Probability, Rational Single-Case Decisions and the Monty Hall Problem

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Abstract

The application of probabilistic arguments to single cases and decision-making is a contentious philosophical problem arises in various contexts. This paper focuses on the validity of probabilistic arguments in the Monty Hall problem and a variation thereof. Two claims are made and defended. First, preferring a certain strategy to another in the Monty Hall Problem does not need any recourse to long-run success frequencies. Second, recent attempts to refute the standard solution of the Monty Hall Problem fail.

1 Introduction: The Monty Hall Problem

The application of probabilistic arguments to single cases is a non-trivial problem which arises in various contexts, e.g. with respect to interpretations of probability. This paper, however, focuses on the validity of probabilistic arguments in the Monty Hall problem and a variation thereof. The examination of this special case teaches us a general lesson about the normative force of probabilistic arguments, too.

The Monty Hall Problem arises in the following situation: You are a candidate in a TV show. You stand in front of three closed doors, one of them hiding a prize. The other two doors are empty. You do not know which door hides the prize and which doors are empty, but every door is equally likely to hide the prize. Monty Hall, the show moderator, asks you to guess a door. You make a guess, and afterwards Monty opens a door at random, but he will neither open the door you have chosen nor the door that hides

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the prize. So he will open an empty door. Then you have the opportunity to stick to your initial choice or to switch to the other closed door. You get the prize if you correctly guess the door where the prize is hidden. Which of the two actions should you take, given that you are rational, self-interested and perfectly know the rules of the game?

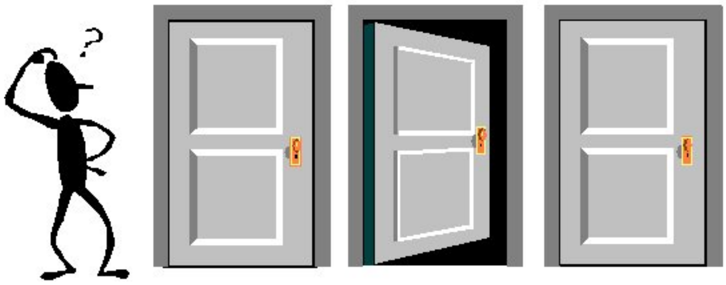

Figure 1: The Monty Hall Problem after the opening of a door.

Bayes' Theorem implies that you should switch to the other remaining closed door. While this may appear counterintuitive, it is not hard to see that it is sound. You are forced to bet on a particular door, but you would prefer to bet on two doors at once, in order to double your chances. In other words, you would like to guess that the prize is *not* behind a particular door. Such a guess is not allowed by the rules of the game, but the switching strategy indirectly provides it: Switching wins whenever the prize is not behind the initially chosen door. So the switching strategy has a success chance of 2/3, compared to 1/3 when you stick to your initial choice.[1] By holding up their initial guess, stickers fix their success chance on 1/3 whereas switchers have a success chance of $1-1/3 = 2/3$: they win unless the initially chosen door was actually the winning one. Those who are not yet convinced might imagine a Monty Hall Problem with thousand doors, all of which but two are opened.[2] Moreover, whether door 1, 2 or 3 was initially chosen does not affect the superiority of switching. I would like to call this the *standard solution* of the Monty Hall Problem.[3]

[1]In this paper, the term "success chance" refers to objective probabilities as well as to rational degrees of beliefs.

[2]Cf. vos Savant (1991).

[3]Cf. Bradley and Fitelson (2003).

2 Trouble with the standard solution

The above argument leaves little doubt that if the Monty Hall game were played infinitely often, switchers would fare better than stickers. But it is less clear whether we ought to switch in an individual game, too. Generally, one can doubt the relevance of limiting relative frequencies for single-case decisions (cf. Howson and Urbach 1993, Albert 2005). Let us perform infinitely many runs of the Monty Hall game. The argument brought up in the preceding section entails that the success frequency of switching converges to 2/3. But by itself this does not entail anything about the degrees of belief which I should assign to the success chance of switching in an individual game. For instance, I could assign degree of belief 1/2 in the success chance of switching *for the first element* of an infinite sequence of Monty Hall games. Since any finite segment of the success-noting sequence will eventually have zero weight, that probability assignment would not alter my degrees of belief about the long-run success frequency of switching. But evidently, it would make a big difference for a decision in the single case. So there is no logical point which ensures that beliefs about relative frequencies in the long (i.e. infinite) run transfer to subjective degrees of belief. More generally, assigning an *arbitrary* degree of belief in the success of switching in an individual game would leave intact the degrees of belief about the long-run success frequency of switching. Hence, rational decision in the single case cannot build on degrees on belief about long-run success frequency.[4] I will come back to this general concern in section 3, after discussing the other objection.

A more specific objection to the standard solution of the Monty Hall Problem has recently been put forward by Baumann (2005, forthcoming). Baumann accepts the superiority of switching in the long run, too, but claims that epistemic single-case probabilities cannot be meaningfully applied in a variation of the Monty Hall Problem. To bring this out clearly, he sets up a modified Monty Hall Problem with two players instead of one. Both players privately guess a door and submit their choice to Monty Hall. Again, Monty is not allowed to open any door that was selected by one of the players. In particular, when the players initially guess different doors and are both mistaken, Monty has no choice but to open the winning door. After the opening of a door, the players are asked whether they want to stick or to switch. So they are allowed to switch to an open door with the prize behind it. Note that both players get the full prize if they both succeed so that there is no real competition between them. All this is common knowledge between

[4]This argument varies a point made by Albert (2005) with regard to frequentist interpretations of probability.

A's choice	B's choice	opened door	switching wins	sticking wins
1	1	2 or 3	no	yes
1	2	3	no	yes
1	3	2	no	yes
2	1	3	yes	no
2	2	3	yes	no
2	3	1	yes	yes
3	1	2	yes	no
3	2	1	yes	yes
3	3	2	yes	no
			6/9	5/9

Table 1: The success chances for switching and sticking in the modified Monty Hall Problem for player A if door 1 is the winning door.

the players, but no player knows the initial choice of the other player. Neither player has a preference for a specific door, and this is common knowledge, too.

Baumann goes on by comparing two different strategies: the *modified sticking* and the *modified switching* strategy. In the first strategy, the player sticks to his initial choice unless the winning door is opened (then he switches to the open door). In the second strategy, the player switches to the other closed door unless the winning door is opened (then he switches to that door). So both strategies agree if Monty discloses the prize and differ in any other case.

It can be shown that even in the modified Monty Hall Problem, the (modified) switching strategy is superior. Assume without loss of generality that door 1 is the winning door. Then there are nine equiprobable cases as a function of the players' first choice (e.g. A chooses door 1, B chooses door 2, etc.). Table 1 compares the success chances of switching and sticking in the modified Monty Hall problem for player A and leads to the result that switching has a success chance of 6/9, as opposed to 5/9 when sticking.

Baumann continues as follows: When Monty opens an empty door, two lines in table 1 (the sixth and the eighth) cancel out and only seven possible cases remain.[5] The players count the remaining cases, assign a success chance of 4/7 to switching as opposed to 3/7 when sticking. More specifically, assume that player A has chosen door 1 and player B door 2 at the beginning. So Monty is forced to open door 3 which turns out to be empty. By conditioning on that information, both players arrive at the probabilities displayed in table

[5] Cf. Baumann (2005, 72).

	"the other door wins"	"door 2 wins"
A's rational degree of belief	4/7	4/7
B's rational degree of belief	4/7	3/7

Table 2: Rational degrees of belief of A and B in the propositions stated in the first line, given that A initially chooses door 1, B initially chooses door 2 and door 3 turns out to be empty.

2. (A more rigorous derivation is given in the appendix.) The table reveals a difference in the probabilities A and B assign to the proposition "door 2 wins".

Now, the essential argument begins. We unanimously accept the

> **Principle of Non-Arbitrariness**:
>
> - if two agents' degrees of belief are fully determined by the available evidence *and*
> - if they share all relevant information
>
> then their rational degrees of belief should be the same.

Baumann thinks that this principle is violated with regard to the proposition "door 2 wins" where the players assign different probabilities (cf. table 2). So we have to check whether the antecedent conditions are satisfied. For the first condition, this is unanimous – the rules of the game and Monty's opening of the empty door 3 fix the players' rational degrees of belief (proof in the appendix). Whether the second condition is also satisfied is more contentious. The only candidate for a relevant informational difference between the players is the privately made first guess which the other player cannot know. Baumann objects, however, that this private information

> "[...] does not matter at all when it comes to one's probabilistic reasoning about sticking and switching."[6]

so that both players share the same relevant information. Hence, the antecedent conditions of the Principle of Non-Arbitrariness are satisfied and the principle is violated, or so Baumann argues. This constitutes a *reductio* against the probabilistic argument for switching. Hence, epistemic single-case probabilities cannot be applied in the modified Monty Hall Problem without invoking arbitrariness. This threatens, of course, the general validity of probabilistic arguments in single cases since similar counterexamples might be found.

[6]Baumann (2005, 76).

3 In defense of epistemic single-case probabilities

There are several ways to argue against Baumann's reductio, but not all of them are satisfactory. First, one might be inclined to think that the introduction of a second player changes the situation in a way that switching is no longer superior to sticking. This is the road taken by Ken Levy (2007) who argues that the success chances of switching and sticking are equal in the modified Monty Hall Problem. The starting point of Baumann's *reductio* is the fact that A and B assign different success chances to door 2 (cf. table 2). Levy observes that the entire situation is completely symmetric with regard to door 1 and door 2, unlike in the original Monty Hall Problem (cf. section 1). The upshot of the standard solution was the opportunity to bet *against* the initially chosen door by always switching to the other closed door, thus achieving a success rate of 2/3. This is no longer possible in the modified problem because the second player's choice interferes with Monty's reaction to the first player's initial guess. Hence, the original argument for switching does no longer apply, and due to the complete symmetry between door 1 and door 2, there cannot be an advantage for either sticking or switching, or so Levy argues. Both players should thus assign success chance 1/2 to the two remaining doors, saving the meaning of single-case probabilities:

> "So the proper conclusion is not Baumann's – i.e. that probabilities cannot meaningfully apply to single games [...]. Rather, the proper conclusion is that a player has no reason to switch [...] because the presence of the second player *changes* the situation in such a way that the probabilities for the remaining closed doors are not 1/3 and 2/3 but rather 1/2 and 1/2."[7]

If that argument worked, it would solve the problem. Unfortunately, this is not the case. Assume that the empty door 3 is opened. Now, let us distinguish two cases: either the players have chosen different doors or they have chosen the same door. In the former case, the situation is indeed completely symmetric with regard to door 1 and door 2. Monty's opening of door 3 was determined by the player's choice alone – where the prize is hidden did not affect his decision. Therefore there is no reason to prefer one door over another or switching over sticking. This argument from symmetry would, however, not apply if the players had initially chosen the same door. Instead, Monty would act precisely as if there was only one player. Hence, the situation would be the same as in the original Monty Hall Problem and

[7]Levy (2007, 144). Italics in the original.

there would be a probabilistic argument for switching, as shown in section 1. The players do not know which of the two situations is the case, but they know that both are real options. Recall that switching is more likely to yield success in one situation while the success chances are equal in the other situation. Thus, the players conclude that switching is the dominant strategy and that it should be strictly preferred to sticking. Hence, Levy's account of the problem is untenable. He seems to miss that the players, unlike ourselves, do not know whether the other player has chosen a different door.

That argument gives an elegant defense of switching in the modified Monty Hall Problem. Still, it does not directly address Baumann's challenge with regard to the Principle of Non-Arbitrariness. As Baumann admits himself, everything hinges on the second premise of the principle – the question whether there are relevant informational differences between both players. In my rebuttal, I am going to argue that such informational differences exist.

Baumann notices that the private information – the initial guess of the player – is irrelevant with regard to the success chances of the two strategies. This is right in so far as the success chances of switching and sticking are invariant over the initial choice of a door. Each player could have forgotten her initial choice, but for playing the game, it would be sufficient just to tell Monty whether to switch or to stick.[8] Therefore the names of the chosen doors are irrelevant to the success chances of switching and sticking. But this observation does not substantiate that both players should agree on the success chances of door 2 because probability assignments are statements about degrees of belief, "to believe" being an *intensional* verb. Note that the reference of "the other door" changes from player to player. Hence, the players can assign different rational degrees of belief to the proposition "door 2 wins" while agreeing on "the other door wins". To give a more mundane illustration: Assume that both players believe the other player to be good-looking while finding themselves unattractive. The believer has to know, however, whether she is player A or player B in order to find out whether she actually believes that player A is good-looking. Consequently, both players disagree on the proposition "player A is good-looking" although they agree on "the other player is good-looking". The same step is performed in the modified Monty Hall Problem: By making use of the mathematical fact that switching to the other closed door has a success chance of 4/7 and the private, context-relative information about the reference of "the other door", the players arrive at non-arbitrary, though different winning probabilities for door 2. For determining the reference of "the other door" and evaluating the success chances of door 1 vs. door 2, both players have to remember their

[8] Cf. Baumann (forthcoming, 4).

initial choice. Indeed, A gives door 2 a success chance of 4/7 while B merely assigns a success chance of 3/7, in agreement with Baumann's table 1. In the appendix, I derive that result solely from a mathematical description of the problem so that no distinction between single cases and long run frequencies is required. Baumann's argument fails because the second premise of the Principle of Non-Arbitrariness is not satisfied.[9]

This point answers the more general scepticism about rational single-case decisions, too. We have argued in section 2 that degrees of belief about single cases are not justified by degrees of belief about relative frequencies in the long run. But it suffices just to apply the Principal Principle – rational degrees of belief should be adapted to objective chances.[10] Those chances, by contrast, are determined by their *meaning* alone. Take, for instance, the following statement:

> (%) If I toss a fair coin three times (independently), the probability of observing "heads" in all tosses is $\frac{1}{8}$.

Propositions as (%) are true under any interpretation of objective probability since it is the *meaning* of a fair coin that the probability of "heads" in any single toss is $\frac{1}{2}$ (given independence), thereby implying the truth of (%). Proposition (%) is therefore an *analytical truth.*[11] Whatever interpretation of objective probability we choose, statements as (%) remain true solely in virtue of the mathematical description. The same holds of statements as

> (+) In the modified Monty Hall Problem, the probability that an empty door is opened in the first place is equal to 2/9 if the players are indifferent between the doors (cf. table 1).

which is just a mathematical truth, given the rules of the game.[12] By assumption, the players are indifferent between the doors (this is just a presupposition of the problem) so that we may infer to the above probability. By means of the Principal Principle, we also adapt our subjective degrees of belief. In particular, (%) and (+) transfer to rational degrees of belief in a single case as well as to degrees of belief about long-run frequencies. Note that all this is entirely independent of the problem of probability interpretation and the debate between subjective, frequency and propensity theories. We only require that the relevant probability statements in the Monty Hall

[9] Baumann's "counterfactual argument" (2005, 76-77) can be rebutted in a similar vein.

[10] This very rough and simplified version of the Principal Principle is sufficient for our purposes, cf. Lewis (1981).

[11] Cf. Good (1952, 108) and Levy (2007, 146).

[12] Cf. the appendix for more details.

Problem are either analytical (such as (+)) or given by assumption (such as "all doors are equally likely to hide the prize"). It can then be shown that the rational degrees of belief in the success of certain actions (sticking vs. switching, door 1 vs. door 2) are exactly determined and can be calculated straightforwardly by means of Bayes' theorem – proof given in the appendix. Thus, the computed degrees of belief in "door 2 wins" or "the other door wins" are indeed binding for the players if they want to make a rational choice.

4 Conclusions

The normative force of probabilistic arguments with regard to rational single-case decisions has always been a contentious philosophical issue. This paper tries to shed new light on the problem by carefully scrutinizing a particularly famous case: the Monty Hall Problem. I make two points: First, rational single decisions and degrees of belief about long-run success frequencies should be disentangled. Second, attempts to refute the standard solution of the Monty Hall Problem fail. Below I summarize my arguments.

Baumann's reductio of the standard solution would go through if he managed to show that both players can rationally assign different success chances to the switching strategy in the modified Monty Hall Problem. Assigning different probabilities to the same strategy would indeed violate non-arbitrariness because the information situation of the players is completely symmetric with regard to the two strategies. However, they only assign different probabilities to particular doors. As I have argued in section 3, there are relevant informational differences between the players with regard to the success chances of particular doors. Therefore Baumann's argument fails.

More generally, it is neither necessary nor advisable to defend the normative force of probabilistic arguments in a single cases by means of rational degrees of belief about long-run frequencies. The analyticity of the relevant probability statements does the job, together with some uncontentious presuppositions about the scenario. Thus, epistemic single-case probabilities can be applied meaningfully, in agreement with the standard solution of the Monty Hall Problem.

Appendix: Calculational Details

We calculate the probability that door 2 is the winning one ("W_2") in the modified Monty Hall Problem, conditional on

- the rules of the game ("R") (cf. section 2)
- the initial guess of player A, namely door 1 ("A_1")
- the fact that the empty door 3 was opened ("$O_3 \neg W_3$")

This describes the complete information available to A. The rules of the game contain that the winning door does not depend on the players' choice, that all doors are equally likely to win, that Monty must not open a door selected by one of the players, that the players do not have preference for a specific door, and so on. Bayes' theorem then gives

$$P(W_2|A_1 O_3 \neg W_3 R) = \frac{P(W_2|A_1 R)\, P(O_3 \neg W_3|A_1 W_2 R)}{P(O_3 \neg W_3|A_1 R)} \tag{1}$$

We separately calculate the three terms on the right hand side of (1). First,

$$P(W_2|A_1 R) = 1/3 \tag{2}$$

because the number of the winning door is not affected by the choice of player A. Second, with B_j denoting player B's first choice,

$$\begin{aligned}
P(O_3 \neg W_3|A_1 W_2 R) &= P(\neg W_3|A_1 O_3 W_2 R) P(O_3|A_1 W_2 R) && (3)\\
&= \sum_{j=1}^{3} P(B_j|A_1 W_2 R) P(O_3|A_1 B_j W_2 R) && (4)\\
&= 1/3 \sum_{j=1}^{3} P(O_3|A_1 B_j W_2 R) && (5)\\
&= 2/3 && (6)
\end{aligned}$$

Some explanations are due. (3) merely applies an axiom of probability. (4) notices that door 3 cannot win when door 2 is actually winning. Furthermore, a conditionalization on the initial choice of player B is introduced. Since B's choice is independent of A's choice and the winning door and since B has no preference for a specific door, $P(B_j|A_1 W_2 R) = 1/3$ in (5). Finally, door 3 is opened if and only if B chooses either door 1 or door 2 (6).

Now, we calculate the last remaining term

$$\begin{aligned}
P(O_3 \neg W_3|A_1 R) &= \sum_{k=1}^{3} P(W_k|A_1 R) P(O_3 \neg W_3|A_1 W_k R) && (7)\\
&= 1/3\,[P(O_3|A_1 W_1 R) + P(O_3|A_1 W_2 R) - 0] && (8)\\
&= 1/3 \left[1/3 \sum_{j=1}^{3} P(O_3|A_1 B_j W_1 R) + 2/3 \right] && (9)\\
&= 1/3 \cdot 1/2 + 2/9 \;=\; 7/18 && (10)
\end{aligned}$$

Again, some explanations are in order. (7) applies the axioms of probability, (8) makes use of the fact that the winning door is independent of the players' choice. (9) cites equation (4) and decomposes $P(O_3|A_1W_1R)$ once more according to the laws of probability. The rest is simple arithmetic.

Combining (2), (4) and (10) and inserting those results into (1) finally yields $P(W_2|A_1O_3\neg W_3R) = 4/7$. This is the rational degree of belief in the success chance of door 2 for someone who knows the rules of the game, that A has chosen door 1 and that the empty door 3 was actually opened. This is precisely the information available to A. Hence A assigns the uniquely rational degree of belief 4/7 to the winning chance of door 2 in the situation discussed by Baumann. A similar argument can be given for B with regard to door 1. As the calculations make clear, any precise probability assignment followed solely from the meaning of the probability statements involved and the specification of the problem.

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