PHILOSOPHIES AND PEDAGOGIES OF MATHEMATICS

Boris Handal

The University of Sydney

Handal(at)ans.com.au

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ABSTRACT

The paper discusses major philosophical stances on the nature of mathematics as held by foundationalists and quasi-empiricalism supporters. It is argued that the contrasting philosophical views between the two groups parallels in many respects the pedagogical debate between behaviourism and socio-constructivism. It is also argued that behaviourism has been influenced by foundationalist conceptions of mathematics while socio-constructivism has been influenced by quasi-empirical philosophies.

Introduction

Mathematical beliefs can be studied in the light of major philosophical and pedagogical stances on the nature, teaching and learning of mathematics. The philosophical and pedagogical stances portray well-structured representations that have been sometimes the result of hundreds of years of collective reflection. These macro stances are useful given their capacity to articulate a background from which other relatively minor issues can be discussed. On the other hand, each individual holds his or her own conception of mathematics teaching and learning. These conceptions are unique in that they are the results of their own formal or informal contemplation of reality. Both macro and micro conceptions of mathematics are significant because they represent human beliefs that influence instructional behaviour.

The Philosophy of Mathematics

The philosophy of mathematics, as a discipline, has dealt for many centuries with the issue of what is the nature of mathematics. This age-old discussion is far from being conclusive, rather it is evolving as each thinker contributes his or her view of looking at the different facets which mathematics presents as a discipline. This philosophical debate is indispensable since teaching and learning mathematics is influenced by the perspective adopted, and because mathematics has had such a central role in the advancement of societies that defining its nature, role and methodology has become a central, ideological and cultural issue.

Early attempts to develop a methodological foundation of mathematics attempted to vindicate it as a discipline free of error, that did justice to its arrogant and secular epithets as the "most perfect of all sciences" (Lakatos, 1986, p.31), the "mother" (Mura, 1995, p. 390), the "queen of all sciences" (McGinnis, Randy, Shama, McDuffie, Huntley, King, & Watanabe, 1996, p. 17), "a science in its own right" (Mura, 1995, p. 390). Others began to doubt the dogmatic assumption that mathematics was actually an *a priori*, infallible enterprise, whose methodology could be perfectly delineated and whose development was amenable to being formulated through a formal and universal system. An alternative conception therefore began to evolve in which mathematics was conceived as a fallible, empirical or quasi-empirical discipline.

In the last century, the nature of mathematics became a central issue for educationalists as it had been before for the philosophers. An individual's philosophy of education was thought to determine how we live our lives. A personal philosophy of mathematics education ascertains the way we learn and teach mathematics within the classroom and the school environment (Southwell, 1999). If mathematics is, as the Platonist tradition suggested, just an entity out there waiting to be discovered then it will be enough for schools to present the curriculum instruction as a mere collection of facts, definitions and algorithms. In that regard, teaching mathematics would be like just transmitting an immutable body of knowledge that students have to accept as a perennial fact without any reasoning. However, if mathematics is a cultural, creative and empirical activity then learners are in the position of constructing their own mathematical knowledge regardless of how different their methodology may be from the canon of orthodox and classical mathematics.

We owe the first attempts to secure an error-proof methodology of mathematics to the Ancient Greeks. It was Euclid (365-275 B.C.) who dared to explain mathematical reasoning through a consistent network of postulates, corollaries, axioms and theorems. For nearly two millenniums, the academic community used Euclid's reasoning model to advance mathematical knowledge. However, it was mainly Lobatchevsky (1793-1856) who dethroned Euclid's infallibility by proving that the fifth of Euclid's five postulates was not absolutely true (Baldor, 1984). Subsequent developments in mathematics showed that conventional methods of mathematical proof led to other paradoxes and therefore the search for an alternative infallible method became central at the beginning of this century. Consequently, three paradigms were advanced to secure the foundation of mathematics, namely, logicism, symbolism and intuitionism, and they become known as the foundationalist movement.

Logicism is basically a form of Platonist realism in which mathematics is seen as a set of abstract realms that exist externally to human creation. According to logicists, all mathematical concepts can be reduced to abstract properties that can be derived through logical principles. Logicism has been criticised because of its failure to enunciate an unerring system of mathematical truth, its discourse lacking appropriate discussion of basic mathematical concepts such as plane, line, sets and so on. Logicism has also been criticised for its obsession for strict logical reasoning, leaving little room for intuition and conjecture which many see as powerful generators of creative thinking (Goodman, 1986).

Formalists share the logicist view that logic is necessary, however they argue that mathematical knowledge is brought about through the manipulation of symbols that operate by prescribed rules and formulas and whose understanding should be accepted *a priori*. Formalism has been criticised because of the little space left for creative thinking, the unfeasibility of creating an inclusive mathematical system due to the need for a concomitant extensive list of definitions, properties, rules and the like, and the reifying of the mastery of mathematical symbolism over meaningful inference and intuition.

In the intuitionist tradition, mathematics is conceived as an intellectual activity in which mathematical concepts are seen as mental constructions regulated by natural laws. These constructions are regarded as abstract objects that do not necessarily depend on proofs. Brouwer, the founder of Intuitionism, rejects the classical stance of categorising proofs as either true or false and instead argues that other possibilities for claiming mathematical truth should be allowed as academically acceptable. For Brouwer, mathematical induction comes before and it is independent of logic. Likewise, intuition and imagination are seen as early and necessary psychological stages in the process of invention. The main critics to intuitionism argue that mathematical constructions are not only mentally but also socially constructed. These critics also argue that absolute freedom of thought is detrimental to mathematical rigor. It has also been said that intuitionists' biggest downfall lies in enunciating their theory using formalist methods (Goodman, 1986).

The crisis and failing of the three traditions described above in securing mathematics as an abstract, absolutist, universal and infallible system was followed by an increasing interest in exploring mathematics as an activity which was practical, fallible, situated and socially and personally constructed. The movement was labelled "quasi-empirical" because it proposed that mathematics did not actually belong to the category of hard sciences such as physics in which something out there is to be discovered. Instead, mathematics is a human creation born of and nurtured from practical experience, always growing and changing, open to revision and challenge, and whose claims of truth depend on "guessing by speculation and criticism, by the logic of proofs and refutations…" (Lakatos, 1976, p. 5). According to Polya (1986), mathematics is both demonstration and creation. Demonstration is achieved by proofs while creation consists of plausible reasoning that includes guessing. Mathematical methods therefore are not perfect and cannot claim absolute truth. Mathematical truth is not absolute but relative because in fact truth is time dependent (Grabiner, 1986) and space dependent (Wilder, 1986). Time dependent because what is scientifically true today, might be a falsehood in the future as theoretical assumptions change, as occurred with the theories of Euclid and Ptolomeus. Mathematical methods are also space dependent because different peoples and different cultures have different ways of doing and validating their mathematical knowledge (Ascher, 1991).

The transition from the foundationalist approach, with its emphasis on pure mathematics, to the quasi-empirical approach was followed by a renewed interest in the application of mathematics. As seen above, for the foundationalists the realm of mathematics was made of abstract constructs, a fact that took them away from an emphasis on application of mathematics (Robitaille & Dirks, 1982; Rogerson; 1989). If pure mathematics is to have any value by itself, however, then it cannot be attained by sacrificing the value of and engaging in the application of mathematics.

Foundationalists' overvaluing of pure mathematics neglected the fact that the origin and goal of mathematics was the search for solutions to humanity's proximal environment. In fact, one of the merits of Euclid's geometry is that he designed his deductive method from empirical evidence (Baldor, 1984). Mathematics therefore had grown parallel to and serving the so-called hard sciences and it is this practical and interactive experience to which mathematics owes most of its greatness (Putnam, 1986). For Putnam (1986), the greatness of mathematics did not reside only in its ability to go beyond the realm of concrete entities, nor in the beauty of their proofs, but in its concomitant power in providing utilitarian solutions to the bewildered *homo sapiens* in their settlement on earth.

Influence of the Philosophy of Mathematics on the Pedagogy of Mathematics

The formalist and logicist paradigms, as Hersh (1979) and Rogerson (1994) have argued, have had a strong influence on mathematics education in this century and therefore have shaped the way teachers and students have learned what mathematics is. The *New Mathematics* wave, set theory, the emphasis on notation, symbolism, functions and relations, more stress on analytical rather than descriptive geometry, and behaviourist perspectives on education have certainly been part of the foundationalist legacy which influenced the school mathematics curriculum and models of teacher education in the world (Laurenson, 1995; Moreira & Noss, 1995; Robitaille & Dirks, 1982; Thom, 1986).

As the second half of the last century continued to evolve, the international mathematics education community was keener to consider and adopt the quasi-empirical conception of mathematics, no matter how eclectic this view was. Major reform documents such as the U.S. *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, NCTM, 1989), *Professional Standards for the Teaching of Mathematics* (NCTM, 1991), *Assessment Standards for School Mathematics* (NCTM, 1995), *Principles and Standards for School Mathematics* (NCTM, 2000), the U.K. *Cockcroft Report* (Cockcroft, 1982), the *National Statement on Mathematics for Australian Schools* (Australian Education Council, 1991), the *Statement of Principles for Mathematics K-12* and *The Nature of Mathematics Teaching and Learning* (Board of Studies New South Wales, 1996) were inspired in different degrees by the principle of "knowing mathematics is doing mathematics" (NCTM, 1989, p. 7) thus reflecting the quasi-empirical approach.

The quasi-empirical approach parallels in many respects the main tenets of the socio-constructivist theory, although it is worthy to note that while the former constitutes a philosophical view on the nature of mathematics, the latter focuses its attention on the psychological underpinnings of teaching and learning mathematics.

For many years, there has been a debate in education on the advantages and disadvantages of socio-constructivism and behaviourism. These two philosophies on teaching and learning mathematics can be depicted as two contrasting views and both have influenced the way mathematics is being taught in schools (Marland, 1994).

Socio-constructivism, or constructivism in shorter terms, as opposed to behaviorist models of teaching and learning, claims that knowledge should not be transferred from one individual to another in educational environments. For constructivist educationalist, knowledge must be actively constructed as the learner is an entity with previous experiences that must considered as a "knowing being". Learning is therefore seen as an adaptive and experiential process rather than a knowledge transference activity (Candy, 1991). As learners encounter new situations, they look for similarities and differences against their own cognitive schemata. These contrasts, also called cognitive perturbations, are the end-product of conflictive knowledge waiting to be resolved through reorganizing schemes of knowledge (Phillip, 1995).

In constructivist terms, learning depends on the way each individual learner looks at a particular situation and draws his/her own conclusions. People therefore determine their own knowledge based on their own way of processing information and according to his/her own beliefs and attitudes towards learning (Biggs & Moore, 1993). Constructivism therefore gives recognition and value instructional strategies in which students are able to learn mathematics by personally and socially constructing knowledge. Constructivist learning strategies include more reflective oriented learning activities in mathematics education such as exploratory and generative learning. More specifically, these activities include problem solving, group learning, discussions and situated learning (Murphy, 1997; Wood, Cobb, & Yackel, 1991).

According to behaviourism focuses on the manipulation of external conditions to the learner to modify behaviours that eventually lead to learning. In a behaviourist oriented environment completion of tasks is seen as ideal learning behaviour and mastering basic skills require students to move from basic tasks to more advanced tasks. In addition, learning is considered a function of rewarding and reinforcing student learning. Likewise, the emphasis is on correct answers rather than of partially correct answers (Elliot, Kratochwill, & Travers, 1996). Inspired by linear programming theories developed particularly during the Second World War, learning and teaching in behaviourist terms is a matter of optimizing and manipulating the instructional environment towards the fulfilment of rigidly and specifically designed educational objectives.

In addition, behaviourists saw the student's affective domain as different from the cognitive domain. The Bloom Taxonomy, for example, classifies educational objectives in cognitive, affective and psychomotor domains (Krathwohl, Bloom, & Masia, 1964). They categorised emotions "as imaginary constructs" that are causes of behaviour (McLeod, 1992). Consequently, behaviourists assume that certain emotions and attitudes can influence behaviour, although, in general, affective issues are neglected (McLeod, 1992). Teachers' and students' minds were seen as "black-boxes" or machines

(Shavelson & Stern, 1981) in which attitudes and behaviour occur somehow or even are not relevant (Nespor, 1987).

It has been said that behaviourism emphasizes a process-product and teacher-centredness model of instructions that have been prevalent in classroom teaching and in teacher education programs in the twentieth century (Marland, 1994). A behaviourist teaching style in mathematics education tends to rely on practices that emphasize rote learning and memorization of formulas, one-way to solve problems, and adherence to procedures and drill. Repetition is seen as one of the greatest means to skill acquisition. Teaching is therefore a matter of enunciating objectives and providing the means to reach those objectives and situated learning is given little value in instruction (Leder, 1994). This over emphasis on procedures and formulas resembles traditional formalist and logicist ideas.

It is worthy to add that while most of the literature on mathematics education revolves around the dialogue between the constructivist and the behaviourist movements, it is apparent that their differences have been described by educationalists and reform documents under other educational terms. These terms basically discriminate between the teaching of specific facts and a type of instruction that fosters independent thought (Schmidt & Kennedy, 1990). It must be noted that, like any other theoretical model, these representations are oversimplification of reality and therefore many educational variables are excluded. Figure 1 shows the different terms used in those discussions.

Figure 1
Divergent Views in Mathematics Education

Behaviourist Perspective	Constructivist Perspective	Source
Behaviourism	Constructivism	Candy (1991)
Traditional	Progressive	O'Laughlin & Campbell (1988)
Mimetic	Transformational	Jackson (1986)
Basic skills	Higher order thinking	Schmidt & Kennedy (1990)
Content	Process	Schmidt & Kennedy (1990)
Positivist	Relativist	Laurenson (1995)
Subject-centred	Child-centred	Sosniak, Ethington, & Varelas (1991)
Transmission of factual and procedural knowledge	Emphasis on qualitative transformations in the character and outlook of the learner	Sosniak et al. (1991)
Euclidean	Quasi-empirical	Lerman (1983)
Absolutist	Fallibilist	Lerman (1983)
Technical-Positivism	Constructivism	Taylor (1990)
Traditional	Nontraditional	Raymond (1997)
Transmission	Child-centredness	Perry, Howard, & Tracey (1999)
Transmission	Constructivist	Nisbet & Warren (2000)

Summary

This paper has reviewed the debate between foundationalism and quasi-empirical supporters on the nature of mathematics and between constructivist and behaviourist proponents on the nature of teaching and learning mathematics. Foundationalism is represented by the logicism, formalism and intuitionism movements that were very popular in the first half of the last century. These philosophical and psychological stances, acting as macro beliefs, have in turn influenced the way students, teachers, schools and the education system in general have thought about what mathematics is and how it should be taught and learned. It was also argued that educational processes have been largely influenced by foundationalist and behaviourist ideas. Consequently, many teachers may perceive mathematics as a discipline firmly grounded in a world of rules and procedures and disembodied from personal experience. Such a view, once translated to the classroom environment, leads the way to a type of instruction that might have little to do with current constructivist oriented reform principles.

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