory.

First, where do candidate axioms for quantum theory come from and how does one judge which statements that can plausibly be taken as axioms? Prior to pronouncing such a judgment, one must develop an intuition of what is plausible about quantum theory and what is not. This can only be achieved by *practicing* the theory, i.e. by taking its prescriptions at face value, applying them to systems under consideration in particular tasks, and obtaining results. In short, one needs to acquire a real 'know-how' above and beyond the theoretical knowledge that quantum mechanics could solve such and such problems. Researcher's intuition develops from experience; it cannot arise from abstract knowledge 'in principle.'

However, taking the prescriptions of quantum theory at face value, applying them and obtaining results will not yet make things *clear* about quantum mechanics. Indeed, one can possess the knowledge about how to apply a certain tool without caring about the structure of the tool nor its meaning. The quantum mechanical know-how serves purely as such a tool for developing one's intuition about which candidate idea is a plausible foundational principle and which other candidate idea will not pass the test. Candidate foundational principles need not even be theorems of the already existing quantum mechanics: one's judgment may be such that a new statement—false or only conditionally true in quantum theory—will be taken as axiom in reconstruction of a new theory. Examples of such principles will be investigated in Section 3.5.

Second, what shall we require from first principles? They must be simple *physical statements*, i.e. assertions whose meaning is immediately, easily accessible to a scientist's understanding. They must also be such as to permit a clear and unambiguous translation of themselves into mathematically formulated axioms. A derivation of quantum theory will then rely on these axioms.

2.3 Status of the first principles

A reconstruction program includes a *derivation* of quantum theory, but in the previous section one was told to *apply* and *use* it in order to motivate the derivation. Is there a vicious circle here? We submit that there is none, and this thanks to the status of the first principles. Namely, they should not necessarily be viewed as ultimate truths about nature. Independently of one's ontological commitments, the first principles have only a minimal epistemic status of being postulated for the purpose of reconstructing the theory in question. As with the 19th-century mathematics, in theoretical physics the axiomatic method is to be separated from the attitude which the Greeks had toward axioms: that they represent the truth about reality. Much of the progress of mathematics is due to understanding that an axiom can no longer be considered an ultimate truth, but merely a basic structural element, i.e. an assumption that lies at the foundation of a certain theoretical structure. In mathematics, after departing from the Greek concept of axiom, "not only geometry, but many other, even very abstract, theories have been axiomatized, and the axiomatic method has become a powerful tool for mathematical research, as well as a means of organizing the immense field of mathematical knowledge which thereby can be made more surveyable" [26]. A similar attitude is to be taken with respect to axioms used for the formal derivation of a physical theory. To give a concise formula, a methodological prescription that gives the minimal status of the first principles in a reconstruction program, runs as follows:

- If the theory itself does not tell you that the states of the system, or any other variables, are ontic, then do not take them to be ontic.

To explain the above prescription, return first to the idea that, in developing an intuition with respect to the plausibility of the foundational principles used to derive a theory, one takes this theory as a given and applies it practically, so as to acquire a know-how that would justify the choice of principles. Now, when one is working with several physical theories, ideas that have previously been used as foundational in theory I, may turn out to be derivative (i.e., theorems) in theory II. Examples include the case of thermodynamics and statistical physics or the relation between macroscale hydrodynamics and the low-level molecular theory of liquids. Such considerations show the limits of philosophical assumptions that one can make about the status of the first principles used in the reconstruction of a given physical theory. Indeed, generically nothing can be said about their ontological content or the ontic commitments that arise from the principles. It is more economical and would amount to a certain *epistemological modesty* to treat the foundational principles as axioms *hic et nunc*, i.e. in a given theoretical description. Epistemological modesty requires that one brackets his or her personal motivation for the choice of first principles and reconstructs the theory based on the principles themselves. Reconstruction becomes meaningful solely thanks to the sense of the first principles on which it is based.

Reconstruction of a physical theory has its main advantage compared to philosophical interpretation of the theory in the fact that it moves a number of questions, previously thought of as philosophical, to the realm of science, and this in virtue of the mathematical derivation which the reconstruction program operates. However, philosophical problems do not altogether disappear; they still apply to the first principles and take the form of a problem of their *justification*. Evidently, it is a minimal logical condition that such a justification should not be seen as a mathematical deduction of the principles from the theory in whose very foundations they lie. Once one has obtained a full formalism of the theory in an epistemologically modest reconstruction, it is then possible to ask the reconstructed theory itself if it allows a realist interpretation of the first principles from which it has been derived or, perhaps, it imposes constraints on possible ontological commitments. And while in general the status of the first principles as ultimate truths about reality is not a necessity, certain reconstruction programs are such that this status can be safely, or almost, attributed to the principles within a particular reconstruction in question. The task of justification is therefore external to the reconstruction program and must be executed by one with a different set of presuppositions, i.e. by taking the theory as a given and motivating from there why the principles that were involved in the reconstruction are simple, physical, and plausible. Therefore, philosophy is not fully chased out of physics. On the contrary, by demarcating the frontier between what can be treated as a scientific question and what belongs to metatheory, one contributes to a better understanding of the structure of the theory and of those of its foundational postulates which require a metatheoretic interpretation and justification.

It is often claimed that applying to theoretical physics the same methodology as in axiomatizations in mathematics leads to a problem, exposed by Einstein. While supporting the axiomatic move in mathematics in that it "dispels the obscurity which formerly surrounded the principles," Einstein argues that if one wants to apply a similar move in physics, then one has to face the difficulty of connecting "conceptual schemata" with "real objects" [13]. Applied to an epistemologically modest reconstruction, this problem is no more than apparent, if the status of the first principles is properly freed from ontological commitments. The latter do have a bearing indeed, but only on justifying the choice of postulates. Starting from a particular set of principles (stage 1) represented formally (stage 2), mathematical derivation (stage 3) proceeds in exactly the same way as in mathematics. Reconstruction understood as stages 1-3 is therefore analogous to an axiomatization in mathematics. However, unlike mathematical axiomatization, the reconstruction program also invokes the problem of justification of the choice of first principles. But when the first principles are formulated in an epistemologically modest way, Einstein's "conceptual schemata," or structural elements of physical theory, become the only building blocks of the latter. Unambiguous derivation of the theory's formalism is detached from the question of reality of the world that the theory describes, with respect to which one is free to take different viewpoints. For quantum theory, this detachment amounts to operating a reconstruction of quantum theory from a set of first principles devoid of the necessity of being justified on the ontological grounds.

3 Examples of reconstruction

3.1 Early examples of reconstruction

In the last decade reconstruction has become a major trend in the foundations of quantum mechanics. Before describing this recent work, let us first look further back in the history of quantum mechanics: there too axiomatic derivations occupy an eminent place. The first paper where quantum mechanics was treated axiomatically appeared shortly after the creation of quantum mechanics itself: in 1927 Hilbert, von Neumann and Nordheim stated their view of quantum mechanics as one in which "... [the theory's] analytical apparatus, and the arithmetic quantities occurring in it, receives *on the basis of the physical postulates* a physical interpretation. Here, the aim is to formulate the physical requirements so completely that the analytical apparatus is just uniquely determined. Thus the route is of axiomatization" [27, our emphasis]. It is on this route of axiomatization that von Neumann in collaboration with Birkhoff was led to study the logic of quantum mechanics [6]. Following their work, many axiomatic systems were proposed, e.g. by Zieler [58], Varadarajan [54, 55], Piron [39, 40], Kochen and Specker [31], Guenin [19], Gunson [20], Jauch [29], Pool

[44, 45], Plymen [43], Marlow [36], Beltrametti and Casinelli [4], Holland [28], or Ludwig [33]. Another branch of axiomatic quantum theory, the algebraic approach was first conceived by Jordan, von Neumann and Wigner [30] and later developed by Segal [48, 49], Haag and Kastler [21], Plymen [42], Emch [14] and others; for a recent review, see [8].

However, a vast majority of these axiomatic developments do not fall under our notion of reconstruction, as they were based on highly abstract mathematical assumptions and not, as we require, on simple physical principles. Consider for instance the exemplary work by Mackey [34, 35]. He develops quantum mechanics as follows. Take a set $\mathfrak{B}$ of all Borel subsets of the real line and suppose we are given two abstract sets $\mathcal{O}$ (a to-be space of observables) and $\mathcal{S}$ (a to-be space of states) and a (to-be probability) function p which assigns a real number $0 \leq p(x, f, M) \leq 1$ to each triple x, f, M, where x is in $\mathcal{O}$, f is in $\mathcal{S}$, and M is in $\mathfrak{B}$. Assume certain properties of p listed in axioms M1-M9:

M1 Function p is a probability measure. Mathematically, we have $p(x, f, \emptyset) = 0$, $p(x, f, \mathbb{R}) = 1$, and $p(x, f, M_1 \cup M_2 \cup M_3 \ldots) = \sum_{n=1}^{\infty} p(x, f, M_n)$ whenever the M_n are Borel sets that are disjoint in pairs.

M2 Two states, in order to be different, must assign different probability distributions to at least one observable; and two observables, in order to be different, must have different probability distributions in at least one state. Mathematically, if $p(x, f, M) = p(x', f, M)$ for all f in $\mathcal{S}$ and all M in $\mathfrak{B}$ then $x = x'$; and if $p(x, f, M) = p(x, f', M)$ for all x in $\mathcal{O}$ and all M in $\mathfrak{B}$ then $f = f'$.

M3 Let x be any member of $\mathcal{O}$ and let u be any real bounded Borel function on the real line. Then there exists y in $\mathcal{O}$ such that $p(y, f, M) = p(x, f, u^{-1}(M))$ for all f in $\mathcal{S}$ and all M in $\mathfrak{B}$.

M4 If f_1, f_2,... are members of $\mathcal{S}$ and $\lambda_1 + \lambda_2 + \ldots = 1$ where $0 \leq \lambda_n \leq 1$, then there exists f in $\mathcal{S}$ such that $p(x, f, M) = \sum_{n=1}^{\infty} \lambda_n p(x, f_n, M)$ for all x in $\mathcal{O}$ and M in $\mathfrak{B}$.

M5 Call *question* an observable e in $\mathcal{O}$ such that $p(e, f, \{0, 1\}) = 1$ for all f in $\mathcal{S}$. Questions e and e' are disjoint if $e \leq 1 - e'$. Then a question $\sum_{n=1}^{\infty} e_n$ exists for any sequence (e_n) of questions such that e_m and e_n are disjoint whenever $n \neq m$.

M6 If E is any compact, question-valued measure then there exists an observable x in $\mathcal{O}$ such that $\chi_M(E) = E(M)$ for all M in $\mathfrak{B}$, where χ_M is a characteristic function of M.

M7 The partially ordered set of all questions in quantum mechanics is isomorphic to the partially ordered set of all closed subspaces of a separable, infinite-dimensional Hilbert space.

M8 If e is any question different from 0 then there exists a state f in $\mathcal{S}$ such that $m_f(e) = 1$.

M9 For each sequence (f_n) of members of $\mathcal{S}$ and each sequence (λ_n) of non-negative real numbers whose sum is 1, one-parameter time evolution group $V_t : \mathcal{S} \mapsto \mathcal{S}$ acts as follows: $V_t \left(\sum_{n=1}^{\infty} \lambda_n f_n\right) = \sum_{n=1}^{\infty} \lambda_n V_t(f_n)$ for all $t \geq 0$; and for all x in $\mathcal{O}$, f in $\mathcal{S}$, and M in $\mathfrak{B}$, $t \rightarrow p(x, V_t(f), M)$ is continuous.

In Mackey's nine axioms all essential features of the quantum formalism are directly postulated in their mathematical form: the Hilbert space structure in M5-M8, the state space and probabilistic interpretation in M1-M4, and the time evolution in M9. The list of axioms is long and their meaning is far from transparent; indeed, at no point is one given an intuition as to where these mathematical definitions come from or how one justifies them on physical rather than formal grounds. In fact, Mackey's concern in the early 1950s was with a precise mathematical axiomatization of quantum mechanics rather than with the question of what quantum mechanics tells us about the world or with reconstructing its formalism from the set of such fundamental ideas. Thus, the first stage of the reconstruction schema, at which one formulates physical principles, is absent from Mackey's work, and instead one starts directly at the second stage where the first principles appear in mathematical form.

Mackey's axioms M5-M8 were consequently reformulated in the language of quantum logic, thereby rephrasing the assumptions that underlie the Hilbert space structure. This has been the case, most prominently, in [29, 39, 40] and also in an important state-of-the-art book [4]. Quantum logical assumptions are simple enough to be accessible for direct comprehension, in contrast to Mackey's mathematically formulated axioms, but they tend to be linguistic rather than physical. This means that one typically argues that it makes no sense to *speak* about certain concepts unless some suitable 'trivial' properties of these concepts had been postulated, e.g., the notion of proposition is only meaningful if, as in Ref. [12], negation or partial order, or, as in Ref. [4], implication, are defined. Although we fully acknowledge that linguistic *a priori* arguments can be interesting and powerful, we however separate them from the reconstruction program as introduced above: in the latter, first principles from which the theory is derived should have a *physical* meaning, i.e. tell us something directly and intuitively apprehensible about the world. Such principles, ideally, should be independent of a particular formalism in which one then derives quantum theory, and therefore should not rely on the language of quantum logic as just one among many such formalisms.

3.2 Hardy's reconstruction

It is startling how Mackey's and similar axiomatic sets for quantum mechanics differ from systems of first principles proposed by several contemporary authors. Although some of these latter ones remain very much in the spirit of earlier proposals of sets of abstract mathematical postulates (e.g., [41]), even in such cases the author typically feels the need to give a non-technical, physical motivation for the choice of axioms. Still further on the way to foundational physical principles rather than purely mathematical axioms, one finds an interesting example of reconstruction coming from Hardy's instrumentalist derivation of quantum theory [23]. Unlike Mackey who starts with two large abstract sets and an abstract real-valued function, Hardy's "five reasonable axioms" set up a link between two initially introduced natural numbers, K and N. K is the number of degrees of freedom of the system and is defined as the minimum number of probability measurements needed to determine the state. Dimension N is defined as the maximum number of states that can be reliably distinguished from one another in a single measurement. The axioms then are:

H1 *Probabilities.* In the limit as n becomes infinite, relative frequencies (measured by taking the proportion of times a particular outcome is observed) tend to the same value for any case where a given measurement is performed on an ensemble of n systems prepared by some given preparation.

H2 *Simplicity.* K is determined by a function of N where $N = 1, 2, \ldots$ and where, for each given N, K takes the minimum value consistent with the axioms.

H3 *Subspaces.* A system whose state is constrained to belong to an M dimensional subspace behaves like a system of dimension M.

H4 *Composite systems.* A composite system consisting of subsystems A and B satisfies $N = N_A N_B$ and $K = K_A K_B$.

H5 *Continuity.* There exists a continuous reversible transformation on a system between any two pure states of that system.

Hardy's list of axioms is considerably shorter and simpler than Mackey's, and although four of H1-H5 still use mathematical language in their formulation, the meaning of the axioms in Hardy's instrumentalist setting can be grasped easier than the meaning of Mackey's M1-M9. In fact, this meaning is already suggested by the names given to the axioms by Hardy. One can rephrase H1-H4 into *physical principles* from which one derives the formalism of the theory. This would provide for the missing first stage of the reconstruction schema and thus amount to a complete example of a reconstruction. Thus, at the first stage of reconstruction, the following physical principles are postulated that rephrase Hardy's axioms; they are simple but non-trivial: for H1, assume that probability can be introduced as *relative frequency* and it is a well-defined concept obeying the laws of probability theory; for H2, assume that the number of parameters needed to characterize a state is linked in a minimal way to the number of states that can be distinguished in one measurement, i.e., information carrying capacity of the system; for H3, that systems that have the same information carrying capacity have the same properties; for H4, assume multiplicability of the information carrying capacity. At the second stage of the reconstruction schema, one formulates these principles mathematically (in Hardy's way); and, at the third stage, one uses Hardy's theorems to derive the full-blown formalism of quantum mechanics.

The particular instrumental philosophy does not play a crucial role in the derivation: Hardy himself acknowledges that his axioms can be adopted by a realist as well as a hidden variable theorist or a partisan of collapse interpretations. Thus, choice of the underlying philosophy is not critical to the success of the derivation, and Hardy's reconstruction advances our understanding of quantum theory irrespectively of the justification which one may have for the axioms. What matters are the simple physical principles formulated as axioms H1-H4. This is exactly what one would expect given the status of the first principles. We shall however see below an opposite example, in which a justification used for the fundamental principles will limit the area where mathematical derivation is both applicable and meaningful.

Still, in Hardy's case it is not so clear whether axiom H5 has a *physical* meaning. Because it is this axiom that makes the theory quantum rather than classical, the reconstruction program cannot be said to be completely implemented. To illustrate this point, we distinguish two types of continuity assumptions that are made in axiomatic derivations of quantum theory. Continuity assumptions of type 1 select the correct type of numeric field which is used in the construction of the Hilbert space of the theory; namely, of the field $\mathbb{C}$ of complex numbers. Solèr's theorem [51] or Zieler's axioms [58] are examples of type 1 continuity assumptions. Hardy's case is different and is an example of a continuity assumption of type 2, which is ultimately responsible for the appearance of the superposition principle. Examples of other type 2 assumptions include Gleason's non-contextuality [16], Brukner's and Zeilinger's homogeneity of parameter space [7], Landsman's two-sphere property [32], or Holland's axioms C and D [28] which bear a particular resemblance to Hardy's H5:

(C) Superposition principle for pure states:

- **1.** Given two different pure states (atoms) a and b, there is at least one other pure state c, $c \neq a$ and $c \neq b$ that is a superposition of a and b.
- **2.** If the pure state c is a superposition of the distinct pure states a and b, then a is a superposition of b and c.

(D) Ample unitary group: Given any two orthogonal pure states $a, b \in \mathcal{L}$, there is a unitary operator U such that $U(a) = b$.

We see that various axiomatic systems for quantum theory contain, under one form or another, the assumption of continuity, and it is this assumption which is largely responsible for making things quantum. Whatever the framework of the reconstruction program, bringing in topological considerations is essential. As it is exceedingly difficult to formulate a physical principle which may provide a meaning for the continuity assumptions of type 2, all reconstruction programs that employ them, suffer from intrusion of an element of mathematical abstraction.

In a more recent reformulation of his axioms [24, 25] Hardy suggests a way to avoid the mathematically abstract type 2 continuity assumption that he has previously made in axiom H5. In the new version Hardy maintains H5 without the word "continuous":

H5′ *Reversibility.* There exists a reversible transformation on a system between any two pure states of that system.

It is then hypothesized that this new reversibility axiom, which is strictly weaker than the former H5, if combined together with axiom H3 about the structure of subspaces, will allow one to derive continuity in any finite-dimensional space.

The idea is to build on the fact that a limited number of finite groups is available in low-dimensional spaces. If the precision of a reversible transformation between pure states in H5′ is sufficiently small, then this transformation, by virtue of being an element of group of reversible transformations, will necessarily be continuous, because no finite group will be available for this transformation to belong to, with a number of elements required for the carefully adjusted precision. If further research shows that this conjecture can be carried through and turned into a theorem, then for the first time one will have a set of algebraic assumptions H1-H5′ that can be used to reconstruct either classical or quantum theory, and the latter will be selected by fixing a particular value of the precision parameter in reversible transformations between pure states. Solèr has proved that the topology of quantum theory can be obtained from a set of assumptions formulated algebraically, in the case of infinite-dimensional space [51]. Hardy's conjecture, if proved, will show how to achieve this deduction of topology from algebra in the case of finite-dimensional spaces. Astonishingly, in this reconstruction program the choice between classical and quantum theory will only depend on the value of one numeric parameter.

3.3 Rovelli's reconstruction

The critique expressed in Section 3.2 with respect to Hardy's continuity axiom applies to the example of reconstruction initially proposed by Rovelli [47], that I have developed elsewhere [17, 18]. Here, the reconstruction starts from two information-theoretic axioms:

R1 There exists a maximum amount of relevant information that can be extracted from a system.

R2 It is always possible to obtain new information about the system.

At the first sight it seems that R1 and R2 contradict each other. Indeed, R1 says that the quantity of information is finite, while from R2 it follows that this quantity must be infinite, because it is always possible to obtain some new information. The reason why there is no contradiction lies in the use of the term ‘relevant’ in R1, which does not appear in R2. In Rovelli’s reconstruction relevance of new information must be judged with respect to information that is already possessed by the observer. Bringing about new information can, not only increase the amount of information currently available to the observer, but also reduce it, due to the fact that some previously relevant information may become irrelevant. Therefore, what is relevant depends on the particular sequence of questions asked by the observer, and for a different observer a change in the amount of information brought in by the same new piece of information may vary. In accordance with his general relational approach, information in Rovelli’s reconstruction should be taken as an observer-dependent, rather than objective, notion. It is information in Shannon’s sense, that is indexed by two indices: first related to the observed system, about which this information has been obtained, second related to the observing system, that has obtained information about the first system. It is impossible, in Rovelli’s view, to separate the notion of information from its second index and to speak about objective information independently of the observing system.

From R1 and R2, with the help of a few quantum logical assumptions, one derives the formalism of quantum mechanics. In particular, if one postulates that information is obtained through answers to yes-no questions that can be asked about the system, then supplementary assumptions are that the set of such yes-no questions forms a complete atomic orthocomplemented lattice. From axiom R1 and a formal definition of relevance of yes-no questions with respect to each other one derives that this lattice is orthomodular. If a further assumption is made about the lattice of questions being isomorphic to the lattice of all closed subspaces of a Banach space constructed over a numeric field (i.e. real or complex numbers or quaternions), one then obtains that this Banach space is indeed a Hilbert space. Its quantum rather than classical character follows from axiom R2.

While the supplementary assumptions cast a shadow on the conceptual clarity of the reconstruction much in the same fashion as does axiom H5 for Hardy’s approach, the whole program presents itself differently from Hardy’s instrumentalism. Mathematical derivation being still devoid of ontological commitments, the proposed justification of the first principles does not refer to an ontology. Rather, by reconstructing quantum theory from information-theoretic principles, we point at its epistemological character and at its role as a theory of (a certain kind of) knowledge, one with certain limits on the kind of information one may be dealing with. The most general theory of this kind of information takes the form of quantum theory. Here again reconstruction appears more appealing than a mere interpretation as it leaves room for any justification of first principles, some such justifications being possibly different from ours. Indeed, one may equally well choose to adopt a specific ontological picture to justify R1-R2. At the same time, regardless of a concrete philosophical justification for first principles, the meaning of quantum theory stands clear: it is a general theory of information constrained by several information-theoretic principles.

3.4 The CBH reconstruction

Clifton, Bub and Halvorson (CBH) propose a set of quantum informational constraints from which one derives the basic elements of quantum theory [10]. They postulate three fundamental principles:

CBH1 No superluminal information transfer via measurement.

CBH2 No broadcasting.

CBH3 No bit commitment.

CBH argue that the meaning of axiom CBH1 is that when Alice and Bob perform local measurements, Alice's measurements can have no influence on the statistics for the outcomes of Bob's measurements, and vice versa. They also submit that "otherwise this would mean instantaneous information transfer between Alice and Bob" and "the mere performance of a local measurement (in the nonselective sense) cannot, in and of itself, transfer information to a physically distinct system." Upon reading these statements, we feel strongly that CBH take *distinct* and *distant* to be synonyms. Such a terminological identification might indeed be a tacit assumption among quantum information theorists, whose analysis is limited to finite-dimensional Hilbert spaces; but in the full-blown C^*-algebraic framework, which CBH also employ, the meaning of the two terms is quite different. We have here an example of the way in which the initial quantum informational departure point of the CBH1-CBH3 principles constrains the use of the C^*-algebraic formalism to only the situations where these principles make sense from the point of view of quantum information; in fact, the formalism is at the same time routinely applied to other settings as well. Unlike Hardy's derivation, which is independent of the particular instrumental justification of its first principles, the CBH reconstruction cannot be carried through outside the field of quantum information, because its mathematics, while still valid outside this field, requires a new justificatory language. Besides the problem of synonymy of 'distant' and 'distinct,' the quantum informational departure point also restricts the question of time evolution. The latter is tacitly taken by the CBH to be the usual quantum mechanical time evolution, while in the general C^*-algebraic framework this is typically not the case and a variety of different 'temporal' evolutions are available [11]. This and other problems arising from the generality of the C^*-algebraic framework are avoided by the CBH reconstruction at the price of confining itself to the quantum informational setting.

Axiom CBH2 is used to establish that the C^*-algebras of Alice and Bob, $\mathcal{A}$ and $\mathcal{B}$, taken separately, are non-Abelian. It is interesting to note that non-Abelianness of $\mathcal{A}$ and $\mathcal{B}$ is proved by assuming that they are kinematically independent. This means that quantumness, of which non-Abelianness is a necessary ingredient, is not a property of any given system taken separately, as if it were the only physical system in the Universe; on the contrary, to be able to derive the quantum character of the theory, one must consider the system in the context of at least one other system that is physically distinct from the first one. As a consequence, for example, this forbids treating the whole Universe as a quantum system if one reconstructs quantum theory along the CBH lines.

Axiom CBH3 entails nonlocality: spacelike separated systems must at least sometimes occupy entangled states. It is not proved, however, that actually instantiated states fill the space of *all* entangled states. CBH show that if Alice and Bob have spacelike separated quantum systems, but cannot prepare any entangled state, then Alice and Bob can devise an unconditionally secure bit commitment protocol. From this theorem the authors deduce that the impossibility of unconditionally secure bit commitment entails that "if each of the pair of *separated* physical systems $\mathcal{A}$ and $\mathcal{B}$ has a non-uniquely decomposable mixed state, so that $\mathcal{A} \vee \mathcal{B}$ has a pair $\{\rho_0, \rho_1\}$ of distinct classically correlated states whose marginals relative to $\mathcal{A}$ and $\mathcal{B}$ are identical, then $\mathcal{A}$ and $\mathcal{B}$ must be able to occupy an entangled state that can be transformed to ρ_0 or ρ_1 at will by a local operation." The term 'separated' is essential and, nevertheless, its precise meaning is not given. Once again, this can be compared to the confusion between distinct and distant. When CBH claim that Alice and Bob represent "*spacelike* separated systems," while formally Alice and Bob are just two C^*-algebras, one sees how the way in which CBH apply the algebraic formalism is constrained by the context of quantum information theory. We witness here an interesting situation in which the language and the context used to formulate and to justify the fundamental principles set a limit on the applicability of the mathematical formalism in which these principles are represented. Even if the formalism can be understood more generally than within the initially chosen disciplinary setting, one still cannot make his way out of this linguistic and contextual prison; without the sense of the axioms being lost. If one persists and escapes, and then obtains a new mathematical result, this result will be void of physical meaning until a new, broader justification of the fundamental principles has been given. Philosophical and linguistic justification, and mathematical derivation, play here a game of mutual onslaught and retreat which, ultimately, leads to the advance of science, in the way described in Section 1.

Notwithstanding the difficulties with justification, the CBH result would be a perfect example of reconstruction were it not for a great deal of mathematical structure which is implied by the choice of the C^*-algebraic framework. The assumptions of the algebraic formalism include at the very least, the relations between operators satisfying linearity, the number field being $\mathbb{C}$, and the states giving rise to the Hilbert space representation via the GNS construction. Once one lists all such tacit supplementary assumptions, the CBH reconstruction appears once again to suffer from the defect of incorporating a serious mathematical abstraction, similar to the derivations from axioms H1-H5 or R1-R2.

3.5 Intentionally incomplete reconstructions

All reconstructions that we have discussed until now shared the goal of deriving, at the final end, the full-blown structure of quantum theory. Recently, a new type of information-theoretic reconstructions appeared, *intentionally* not aimed at deriving the whole quantum theoretic structure [1, 2, 3, 22, 46, 50, 52]. Christened pejoratively by their own authors, these "toy models", "fantasy quantum mechanics," or "quantum mechanics lite" employ a methodology of reconstruction that has been overlooked by the previous generations of researchers: it is now claimed as helpful to reconstruct, not the full version but only a certain part of quantum theory. One builds, therefore, a new theory which is, from the very beginning, not intended to be *the* quantum theory; but this new theory allows nevertheless to better understand the structure of the 'true' quantum theory. To the same old ultimate aim of better understanding quantum mechanics toy models provide a variety of new promising insights.

The idea of modifying usual quantum theory is not new. Nonlinear extensions of the Schrödinger equation have been explored by various authors [5, 15, 56], and comprehensive reviews of these attempts can be found in [38, 53]. Nonlinear models are also sometimes analyzed in the context of quantum theories more general than quantum mechanics, e.g., in quantum gravity. Intentionally incomplete reconstructions, however, differ substantively from these attempts to modify quantum theory. While the latter take standard linear quantum mechanics to be incomplete and purport to replace it by a nonlinear, more complete theory, toy models see *themselves* as incomplete. They do not question the validity of quantum mechanics and do not compete with it in explaining empirical phenomena. Methodologically, as this is routinely emphasized in the opening paragraphs of articles introducing toy models, e.g., in [3], toy models focus on important physics principles, which are upheld while other possibilities are modified. This way the consequences of such and such principle are investigated, independently of other principles. Thus, intentionally incomplete reconstructions are not aimless, but allow one to achieve a better understanding of the structure of quantum theory. Incompleteness, then, becomes a *feature* rather than a flaw of toy models.

Most of the existing examples of toy models are based on information-theoretic principles. To compare the toy model with standard quantum theory, one then asks whether the former reproduces quantum computational phenomena available in the latter. Among others, such questions may include:

- Does the toy model allow superluminal signalling?
- Does the toy model allow bit commitment?
- Does the toy model allow teleportation, dense coding, or remote steering?
- Does the toy model allow exponential speed-up relative to classical computation or solving NP-complete problems in polynomial time?

Other toy models are inspired by information-theoretic principles, but they are compared with standard quantum theory in the aspects that do not necessarily relate to computation:

- Does the toy model allow nonlocality and to what extent?
- Is the toy model contextual?
- Does the toy model possess a continuum of states, measurements and transformations between states?

Answers given to all of these questions can be yes or no depending on the model. Investigating then the difference between the first principles of a particular toy model, and of standard quantum mechanics, one learns with precision which fundamental principle is responsible for which element of the quantum theoretic structure. Spekkens's toy model, for example, accommodates such quantum phenomena as noncommutativity, interference, the multiplicity of convex decompositions of a mixed state, no cloning, teleportation, and others [52]. We learn that the continuous state space, the existence of a Bell theorem, or contextuality, all of which are absent from this toy model, go unconnected with the appearance of the phenomena that are reproduced. Analogously, the toy model known as non-local, or PR, boxes [3, 46] allows non-local correlations that are strictly stronger than those allowed by quantum mechanics, while it only slightly modifies the quantum mechanical state space. One then sees that non-locality is not an exclusively quantum feature and, further, that the amount of non-locality in quantum theory is smaller than in some other theories. We can then conjecture, for instance, that the 'true' quantum theory has as much non-locality as it does, and not more or less, due to the continuum of states and to reversible transformations between pure states, both of these being left out of the PR boxes toy model.

These examples show how one obtains the deeper insight into the structure of quantum theory and therefore achieves what was initially promised by the program of intentionally incomplete reconstruction. Toy models, although incomplete, form a very fertile class of reconstructions. Their recent advent in the area of the foundations of quantum theory manifests the fact that the shift from interpretation of quantum theory to its reconstruction gave birth, in this area, to many a new, previously non-existent idea.

4 Conclusion

We have argued that reconstruction is the exclusive way to make things *clear* about quantum mechanics. As such, this idea is not novel but has been in the air for some time, and a concise statement can for example be found in Rovelli [47],

> Quantum mechanics will cease to look puzzling only when we will

> be able to *derive* the formalism of the theory from a set of simple physical assertions ("postulates," "principles") about the world. Therefore, we should not try to append a reasonable interpretation to the quantum mechanical formalism, but rather to *derive* the formalism from a set of experimentally motivated postulates.

What is novel, however, is that an increasing number of researchers work nowadays on reconstructing quantum theory, and the time is ripe to promote this general framework to the status of a widely accepted paradigmatic shift in the body of work in the foundations of physics.

Reconstruction brings in clarity to where interpretation was struggling to make sense of a physical theory. What belongs to physical theory is no more than what is needed for its derivation. All other questions belong to metatheory and are related to the metatheoretic justification task for the choice of first principles. However, completely reconstructing quantum theory remains only a partially solved problem. Notwithstanding, reconstruction has been successfully competing with more traditional interpretations, due to its appealing conceptual transparency and to the clarity that it brings into the structure of the theory. It would be too ambitious to expect that all of modern quantum theory, including field theory and quantum gravity, could be derived from a few axioms. In other words, none of the reconstructions that have been proposed until now is complete, and some are intentionally incomplete. Although to a varying degree, mathematical abstraction is a necessity for each of the currently existing reconstructions. However, if we want to *understand* the meaning of even most advanced parts of quantum theory, and to reach a consensus in this understanding, it is then inevitable that simple physical principles be formulated and put in the very foundation of quantum theory.

Acknowledgements

Parts of this paper were written during visit to the Perimeter Institute for Theoretical Physics. Thanks to Lucien Hardy, Jeff Bub and Bill Demopoulos

for their comments.

References

[1] S. Aaronson. Is quantum mechanics an island in theoryspace? In A. Khrennikov, editor, *Proceedings of the Växjö Conference "Quantum Theory: Reconsideration of Foundations"*. Växjö University Press, 2004, quant-ph/0402095.

[2] S. Aaronson. Quantum computing, postselection, and probabilistic polynomial-time. *Proc. Roy. Soc. Lond.*, A461:3473–3482, 2005, quant-ph/0412187.

[3] J. Barrett. Information processing in non-signalling theories. 2005, quant-ph/0508211.

[4] E.G. Beltrametti and G. Cassinelli. *The logic of quantum mechanics.* Addison-Wesley, Reading, 1981.

[5] I. Bialynicki-Birula and J. Mycielski. Nonlinear wave mechanics. *Annals of Physics*, 100:62–93, 1976.

[6] G. Birkhoff and J. von Neumann. The logic of quantum mechanics. *Ann. Math. Phys.*, 37:823–843, 1936. Reprinted in: J. von Neumann *Collected Works* Pergamon Press, Oxford, 1961, Vol. IV, pp. 105–125.

[7] C. Brukner and A. Zeilinger. Information and fundamental elements of the structure of quantum theory. In L. Castell and O. Ischebeck, editors, *Time, Quantum, Information*, pages 323–356. Springer-Verlag, 2003, quant-ph/0212084.

[8] R. Brunetti and K. Fredenhagen. Algebraic approach to quantum field theory. 2004, math-ph/0411072.

[9] R. Carnap. *Der Logische Aufbau der Welt.* Berlin, 1928.

[10] R. Clifton, J. Bub, and H. Halvorson. Characterizing quantum theory in terms of information-theoretic constraints. *Found. Phys.*, 33(11):1561–1591, 2003.

[11] A. Connes and C. Rovelli. Von Neumann algebra automorphisms and time-thermodynamics relation in general covariant quantum theories. *Class. Quant. Grav.*, 11:2899–2918, 1994.

[12] W. Demopoulos. Elementary propositions and essentially incomplete knowledge: A framework for the interpretation of quantum mechanics. *Noûs*, 38(1):86–109, 2004.

[13] A. Einstein. Geometry and experience. Address to the Prussian Academy of Sciences, 27 January 1921.

[14] G.G. Emch. *Algebraic methods in statistical mechanics and quantum field theory.* John Wiley, New York, 1972.

[15] N. Gisin. Weinberg's non-linear quantum mechanics and superluminal communication. *Phys. Lett.*, A143:1–2, 1990.

[16] A. Gleason. Measures on the closed subspaces of a Hilbert space. *Journal of Mathematics and Mechanics*, 6:885–894, 1967.

[17] A. Grinbaum. Elements of information-theoretic derivation of the formalism of quantum theory. *International Journal of Quantum Information*, 1(3):289–300, 2003.

[18] A. Grinbaum. Information-theoretic principle entails orthomodularity of a lattice. *Foundations of Physics Letters*, 18(6):573–592, 2005.

[19] J. Guenin. Axiomatic formulations of quantum theories. *J. Math. Phys.*, 7:271–282, 1966.

[20] J. Gunson. On the algebraic structure of quantum mechanics. *Comm. Math. Phys.*, 6:262–285, 1967.

[21] R. Haag and D. Kastler. An algebraic approach to quantum field theory. *J. Math. Phys.*, 5:848–861, 1964.

[22] L. Hardy. Disentangling nonlocality and teleportation. 1999, quant-ph/9906123.

[23] L. Hardy. Quantum theory from five reasonable axioms. 2001, quant-ph/00101012.

[24] L. Hardy. Probability theories with dynamic causal structure: A new framework for quantum gravity. 2005, gr-qc/0509120.

[25] L. Hardy. Private communication. April 2006.

[26] A. Heyting. *Axiomatic projective geometry.* North-Holland, Amsterdam, 1963.

[27] D. Hilbert, J. von Neumann, and L. Nordheim. Über die Grundlagen der Quantenmechanik. *Math. Ann.*, 98:1–30, 1927. (Reprinted in J. von Neumann *Collected Works* Pergamon Press, Oxford, 1961, Vol. I, pp. 104–133).

[28] S.S. Holland Jr. Orthomodularity in infinite dimensions; a theorem of M. Solèr. *Bull. Amer. Math. Soc.*, 32(2):205–234, 1995.

[29] J.M. Jauch. *Foundations of Quantum Mechanics.* Addison-Wesley, 1968.

[30] P. Jordan, J. von Neumann, and E. Wigner. On an algebraic generalization of the quantum mechanical formalism. *Ann. Math.*, 35:29–34, 1934.

[31] S. Kochen and E.P. Specker. Logical structures arising in quantum theory. In Addison J. et al., editors, *The Theory of Models*. North-Holland, Amsterdam, 1965.

[32] N.P. Landsman. *Mathematical Topics Between Classical and Quantum Mechanics*. Spinger, New York, 1998.

[33] G. Ludwig. *An Axiomatic Basis for Quantum Mechanics*. Springer, 1985.

[34] G.W. Mackey. Quantum mechanics and Hilbert space. *Amer. Math. Monthly*, 64:45–57, 1957.

[35] G.W. Mackey. *Mathematical Foundations of Quantum Mechanics*. Benjamin, New York, 1963.

[36] A.R. Marlow. Orthomodular structures and physical theory. In A.R. Marlow, editor, *Mathematical Foundations of Quantum Theory*, pages 59–70. Academic Press, 1978.

[37] N.D. Mermin. Could Feynman have said this? *Physics Today*, page 10, May 2004.

[38] P. Nattermann. On (non)linear quantum mechanics. *Symmetry in Nonlinear Mathematical Physics*, 2, 1997.

[39] C. Piron. Axiomatique quantique. *Helvetia Physica Acta*, 36:439–468, 1964.

[40] C. Piron. Survey of general quantum physics. *Found. Phys.*, 2:287–314, 1972.

[41] I. Pitowsky. Quantum mechanics as a theory of probability. 2005, quant-ph/0510095.

[42] R.J. Plymen. C^*-algebras and Mackey's axioms. *Comm. Math. Phys.*, 8:132–146, 1968.

[43] R.J. Plymen. A modification of Piron's axioms. *Helvetia Physica Acta*, 41:69–74, 1968.

[44] J.C.T. Pool. Baer *-semigroups and the logic of quantum mechanics. *Comm. Math. Phys.*, 9:118–141, 1968.

[45] J.C.T. Pool. Semimodularity and the logic of quantum mechanics. *Comm. Math. Phys.*, 9:212–228, 1968.

[46] S. Popescu and D. Rohrlich. Nonlocality as an axiom for quantum theory. *Foundations of Physics*, 24:379, 1994, quant-ph/9508009.

[47] C. Rovelli. Relational quantum mechanics. *Int. J. of Theor. Phys.*, 35:1637, 1996.

[48] I. Segal. Postulates of general quantum mechanics. *Ann. Math.*, 48:930–948, 1947.

[49] I. Segal. *Mathematical Problems of Relativistic Physics.* American Mathematical Society, Providence, 1963.

[50] J. Smolin. Can quantum cryptography imply quantum mechanics? *Quantum Information and Computation*, 5:161–169, 2005, quant-ph/0310067.

[51] M.P. Solèr. Characterization of Hilbert spaces with orthomodular spaces. *Comm. Algebra*, 23:219–243, 1995.

[52] R. Spekkens. In defense of the epistemic view of quantum states: a toy theory. 2004, quant-ph/0401052.

[53] G. Svetlichny. Nonlinear quantum mechanics. quant-ph/0410036.

[54] V.S. Varadarajan. Probability in physics and a theorem on simultaneous observability. *Comm. Pure and Appl. Math.*, 15:189–217, 1962.

[55] V.S. Varadarajan. *Geometry of quantum theory.* Van Norstand, Princeton, 1968.

[56] S. Weinberg. Precision tests of quantum mechanics. *Phys. Rev. Lett.*, 62:485–488, 1989.

[57] S. Weinberg. Sokal's hoax. *The New York Review of Books*, XLIII(13):11–15, 1996.

[58] N. Zieler. Axioms for non-relativistic quantum mechanics. *Pacific J. Math.*, 11:1151–1169, 1961.