CONSENSUS AND COHERENCE IN MATHEMATICS – HOW CAN THEY BE EXPLAINED IN A CULTURALISTIC VIEW?

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ABSTRACT.

In the more recent philosophy of mathematics, mathematics is seen as a cultural product. In this view, it is not easy to explain the high level of coherence of mathematical theories and concepts and the wide spread consensus among mathematicians. As these phenomena are cited in orer to negate the cultural relativity of mathematics, it is worth searching for explanations. In this article, attempts at explanations are given by looking at the historical development of mathematical concepts and theories. In the last section, consequences for teaching and learning mathematics are briefly discussed.

0. Introduction: Mathematics as a cultural phenomenon

> “It is characteristic for mathematics as a scientific domain that it has disconnected from everyday life and the socio-cultural foundation which it has originally come from. Scarcely any other subject regards itself that definitively as being independent of time, values and culture. The exclusive reference on the formal and the abstractable [...] makes it difficult to discuss the relation between mathematics and cultural or social elements. Mathematics is [...] widely seen as the paradigm of the formal, the structural or the algorithmical and contrasted to culture – i.e. the historical, the dynamical, the informal or the intuitive or social: thus mathematics and culture are conceived to be extremes that are not reconcilable.“ (Schroeder 2000, p. 452, author’s translation)

This citation provides us with a very concise characterization of mathematics as it exists in the minds of many non-mathematicians, and even mathematicians and mathematics teachers. However, in the disciplines which reflect on mathematics professionally, this characterization is questioned more and more. Rejecting this old, absolutist image of mathematics, philosophers of mathematics have established humanistic or social constructivist positions, in which mathematics is understood as a cultural product (cf. Ernest 1998, Tymoczko 1985, Restivo et al. 1993 for the social-constructivist view, or White 1993 for the humanistic position). Emphasis is put on both parts: “product“ accounts for the fact that mathematics must not only be discovered but must really be created by humans. These creations always take place in a specific cultural setting, thus it is a “cultural product“.

One of the most prominent exponents of this position is Reuben Hersh. In his book “What is mathematics, really?“ (1997) he describes mathematics as a human activity:

> “From the viewpoint of philosophy mathematics must be understood as a human activity, a social phenomenon, part of human culture, historically evolved, and intelligible only in a social context.“ (Hersh 1997, p. 11)

To understand mathematics as a cultural product means to acknowledge the human influence on mathematics. Nevertheless, every individual is confronted with an apparently impartial theory. Leslie

White has pointed out this fact and given the following explanation:

> "The concept of culture clarifies the entire situation. Mathematical formulas, like other aspects of culture, do have in a sense an 'independent existence and intelligence of their own.' The English language has, in a sense, 'an independent existence of its own.' Not independent of the human species, of course, but independent of any individual or group of individuals, race or nation." (White 1947, p. 295)

Thus, there is a discrepancy between mathematics as a culturally evolved product and the fact that it faces the individual human being as an unchangeable corpus of ideas, notions, and theorems. It is a specific characteristic of mathematics that its cultural roots are hidden more successfully than those of other cultural products (like language).

The disconnection of mathematics and its cultural roots is strengthened by the high coherence of mathematical concepts and theories and a wide consensus among mathematicians. Both phenomena, coherence and consensus, shall be analysed in this paper, because they serve as main arguments opposing a culturalistic view of mathematics. By explaining them in a culturalistic framework, the author wants to contribute to the elaboration of a culturalistic philosophy of mathematics.

1. The opponents' argumentation

For several years, the constructivist sociology of knowledge has disputed about the cultural relativity of mathematics. Sociology of knowledge is founded on the thesis that scientific theories are principally underdetermined. That means, for every description of empirical data there is still a scope left for the influence of social factors in building up theories (cf. Bloor 1991). Although there is a certain agreement about the contingent character of sciences, sociologists of knowledge do not agree whether it also applies to mathematics. David Bloor emphasizes that even in mathematics, concepts and proofs are not timeless but subjects of controversies and negotiations.

An important opponent is the sociologist Bettina Heintz. She cites two indubitable phenomena to grant a "special epistemical status" to mathematics: the high coherence of mathematical concepts and theories and the wide consensus among mathematicians (Heintz 2000). Concerning coherence, she writes:

> "In contrast to other domains that decompose into separate and partly contradictory theories, mathematics is still a connected ensemble. In view of the enormous specialization [...] this coherence is not natural by any means. Mathematics is a collective product but not coordinated centrally. There is no instance which would ensure that the individual results match one another. But although mathematicians operate relatively isolated and restrict themselves to a small domain of work, connections can be discovered again and again between areas which were developed independently." (Heintz 2000, p. 19, author's translation)

For Heintz, the second phenomenon, the high consensus, is even more important:

> "Ludwig Wittgenstein says in a famous passage that in mathematics, there is hardly a controversy, and if there is one, 'it is safe to decide' (Wittgenstein 1983: 571). In contrast to other sciences, mathematics does not provide any flexibility for interpretation. The conclusions of mathematics are mandatory. Whoever follows the rules of the mathematical method will inevitably arrive at the same result." (Heintz 2000, p. 20)

Starting from this estimation, her conclusions seem stringent:

> “Modern mathematics is characterized by features that hardly leave a scope for a sociological analysis. [...] A sociological perspective is legitimate and appropriate where it concerns the reconstruction of the development which led to that epistemical structure being typical for modern mathematics and singular in its coherence and argumentative rationality.“ (Heintz 2000, p. 274/275)

If Heintz was right, mathematics would be inappropriate for a sociological analysis of social factors of influence on the scientific development, and also, even worse, totally immune against human influence. A human factor could only be detected in the historical development of mathematics, during the long processes where humans made decisions, e.g. about the style or the strictness of formal proofs. Due to its mandatory conclusions, contemporary mathematicians could only be creative in their ways of discovering theorems and proofs. In its consequence, Heintz’s thesis of the “special epistemical status“ claims that contingency in mathematics is only located in the *ways* to mathematical contents not in the *contents* itself.

We must emphatically reject this view of mathematics, because in this account, crucial areas of mathematical activities are ignored. The entire process of mathematization (i.e. the question how initial non-mathematical problems are to be translated into mathematics), the concept formation, and the development of theories as well as the criteria of relevance of research questions are missing. How are mathematical concepts found? What influences the process of concept formation? How does the community decide whether a problem is adequately mathematized? Which factors affect the development of a theory? Who decides about the relevance of questions or theorems? In all these fields, the contingent character of mathematics is much more evident than in a simple limitation on proving.

By distinguishing between “the front“ and “the back“ of mathematics as Reuben Hersh does (i.e. the way of presenting finished mathematics or the creating of mathematics, resp.), we can see directly that consensus is substantially restricted to “the front“:

> “There’s amazing consensus in mathematics as to what’s correct or accepted. But just as important is what’s interesting, important, deep or elegant. Unlike correctness, these criteria vary from person to person, speciality to speciality, decade to decade. They’re no more objective than esthetic judgments in art or music.“ (Hersh 1997, p. 39)

Though Heintz addresses these issues in her description of mathematicians’ practice, she completely neglects them in her theoretical conclusions. Just as many philosophers of mathematics, even the sociologist Heintz has taken the easy way out and concentrated her epistemological considerations exclusively on proofs. In contrast, in her practical part, she describes that proofs only appear at the end of the mathematicians’ working process. Moreover, it is not helpful to consider the distinction between the context of discovery and context of justification which is usually drawn in the philosophy of science (see e.g. Lakatos 1976), because it is founded on the same patterns that reduce mathematics to proofs and the discovery of theorems or proofs.

Rejecting Heintz’s thesis of the “special epistemical status“ of mathematics, we need a further analysis of the phenomena coherence and consensus. If we insist on the cultural relativity and the contingency of mathematical knowledge, then the coherence and the absence of real conflicts or revolutions in mathematics cannot be explained easily. In order to find an account for it, Fleck’s philosophy of science and a suited theory of mathematical development are presented in the following section.

2. Approaches to an explanation

In order to explain the phenomena coherence and consensus, let us first consider Ludwig Fleck's philosophy of science he developed in 1935 in his book "Entstehung und Entwicklung einer wissenschaftlichen Tatsache: Einführung in die Lehre vom Denkstil und vom Denkkollektiv" (English edition: Genesis and development of a scientific fact). Today, Fleck is widely recognized as a pioneer of the constructivist-relativist tendencies in the philosophy of science and of the sociologically-oriented approach to the study of the evolution of scientific and medical knowledge. He deserves this recognition and respect all the more as during his lifetime his philosophical achievement, passed completely unnoticed, until the well known historian of science Thomas Kuhn recognized Fleck's main work as a source of inspiration of his "The Structure of Scientific Revolution" (1970). As Fleck is not well known in the philosophy of mathematics, his work shall be described in some detail.

2.1 Fleck's theory of the thought-collectives and thought-styles

Fleck can be called a pioneer of culturalistic epistemologies, because he has not only considered the subject and the object of perception but he added the conditions of perception as a third important component to all epistemology. According to his theory, the conditions of perceptions are determined by the existing standards of knowledge which are not located in the individual but in the collective. He describes an

> "interaction between the perceived and the perception: the already perceived influences the way of new perceiving; perception enhances, regenerates, reinvents the perceived. Thus, perception is not an individual process of theoretical consciousness, it is the result of a social activity, because the particular standard of perception exceeds the individual's limits." (Fleck 1935, p. 54, author's translation)

Fleck outlines this idea in a case study of the changing concept of syphilis in the medical history. With this example he demonstrates, how the development of a scientific fact is influenced by the culturally determined ways of thinking.

In order to describe the intersubjective character of perception and science in general, he has developed the notions of thought-style (*Denkstil*) and thought-collective (*Denkkollektiv*). The thought-collective is defined as a community possessing a common thought-style. This style develops successively, and is connected to its history at every stage. It creates a certain definite readiness and dictates what and how the members of the thought-collective can observe. Thought-style is defined as a directed perceiving by which perceptions are processed. It is characterized by

- "common attributes of the problems of interest for the collective,
- common judgments what is considered to be evident;
- common methods as media of perceiving.
- Eventually, it is accompanied by a technical and literary style of a knowledge system."

(Fleck 1935, p. 130)

The thought-styles in which individuals think are the results of their theoretical and practical education. Passing from teacher to student, they are certain traditional values which are subjected to a specific historical development and specific sociological laws.

If a certain thought-style is sufficiently elaborated, it does not only determine the perception but also what is considered to be true. Therefore, truth is located in the intersubjective dimension:

> "The notion of truth in its classical significance, as a value independent of the subject of cognition and of social forces, compels one to accept truth as an unattainable ideal, and the history of science teaches us besides that we do not approach that ideal, even asymptotically, for the development of science is not unidirectional and does not consist only in accumulating new pieces of information, but also in overthrowing the old ones. Thus classical theories of cognition ought to distinguish between: (1) the ideal, unattainable truth, (2) the official 'truths' which 'should' somehow approach it, (3) illusions and mistakes. At the same time they have to admit that there is no general criterion of truth. [...].The epistemology which is the science of thought-styles, of their historic and sociological development, considers the truth as the up-to-date stage of changes of thought-style." (Fleck 1936, p. 111, translated by the editors)

We can learn a lot from Fleck's theory of thought-collectives and thought-styles for mathematics: In Fleck's view, the sciences are specific, thought-collective ensembles which are especially stable. If we consider mathematics to be this such a thought-style, Fleck gives us interesting answers to our question why there is this wide consensus and this high coherence. According to Fleck, these phenomena give no evidence for a "special epistemical status", but they are to be understood in correlation to the standard of a discipline:

> "The more a field of knowledge is elaborated, the more it is developed, the smaller are the differences [...] It is, as if the scope for development was shortened with the growth of nodes, as if more resistance appeared, as if the room for free thinking was restricted." (Fleck 1935, p. 110)

This idea is followed by the notion 'active and passive joining' (*Kopplung*): "every active part of knowledge corresponds to a passive joining which results mandatorily." (Fleck 1935, p. 110). The more active parts of knowledge belong to a thought-style, the more passive joinings evolve. Thus, according to Fleck, we can understand mathematics as a field of knowledge which is elaborated to a high degree. His short thesis

> "The deeper we go into a field of knowledge, the stronger it is bound to a thought-style (*Denkstilgebundenheit*)." (Fleck 1935, p. 109)

gives a good explanation for the high consensus: Mathematics is a thought-style which is well elaborated and has a long tradition. This enforces constraints of thought (*Denkzwang*). It is characterized by common attributes of the problems which are interesting to mathematicians, common assessment of values and common methods which are used for mathematical cognition.

To sum this up, we do not need the "special epistemical status" as a reason for consensus of mathematicians and coherence of mathematics. The degree of elaboration of the mathematical thought-style supplies a more convincing explanation. More than in other fields of knowledge, the active elements of mathematical knowledge produce passive joinings. Thus the evolution of the mathematical thought-style has superseded contingency to a high degree. Nevertheless, this process can never end in the complete elimination of contingency in mathematics.

In addition to these explanations given by Fleck's theory, historical investigations can help us to understand the phenomena coherence and consensus in mathematics. One important contribution has been made by Philip Kitcher in specifying some interesting characteristics of the development of mathematics.

2.2 Changes of thought-styles in mathematics – historical investigations

In his book "The nature of mathematical knowledge" (1984), Philip Kitcher compares mathematical change and scientific change. In analogy to Fleck's concept of thought-style, Kitcher defines the notion *mathematical practice*:

> "We view a mathematical practice as consisting of five components: a language, a set of accepted statements, a set of accepted reasonings, a set of questions selected as important, and a set of meta-mathematical views (including standards for proof and definition and claims about the scope and structure of mathematics)." (Kitcher 1984, p. 229)

As Fleck investigates the progress in science as transition of thought-styles, Kitcher describes mathematical change as transition from one mathematical practice to the next:

> "The problem of accounting for the growth of mathematical knowledge becomes that of understanding what makes a transition from a practice <L,M,Q,R,S> to an immediately succeeding practice <L',M',Q',R',S'> a rational transition." (Kitcher 1984, p. 229)

He shows in various historical examples that these transitions are mostly initiated by discrepancies of the components of the mathematical practices. By changing one or more components, they can by re-equilibrated. For example, theorems are retained valid by changing the language: Instead of rejecting a theorem when counter-examples were found, mathematics often restricts the concerned notions in such a way that the theorem becomes valid again (this mechanism has been described in detail in Lakatos' book "Proofs and Refutations" 1976).

> "So, where in the case of science we find the replacement of one theory by another [...], in the mathematical case there is the adjustment of language and a distinction of questions, so that the erstwhile 'rivals' can coexist with each other. Mathematical change is cumulative in a way that scientific change is not, because of the existence of a special kind of interpractice transition." (Kitcher 1984, p. 229)

Kitcher considers this mechanism for producing consensus and coherence to be characteristic for mathematics. It helps to avoid explicit discontinuities. Though singular components of the mathematical practices must be revised in order to face inconsistencies, mathematical practices are rarely completely abandoned.

In short: coherence in mathematics emerges, because mathematicians immediately search for solutions to level inconsistencies whenever they appear. In consequence, inconsistencies do not exist in mathematics, because they are not tolerated.

This thesis is supported by the work of Raymond Wilder who has analysed mathematics as a developing, cultural system (Wilder 1969, Wilder 1981). He emphasizes the cultural relativity of mathematics:

> "Because of its cultural basis, there is no such thing as the absolute in mathematics; there is only the relative." (Wilder 1981, p. 148)

Anyhow, mathematics is not arbitrary and real discontinuities can only be found on the meta-level, as he postulates in agreement with Crowe (1975):

> "Revolutions may occur in the metaphysics, symbolism and methodology of mathematics, but not in the core of mathematics." (Wilder 1981, p. 142)

On this fact, Heinz's and Wilder's positions coincide. But for Wilder, the absence of revolutions does not imply that mathematical knowledge grows cumulatively, because the patterns of development are more complicated. When he describes these patterns, he does not focus on standards of strictness for proofs nor on other aspects on the meta-level but he concentrates on central elements in mathematics: mathematical objects, concepts and theories. Over the centuries, they undergo radical changes in their meaning and their role within the theories. On the basis of historical case studies, Wilder tries to specify "laws" of this evolutionary process and figures out different characteristic mechanisms. Besides the mechanisms *abstraction* and *generalization* which have often been described, he attaches importance to *consolidation* by which he means the unifying of theories or concepts (cf. Wilder 1981, p. 87).

By using the notion *hereditary stress*, Wilder characterizes the culturally determined phenomena that initiate the evolution of theory and concepts, such as mathematical or non-mathematical problems, a changing conception of nature, discovered inconsistencies or paradoxes, growing demands for strictness etc. In addition, there is the mechanism of *diffusion* of ideas and methods by which mathematical thoughts are transferred from one domain to the next. It is an important condition for processes of consolidation.

On the whole, Wilder considers these evolutionary processes to be embedded into their cultural background. Therefore, his patterns of change heavily put the singular achievements of individual mathematicians into a culturalistic perspective (instead of celebrating them as genius discoverers, as usual in the historiography of mathematics). Starting from the observation of multiple discoveries and the "before his time"-phenomenon (i.e. concepts or ideas which fail to attract attention at their time and are rediscovered and appreciated later), Wilder describes to what degree individual thinkers rely on the cultural environment. He concludes the following:

> "The individual mathematician cannot do otherwise than preserve his contact with the mathematical culture stream; he is not only limited by the state of its development and the tools which it has devised, but he must accommodate to those concepts which have reached a state where they are ready for synthesis." (Wilder 1981, p. 145)

Thus, according to Wilder, the cultural influence on every individual thinker provides another explanation for the phenomena consensus and coherence: If all further developments in mathematics are based on the same cultural background, they coincide significantly in most cases. And when inconsistencies appear, they initiate processes of consolidation which ensure consistency again.

Paul Ernest has described these patterns of mathematical change in his "generalized logic of discovery" (built on Lakatos' "logic of discovery", Ernest 1998). He considers this process of discoveries to be a dialectical, cyclic process in which definitions, proposals and relations are discussed in the community. Along this social process, the proposals are accepted or rejected. Rejection initiates modifications of the original proposal (cf. Ernest 1998, p. 149-160). The community always acts in a scientific and epistemic cultural context, "including problems, concepts, methods, informal theories, proof criteria and paradigms, language, and metamathematical views" (Ernest 1998, p. 151).

Just as Kitcher, Wilder and Ernest emphasize the important role of well-working mechanisms that re-establish coherence in mathematics. Thus, they do not consider coherence to be a surprising phenomenon that legitimize the hypothesis of the “special epistemical status“ but to be an aim which mathematicians consequently strike for again and again.

Against the background of these historical investigations, we must pose the question for coherence in a different way: In a culturalistic perspective, we do not need to ask why mathematical theories are coherent but why it is always remade coherent and how this is possible. Without being able to solve these questions finally, some approaches can be sketched how to find an answer:

2.3 Ontological and socio-philosophical approaches

Why can coherence be re-established more easily in mathematics than in other sciences? One important reason is the ontological nature of mathematics. Whenever necessary, mathematical theories have been detached from reality. In this way, refutations of theorems can be answered by changing (mostly restricting) the concerned mathematical concepts. Hence, one reason for the possibility of coherence is the convertibility of mathematical concepts, not the a priori nature of mathematical objects. Whereas scientific concepts must correspond to reality, mathematical concepts can be detached from reality in order to avoid complete refutations. Mathematicians do prefer coherence within the theory to conformity with reality. This preference has even been formalized in Hilbert’s notion of truth as freedom of contradiction, in short: consistency (cf. e.g. Thiel 1995).

At this point, we come to the second level of explanation: Asking why mathematicians prefer coherence to conformity with reality, we must consider the prevalent view of mathematics. When a community is convinced (as platonist or others) that inconsistencies *cannot* appear, the participants will make great efforts to remove them whenever they *do* appear. Following this line, the prevalent view of mathematics has proved to be a “self-fulfilling prophecy“ again and again.

Beyond, we can find the typical human need for certainty and truth. As long as the objectivist view of mathematics is not questioned, this want can be perfectly satisfied by mathematics.

Last but not least, we must mention the mechanism of out-sourcing as a mechanism to assure coherence. Over the centuries, mathematics has outsourced many (usually applied) sub-domains when they developed their own ways of thinking and working (cf. Laugwitz 1972). By considering them not to be a part of mathematics anymore, inconsistencies or conflicts could be removed in an easy way. Even today, there are disciplines of mathematics (like scientific computing or other parts of experimental mathematics) whose standards have been removed from the widely accepted mathematical standards. There are lively discussions whether they are still parts of mathematics or whether the mathematical community can begin to accept such differences (cf. Heintz 2000).

3. Conclusions and Consequences for the Learning of Mathematics

It turned out to be characteristic for mathematics that its cultural relativity can be hidden more easily than those of other cultural achievements (like language). Because of the high coherence and the wide consensus, the human dimension and cultural origin of mathematics is obscured more successfully than in other sciences. Accounts for these phenomena can be found by launching historical investigations as the argumentations of Fleck, Wilder and Kitcher prove.

Searching for reasons of these dehumanizing mechanisms, we have found the basic human need for certainty: It is just the human desire for security that produces a dehumanized discipline leaving no place for the individual. Here, we encounter a contradiction which is not only typical for the relation of humans and mathematics but an intrinsic part of the human nature.

This tension between the human origin of mathematics and its dehumanization has important impacts on all processes of learning mathematics. It becomes obvious whenever mathematics is presented to learners as a finished theory, whenever learners cannot find an individual approach to mathematics.

Although it is very important to experience the fascinating coherence of mathematical theories (otherwise mathematics would not be that intriguing), there is hardly an individual access when mathematical theories are always presented as completely consolidated, perfect, and coherent. Then, all the difficulties and conflicts which have appeared in the process of concept formation and mathematization are hidden.

In the light of the complicated ways of mathematical concept formation, we should not be astonished that learners must also undergo these difficulties, conflicts, processes of generalization and consolidation (without postulating the misleading idea of phylogenesis equals ontogenesis!). What we need is a classroom culture in which these processes can take place instead of completely polished lessons without any difficulties or mistakes (cf. Prediger 2001). Students should have the opportunity to experience the development of mathematics with its changes and processes of consolidation, otherwise they can not acquire an adequate view of mathematics. There is no place for individual access in a perfectly presented mathematical theory.

On the whole, the problem of the subjective relevance of mathematics is one of the most difficult problems, at least in German mathematics education (cf. Bauer 1995). The prevalent change in the philosophy of mathematics brings with it important chances for mathematics education: If the human dimension of mathematics is not denied any longer, there is a better chance of allowing it to emerge more consequently in mathematics classrooms.

As early as 1983, Lerman has asked how the change in the philosophy of mathematics could change mathematics education (Lerman 1983). He contributes some convincing suggestions in the field of problem-solving, but the approach should be widened to a holistic, integrated vision of a human oriented, subjectively accessible mathematics education. I have made some first approaches in my articles "Mathematiklernen als interkulturelles Lernen. Entwurf für einen didaktischen Ansatz" and "Mathematics learning is also intercultural learning" (Prediger 2001a and 2001b). But there is still a vaste area of important work to do.

Literature

Bauer, L.: 1995, 'Objektive mathematische Stoffstruktur und Subjektivität des Mathematiklernens', in: Steiner, H.-G. & Vollrath, H.-J. (eds.): *Neue problem- und praxisbezogene Forschungsansätze*, Aulis, Köln, 9-16.

Bishop, A. J.: 1991, *Mathematical enculturation. A cultural perspective on mathematics education*, Kluwer Academic Publ., Dordrecht.

Bloor, D.: 1991, *Knowledge and social imagery* (second edition), University of Chicago Press.

Ernest, P.: 1998, *Social Constructivism as a Philosophy of Mathematics*, State University of New York Press, New York.

Fleck, L.: 1935, *Entstehung und Entwicklung einer wissenschaftlichen Tatsache: Einführung in die Lehre vom Denkstil und vom Denkkollektiv*, Benno Schwabe Co., Basel (English edition: *Genesis and Development of a Scientific Fact*, The University of Chicago Press, 1979).

Fleck, L.: 1936, 'The Problem of Epistemology', in Cohen, R.S. & Schnelle, T. (eds.): *Cognition and Fact - Materials on*

Ludwik Fleck, Reidel, Dordrecht 1986.

Heintz, B.: 2000, *Die Innenwelt der Mathematik. Zur Kultur und Praxis einer beweisenden Disziplin*, Springer, New York.

Hersh, R.: 1997, *What is Mathematics, really?* Jonathan Cape, London.

Kitcher, P.: 1984, *The nature of mathematical knowledge,* Oxford University Press, New York.

Kuhn, T.S.: 1970, *The Structure of Scientific Revolutions* (second edition), University of Chicago Press, Chicago.

Lakatos, I.: 1976, *Proofs and refutations: the logic of mathematical discovery*, Cambridge University Press.

Laugwitz, D.: 1972, ’Anwendbare Mathematik heute‘, in: Meschkowski, H. (ed.): *Grundlagen der modernen Mathematik*, Wiss. Buchgesellschaft, Darmstadt, 224-252.

Lerman, S.: 1983, ’Problem-solving or Knowledge-centred: The Influence of Philosophy on Mathematics Teaching‘, *International Journal Math. Edu. Sci. Technol.* 14(1), 59-66.

Prediger, S.: 2001a, ’Mathematiklernen als Interkulturelles Lernen. Entwurf für einen didaktischen Ansatz‘, in: *Journal für Mathematikdidaktik* 22(2), 123-144.

Prediger, S.: 2001b, ’Mathematics Learning is also Intercultural Learning‘, in: *Intercultural Education* 12 (2), 163-171.

Restivo, p. & van Bendegem, J.P. & Fischer, R. (eds.): 1993, *Math Worlds. Philosophical and Social Studies of Mathematics and Mathematics Education*, State University of New York Press, New York.

Schroeder, J.: 2000, ’Mathematik‘, in: Reich, H. & Holzbrecher, A. & Roth, H.J. (eds.): *Fachdidaktik interkulturell. Ein Handbuch*, Leske + Budrich, Opladen, 451-468.

Thiel, C.: 1995, *Philosophie und Mathematik. Eine Einführung in ihre Wechselwirkungen und in die Philosophie der Mathematik*, Wiss. Buchgesellschaft, Darmstadt.

Tymoczko, T. (ed.): 1985, *New Directions in the Philosophy of Mathematics*, Birkhäuser, Boston.

Wilder, R. L.: 1981, *Mathematics as a cultural system*, Pergamon Press, Oxford et al.

White, L. A.: 1947, ’The Locus of Mathematical Reality: An anthropological footnote‘, *Philosophy of Science* 14, 289-303.

White, A. M.(ed.): 1993, *Essays in humanistic mathematics*, MAA Notes, Mathematical Association of America, Washington, DC.