An Analysis of Errors Made in the Solution of Simple Linear Equations

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ABSTRACT

This is an investigation into the errors made by pupils when solving simple linear equations. Data was collected from a final examination and analyzed with reference to recent literature. After finding three error types identified in the literature, six new error types not discussed in the literature were identified by the researcher. This constituted the pilot study. An expanded large-scale study employing the same methodology was carried out on a sample of 246 pupils' answers to between three and six linear equation questions, both to test the robustness of these six new error types and to find examples of the Transposing error mentioned in the literature. During this process the new data, and the 166 errors identified, was analyzed and, because of the nature of the new examination questions in this large-scale study, examples of the Transposing error were found and the unidentified errors were reduced to five. Thus nine error types appear in the analysis of the large-scale study. Three of these errors, Transposing, Switching Addends, and Division, are found to account for approximately three-quarters of the total number of errors. Transposing and Switching Addends errors are classified as structural errors, and mechanisms are suggested for their commission, both from the literature and from the experiences of the researcher. An interesting finding of this study, and one that may deserve further study, is that Transposing errors occur due to what may be oversimplification of the transposing process. Also, the Switching Addends error appear more frequently in 'algebraic' than in 'arithmetical' equations, and the resulting differing success rates confirm the findings of the literature on the difference in difficulty between these two types of equation. The findings of the large-scale study also highlight the importance of subordinate skills such as division, especially among younger pupils.

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Chapter 1 – Introduction

The purpose of this inquiry is to identify, and classify by relative frequency, the most common errors made by a sample of pupils in their attempts to solve simple linear equations. The further analysis of the possible mechanisms responsible for these errors may facilitate the design and development of improved teaching strategies in the classroom.

Linear equations are a central part of any Mathematics course, especially at Lower Secondary level. Not only do they appear in their own right, but they are also an integral part of a wide variety of Algebraic, Geometric, and Trigonometric problems. In Physics especially, as well as in other sciences and social sciences, linear equations are used in the solution of problems. For example, Density is defined as Mass divided by Volume, which leads to the simple linear equation D = M / V. It is this equation, and other homogeneous equations from Physics such as R = V / I (Ohm's Law), S = D / T (speed, distance and time), and I = W / V (power equation for determination of fuses in an electrical circuit), to name but a very few, where pupils have a rich opportunity to apply their mathematical knowledge in a physical situation in the real world, in other words, to use mathematics as a tool. It follows that any attempt to analyse and improve the teaching and learning of this topic is important.

In referring to a study by Livingstone & Borko (1990) into differences in actual teaching between novice and expert teachers, Koehler & Grouws observe that novice teachers "had little knowledge of student misconceptions. Their schemata was adequate for their own understanding, but was insufficiently developed....to enable them to be responsive, flexible teachers" (1992, page 121). Armed with knowledge of errors likely to be made by pupils, novice teachers may be able to improve the quality of their teaching of mathematics not only by being aware of errors or misconceptions, but also by alerting pupils to such errors while the initial teaching is taking place. Further, this potential for improvement in teaching may not be limited to novice teachers because, as Koehler & Grouws (1992, page 121) point out, "experts become in some ways like novices when teaching new content" i.e. it is hoped that the findings of this dissertation may be helpful to novice and expert teachers alike, in improving the initial teaching of linear equations in four ways.

First, the analysis of the range of errors made by a large sample of pupils might provide information to enable the development of teaching strategies, which anticipate probable, documented difficulties. Second, this analysis could also give guidance to the teacher as to the development of the pupil's initial thought processes during the learning of the material, although the drawing of such inferences would have to be done with caution. Writers of textbooks could use third, such information in that instead of presenting only correct material, common misconceptions and errors could be discussed at relevant points in model solutions (although this would have to be done carefully to avoid confusion). Fourth, the teacher could match a certain pupil's error with one from the list in order to better understand what the pupil did wrong.

To make it more useful, an informative error analysis could include the relative frequency by type of each classified error. It may then be possible to establish a frequency hierarchy of common errors. This would be of special help to the teacher of a large class, allowing her to both anticipate the more frequent errors and to enable her to identify the most important intervention points. For instance, if error X is the most frequently-made in the study, the teacher could demonstrate to the class during the initial presentation of simple linear equations exactly what many other pupils believe (erroneously) to be the next step in the solution. Conversely, class time might not be wasted in attempting to forestall what appears to be, according to the studies, relatively infrequent errors.

This dissertation provides a brief overview of the current literature pertaining to error analysis in Algebra, and presents and discusses the findings of two analyses of errors found in linear equation questions, based on a pilot study and a large-scale study. There are already examples in the literature of large-scale studies in this particular field. It is hoped that this study will add to the results of researchers such as Kieran (1984, 1989, 1992) and Matz (1981) who identify and classify errors, by introducing relative frequency of errors made. However it will not go into the exhaustive detail as did Carry et. al. (1980), who analysed in such detail that it is not in a concise enough form to be used by the classroom teacher. The aim is that the study will build on the work and ideas of these and other researchers in such a way that the practicing teacher may quickly form an opinion on what to look out for in the teaching of this topic, while being careful not to assume unwarranted generalisability, for reasons including the fact that the study was conducted using samples of students only from Bermuda.

Chapter 2 - Review of Literature

2.0 For the sake of clarity, some findings of this study, as they relate to the literature, will be anticipated in this chapter, with full details to follow in subsequent chapters.

2.1 Error Analysis in Mathematics.

There is an extensive literature on error analysis in general, and in linear equations in particular. Hiebert & Carpenter (1992, page 88) have summed the former up in the following way: "One of the potential implications of research on students' errors is that instruction might be designed to address directly the specific deficits that the error analysis helps us to diagnose."

However, it may be worth noting that researchers are well aware of the possibility that too much can sometimes be inferred by the teacher from an analysis of errors. Bell-Gredler (1986, page 211).) points out that "Childhood logic (according to Piaget) is transductive; it does not follow the rules of deductive or inductive thinking". Kieran (1989, page 44) reinforces this idea: "With beginning algebra students on tasks involving algebraic expressions, Greeno (1982) found that students' performance appeared to be quite haphazard, for a while at least". These two statements would appear to mitigate against elevating error analysis to an exact science. In support of this, Ernest & Bayliss (1995, page 40) observe that "We can never know what a learner knows and/or understands. We can at best make inferences based on a limited number of performances and behaviours". The natural question which follows - why analyse what may be a “haphazard” situation? - may be answered by Carry et. al. :

> The reader should therefore attend to the errors themselves as much as to the classification, treating the classification as a suggestion concerning mechanism behind the error that might by explored more fully. Also, the grouping in many cases reveals a pattern of similarities among errors that may be important in identifying mechanisms. (1980, page 42)

One of the main concerns of mathematics teachers is the improvement of pupils' learning and understanding. Error analysis can prove to be a useful tool in this process as, according to Labinowicz, "a child's errors are actually natural steps to understanding" (quoted in Brooks, 1993, page 83). Perhaps this development of understanding through error-making is more common in the study of mathematics (and the sciences) than in other school subjects. Mathematical errors can provide valuable insight for the teacher into the pupils' thinking as well as for the students themselves, for "unless students are forced to confront explicitly the conflict between their misconceptions and the scientific principles they have learned, the connections may never by made; the misconceptions and the scientific principles may coexist as separate islands of knowledge" (Hiebert and Carpenter, 1992, page 89).

2.2 Procedural and Structural Views of the Nature of Algebra.

There are several features of linear equations which are not immediately apparent to the pupil and may even not be appreciated by novice teachers. Prior to the study of linear equations, solution methods for such tasks are of a procedural nature in that "arithmetic operations are carried out on numbers to yield numbers" (Kieran, 1992, page 392). An example of such a procedural approach would be found in the solution to $2x + 1 = 5$ where various numerical values for x might be tried until a solution is found. As Kieran notes, "the objects that are operated on are not the algebraic expressions, but their numerical instantiations" (1992, page 392). However, at this stage of a pupil's mathematical education, the introduction of formal algebra requires, according to Kieran (1992, page 392), a teacher-directed move away from a procedural approach towards a structural approach. Here the term structural "refers to a different set of operations that are carried out, not on numbers, but on algebraic expressions" (Kieran, 1992, page 392). For example, the solution to

$$3x + 2 = 15$$

can involve the step

$$3x + 2 - 2 = 15 - 2$$

which has nothing to do with either the final solution or any numerical instantiation. This step, which requires a grasp of the structure of an equation, must normally be understood if there is to be success in questions of the form $ax + b = cx$. It should be noted that there does exist the possibility of solving an equation of this type (defined below and referred to throughout as an ‘algebraic’ equation as opposed to an ‘arithmetical’ equation) by a procedural approach such as trial and improvement.

If, as Kieran states "the implicit objectives of school algebra are structural" (1992, page 392), then an effective teacher should have a clear understanding of the shift from the procedural towards the structural.

2.3 Arithmetical Equations and Algebraic Equations.

Filloy & Rojano (in Kieran, 1992, page 402) refer to an "arithmetical" equation having the form $ax + b = c$, an "algebraic" equation having the form $ax + b = cx$, and the transition from one to the other.

In this study, the two questions $5x - 1 = 38$ and $4x - 2 = x - 1$ (Questions 4 and 5 respectively) were suggested by Filloy and Rojano's classification of linear equations, and an analysis of these two questions in the light of their theories will be attempted.

2.4 Syntax and Semantics.

Having recognized that a particular equation is ‘algebraic’ rather than ‘arithmetical’, to use the terms as defined above, the pupil is now faced with a further choice because of the dual nature of the equals sign as detailed by Matz :

> Syntactic similarity between semantically different statements like:
>
> $$4x + 12 = 4(x + 3)$$
>
> and
>
> $$3x + 3 = 2x + 7$$
>
> presents a serious obstacle....When students first study a sample solution to a non-trivial equation, they are often still working under the preconception that each individual line is a tautology; i.e., habit leads them to 'read' across each line of the solution. As a result they expect that some transformation has turned the expression on the left-hand side of the equal sign into the expression on the right-hand side. This expectation leads to total confusion. Very poor students conclude that the equality statement embodies an obscure new transformation and so invent some explanation for the collection of lines that constitute the solution. Most realize (when the problems become nontrivial) that the line is obviously not even an equality in the original tautological sense and begin to look in another direction for correspondences. Unlike tautologies, constraint equations are not universally true statements. The equals sign in an equation does *not* connect equivalent expressions. Instead the left- and right-hand side expressions are two distinct descriptions, neither

> of which is really a property of the unknown, *but their equality constrains the unknown.* (1981, page 140)

To clarify, Matz's first equation $4x + 12 = 4(x + 3)$ is a tautology because the left hand side and the right hand side are syntactically equivalent, whereas the second equation $3x + 3 = 2x + 7$ is not a tautology but rather requires a solution using new procedures. Here Matz has described a major structural confusion which can arise in solving linear equations - a point that may be well worth making to the beginning teacher. Indeed, it may valuable to stress to students as well according to Mevarech and Vitschak (1983) "who showed that the (college) students they tested had a poor understanding of the meaning of the equals sign" (Kieran, 1992, page 399). This will serve to clarify the alternative uses of the equals sign and, if it is done in an explicit way, may help to give many pupils a more structural understanding of linear equations.

Confusion in going from arithmetic to algebra can also arise at a less esoteric level. For instance, "Matz (1982) argues that as 3¾ is to be interpreted as 3 + 3/4 it is not unreasonable that the student should interpret the algebraic expression, $3x$ as $3 + x$." (Sleeman, 1982, page 392). Thus there is plenty of room for confusion and misinterpretation in the initial study of linear equations. We shall now examine frequent errors both from the literature, denoted throughout by (Lit), and from the pilot and large-scale studies conducted by the researcher (New).

2.5 The Deletion Error (Lit).

An example of an error that has frequently been seen is simplifying, say $2yz - 2y$ to z (because $2yz – 2y$ may be wrongly equated to $2y + z - 2y = z$). That these errors are not restricted to novice algebra students is indicated by the research of Carry, Lewis, and Bernard (1980). In their study of the equation-solving processes used by college students, they found that this type of error, which they called the Deletion error, was the most prevalent one that students made when simplifying expressions at various steps in the equation-solving process. In discussing this error, Carry et. al. suggested that "some students are overgeneralising certain mathematically valid operations, arriving at a single generic deletion operation that often produces incorrect results" (Kieran, 1992, page 398). This Deletion error of Carry et. al. is corroborated by Matz (1981, page

99) who cites error number 29, in her list of thirty-three algebra errors, as of the type 3x + 5 = y + 3 yielding x + 5 = y (in which 3x - 3 has become simply x). "Matz (1979) has pointed out that in working with algebraic expressions, students tend to 'slap a veneer of names on an arithmetic base, but all the work remains in the arithmetic'." (quoted in Kieran, 1992, page 398). This is a straightforward explanation of the possible source of the Deletion error.

There may be another, more structural mechanism for the production of the Deletion error, which actually ties in with a previously-uncatalogued error, the Other Inverse Error.

2.6 The Other Inverse Error (New).

From roughly the same mechanism that may produce the Deletion error of Carry et. al. (1980) might come the Other Inverse error, as named by this researcher, in which 4x = 1 becomes x = 1 - 4. In this example the additive inverse has been employed instead of the multiplicative inverse i.e. the pupil used the Other Inverse. This may be similar to what could be happening to produce the Deletion error, where 3x - 3 becomes simply x, as the pupil may see 3 and -3 as "inverses" and so she cancels them out, exactly as she would if they were side by side and the "x" was not there. Another interpretation of a possible mechanism at work here stems from the pupil seeing 3x, as Sleeman (1982) points out, as 3 + x. That they are not inverses in this case is a fact that needs to be made clearer to beginning algebra students. In fact, both the Deletion errors and Other Inverse errors may possibly be reduced in frequency by heightened teacher emphasis in just one area i.e. that of reinforcing the idea of inverses. The Other Inverse error is not present in the literature, the main justification for its inclusion in this study being that it appears on eight occasions in the large-scale study reported here.

2.7 The Redistribution and Switching Addends Errors (both Lit).

When the pupil tries to follow the taught process of doing the same to both sides of an equation, two other errors may appear:

> The Redistribution error, where x + 37 = 150 is judged to have the same solution as x + 37 - 10 = 150 + 10, and the Switching Addends error, where x + 37 = 150 is judged to have the same solution x = 37 + 150 (Kieran, 1992, page 402).

In her study into algebra novices' views of the equivalency of equations, Kieran (1984) found that pupils who made these two errors (Redistribution and Switching Addends) were more likely to be "transposers", as opposed to "trial and improvers". Transposing, as its name suggests, is a change side-change sign technique. This suggests that the transposers may actually by less sure of the underlying structure of an equation than those used to using what might appear to be the less sophisticated trial and improvement method. We may be seeing here confusion between two points of view. That is, the pupil may know about and have, in the past, used two methods: “transposing” and “trial and improvement”.

In this case an interesting dichotomy may be explored by this observation, and emphasised by another study: Greeno (1982) pointed out that novice algebra students often cannot check a finished equation except by doing the same work all over again. They appear to be unaware that the solution may be inserted into the original question (or, for that matter, in any of the equations in the equation-solving chain). This is consistent with Bloom’s hierarchy in which checking/evaluation is viewed as a higher order skill than applying procedural knowledge. This piece of research may yield more than the obvious implication i.e. that it reinforces the view that the novice pupil lacks a true structural understanding of the linear equation. Also, from a purely psychological viewpoint, it may imply that the pupil is being expected to take two differing views during the solution of one problem, one view during solution and another during the check, so that there may be a case for treating this a mild example of cognitive dissonance in the learning of algebra i.e. having been taught to solve an equation one way (by some form of transposing), and therefore rejecting the trial and improvement technique, the pupil is then expected to re-adopt the trial technique (and consequently reject the transposing technique) to check the answer. Perhaps it could be argued that cognitive dissonance could help to explain why Greeno's pupils could not check their work by numerical instantiation having arrived at the solution through some other taught method. However, a stronger case could be made that this conflict between “transposing” and “trial and improvement” may help to explain the difficulty pupils have in shifting between two perspectives for different purposes within the same problem.

2.8 The Transposing Error (Lit).

In looking at this error, Kieran points out that the "emphasis (on symmetry) is absent in the procedure of transposing....There is some evidence to suggest that many students who use transposing are not operating on the equation as a mathematical object but rather are blindly applying the Change Side-Change Sign rule." (1992, page 400). The researcher has been aware of the "Change Side-Change Sign" approach, in which this error surfaces in equations involving a denominator such as:

$$\frac{x}{2} + 3 = 5$$

$$\Rightarrow \quad x + 3 = 10$$

The Transposing error was not detected in the pilot study reported here. This may be because of the nature of the test item: $5x + x + 2 = 3x + 12$. In this question there is no denominator.

Thus, in order to investigate the existence of the transposing error, a new test item

$$5 + \frac{x}{2} = 2$$

was employed in the large-scale study. As will be seen later in an analysis of the data, since this transposing error was made by one quarter of the fourth year, it may be inferred that the taking of this short-cut (simply taking the '2' across to yield $5 + x = 4$ and hence a solution of $x = -1$) falls within the range of what is suggested by Hiebert and Carpenter: "In school mathematics, students rely on invented strategies to solve a variety of problems."(1992, page 74). In this case, especially in the fourth year, pupils appear to be over-generalising an often-used "rule" which works perfectly well in simple equations such as

$$\frac{x}{2} = 3$$

$$\Rightarrow \quad x = 6$$

and which may be reinforced in trigonometry, during mathematics studies in the third and fourth years of secondary schooling (Bermudian syllabus), in such examples as:

$$\frac{x}{3} = \sin 60$$

$$\Rightarrow \qquad x = 3 \sin 60.$$

Here the pupil may have constructed her own mechanism, putting the denominator on the other side and thinking this can always be done because it can often be done, confirming Hiebert & Carpenter’s finding that "it is now well accepted that students construct their own mathematical knowledge rather than receiving it in finished form from the teacher or a textbook" (1992, page 74). How students construct this knowledge may thus depend on the students’ previous knowledge and experience, as stated by Ausubel: "The most important single factor influencing learning is what the learner already knows" (quoted in Orton, 1992, page 35). This brings up an important point, outlined in the discussion, which may help explain the large numbers of transposing errors made by older pupils i.e. with so many examples from trigonometry in which simple transposing does work, pupils may begin to think that it always works, no matter what equation is to be solved.

2.9 The Number Line Error (New).

In learning to solve simple linear equations, there are many subordinate skills, one of which is the ability to simplify expressions such as $-3+1$. Obtaining -4, perhaps by attempting to use the real number line or by attending to the numbers first then the sign, was an error was made sufficiently often (especially by the younger pupils) to warrant its own category, which will be called the Number Line error. Care must be taken in teaching about the real number line, for as Orton notes, “Nuffield (1969), in introducing integers, criticized such devices as temperature scales which took children some way into a study of integers but had to be rejected when it came to multiplication and division.” (1992, page 69).

The background to the teaching and learning of the topic, simple linear equations, has been widely written about. Manipulatives and algorithms can be helpful, as can an appreciation of some of the difficulties encountered by pupils. Note that for the purposes of this study, errors using manipulatives and algorithms will not be classified as errors per se.

2.10 The Use of Manipulatives.

As far as subordinate skills are concerned, another of the conceptual models utilised in this particular experiment is an appreciation of the working of a balance. A balance analogy was used by the pupils in this particular algebra course. In this system, which will be explained greater depth later, manipulatives take the pupil from their knowledge of a balance to a method of solving linear equations. Since manipulatives were used in the teaching of the pupils in this study, some analysis of their use may be warranted. One main reason for adopting manipulatives here is suggested by Davis:

> An example may make the point more clearly. Some teachers want a student to begin solving the equation 2x + 3 = x + 8 by thinking, "I'll move the 3 across the equals sign and change its sign." These teachers hope the student will then write 2x = x + 8 -3 but this approach is surely a case of regarding mathematics as a collection of small, meaningless rituals. Why "move the 3 across the equals sign?" And why on earth should such an act "change (the) sign of the 3? Certainly, if we want the student to think of mathematics as consisting of reasonable responses to reasonable challenges, it will be far better if we encourage the student to think, "I can subtract 3 from each side of that equation, without changing its truth set." If the student has seen pictures of balance scales or has worked with actual balances, there can by very straightforward imagery underlying the idea of subtracting the same thing from each side of an equation (1989, quoted in Wagner and Kieran, page 270).

However, a lack of confidence in transferring from manipulatives, metaphors and imagery to the abstraction of Algebra may lead to the production of errors, maybe of the Switching Addends type, as pupils attempt to bring a "structural" approach to the solution of equations.

Using manipulatives or mental imagery specifically in linear equations is discussed by several researchers: "According to Piaget's theory of intellectual development, two different forms of reversibility-negations (inversions) and reciprocities (compensations) - are applicable on the level of concrete operations" (Inhelder & Piaget, quoted in Adi, 1978, page 204). As Adi explains, "negation, or inversion, means reversing an action by undoing it, and reciprocity, or compensation, means reversing an action by cancelling for its effects. For example, given a balance at equilibrium when an object is placed on one of its sides disequilibrium occurs. Equilibrium is restored by either removing the object (inverting the action) or by compensating for its effect by placing an equivalent weight on the other side of the balance" (1978, page 204). This may help explain a possible mechanism for the previously mentioned redistribution error, in which a certain quantity is added to one side of the equation and subtracted from the other side, showing confusion between inversion and compensation. Again, this study is not designed to discern the probable causes of errors.

2.11 Mal-rules.

Mal-rules are rules, perhaps invented by the pupil, which appear to work but in fact work only under certain conditions. Several pieces of literature inform us that just because correct answers are given, it cannot be assumed that it is based on the desired understanding. An example is that of Erlwanger's (1973) study, in which a pupil successfully completed a self-paced course by applying many mal-rules which happened to lead to a preponderance of correct final answers. In both the pilot study and the large-scale study reported here, many of the papers had no working shown so there may have been some undetected mal-rule errors. As Steinberg et al. observed in their study (1990, page 120) "many students who are able to solve equations and perform other algebraic tasks correctly may have misconceptions or lack good understanding of algebra concepts". Thus care must be taken when ascribing good algebra methods to a pupil who solves an equation correctly but leaves out steps of reasoning.

2.12 The Use of Algorithms.

There remains one important error category: one in which the questions are not attempted. There can be many conjectures as to why a pupil writes nothing, but it may be worth noting one factor in particular.

> Matz (1979) has suggested that two kinds of processes are involved in solving first-degree equations with one unknown: "deduction" and "reduction". The deduction process involves performing the same operation on both sides; the reduction process involves replacing one expression by another equivalent expression. The following example demonstrates both processes:

$$3x + 7 = 2x$$

$$3x + 7 - 2x = 2x - 2x \quad \text{(deduction)}$$

$$x + 7 = 0 \quad \text{(reduction)}$$

(quoted in Kieran, 1989, page 48). By adding likely working, a two-step algorithm takes shape:

$$x + 7 - 7 = 0 - 7 \quad \text{(deduction)}$$

$$x = -7 \quad \text{(reduction)}$$

This deduction-reduction algorithm can be confusing to a beginning algebra student. Weaker pupils may get mixed up, perhaps not knowing which step they are working on, nor about what to do next. Indeed, they may not even have appreciated that this algorithm consists of two steps. One reason that pupils omit entire questions on equations could be that the pupil finds algorithmic work so daunting. Explicit intervention by the teacher about this algorithm may be all that is required to raise many pupils' appreciation of the structure of the equation-solving process. However, as “Hart (1981) stated: We appear to teach algorithms too soon…..and assume once taught, they are remembered. We have ample proof that they are not remembered or (are) sometimes remembered in a form that was never taught.” (Orton, 1992, page 31).

2.13 The Innate High Cognitive Demand of the Subject, Algebra.

The literature points to many complex psychological processes involved in gaining an understanding (and avoiding a misunderstanding) of the rules of algebra, and being able to operate correctly in accordance with them.

For example, "Kuchemann found that only a very small percentage of 13- to 15- year old pupils were able to consider the letter as a generalised number" and "the majority of the students he tested (73% of 13-year-olds, 59% of 14-year-olds, 53% of 15-year-olds) either treated letters as concrete objects or ignored them." (Kieran, 1992, page 396).

Also, Kieran points out that "the overall conclusion that emerges from an examination of the findings of algebra learning research is that the majority of students do not acquire any real sense of the structural aspects of algebra." (quoted in Grouws, 1992, page 412). Kieran's statement suggests that many individuals must try to understand a set of ideas and master a procedure for which there may be a poor prognosis of success. The negative ramifications in the affective domain of both teacher and pupil could further mitigate against success. Thus it may be difficult to follow the advice of Daniels and Anghileri who suggest that "classroom approaches must seek to reward effort with success by providing tasks that have the appropriate balance of challenge and support for all the individuals who will be involved."(1995, page 99). This high cognitive demand has been explained by Piaget as being part of a more advanced level of cognitive processing. It is therefore not surprising that some of these errors, like the Deletion error, have been found to persist into college, as indicated by Carry, Lewis and Bernard (1980).

2.14 Confusion Between Expressions and Equations.

Pupils frequently attempt to "solve" expressions, i.e. according to Wagner, Rachlin, and Jensen (1984), "many algebra students tried to add " = 0 " to expressions they were asked to simplify" (Kieran, 1992, page 397). One explanation may lie in the willingness of pupils to accept 'lack of closure' as suggested by Hoyles and Sutherland:

> Previous studies have found that many pupils cannot accept that an unclosed algebraic expression is an algebraic object. So, for example, pupils are unable to accept that an expression of the form x + 3 could possibly be the solution of a problem. (1992, page 216)

e.g.

$$2a + a + 3$$
$$= 3a + 3 = 0$$
$$= 3a = -3$$
$$= a = -1$$

and

$$x^2 + 5x + 6$$
$$= (x + 3)(x + 2) = 0$$
$$x = -3 \text{ or } -2$$

This kind of error indicates an absence of knowledge of the difference in meaning of an expression and an equation. Further confusion may result from the fact that the processes of simplifying expressions and solving equations are taught at roughly the same time. This has an important implication for the solution of linear equations.

In an equation of the form ax + b = cx + d, it is possible to view this as one expression, $ax + b$, being equal to another, $cx + d$. Those pupils who first isolate $ax + b$ and then have difficulty assigning meaning to $ax + b$ (as was found to be the case by Kieran (1983) noted in

Kieran, 1992, page 397) will find difficulty understanding the task of solving the equation from the very beginning. While the reasoning for this might appear tortuous, consideration of two different routes of solving an equation involving fractions may serve to illustrate such expression/equation confusion:

Method A		Method B	
	$\frac{x}{5} + 4 = 7$		$\frac{x}{5} + 4 = 7$
=>	$\frac{x + 20}{5} = 7$	=>	$\frac{x}{5} = 7 - 4$
=>	$x + 20 = 35$	=>	$\frac{x}{5} = 3$
=>	$x = 15$	=>	$x = 15$

In method A, the left side of the equation has been dealt with as an expression in its own right, and initially simplified as such without reference to the right side of the equation. In other words, a pupil using method A has dealt with an expression *within* an equation, giving further scope for confusion between expressions and equations. A direct result of this confusion can be seen in more complicated equations

$$\text{e.g.} \quad \frac{x}{2} + \frac{5x}{3} = 7$$

in which it is possible to perform the addition first to give

$$\frac{13x}{6} = 7$$

in which 6 is the lowest common denominator, or to begin solving the original equation by multiplying throughout by this lowest common denominator of 6 thus:

$$6\left(\frac{x}{2} + \frac{5x}{3}\right) = 7 \text{ X } 6$$

Various mixtures of methods A and B, leading to incorrect results, have been seen often by the researcher. While such an equation is slightly more complex than those in this study, such an example serves to demonstrate Kieran's (1992, page 401) finding that "students have generally been found to lack the ability to generate and maintain a global overview of the features of an equation that should be attended to in deciding upon the next algebraic transformation to be carried out".

Chapter 3 - The Pilot Study

3.1 Rationale.

A pilot study was undertaken before the main study reported here. The plan at the pilot study stage was to have written evidence of linear equation errors made by first year secondary school pupils.

After having read the literature, and before the pilot study was carried out, it was difficult to predict three things: how confidently documented errors could be identified and categorized from written evidence, whether the list of errors in the literature was exhaustive, and what the frequency and extent of error-making among an entire year group would be.

3.2 Methodology.

It appeared that the first two issues could be dealt with in several ways. Interviews could be held with pupils individually, in small groups, or in a class setting, or pupils' work done in a lesson could be analysed. However, it was felt that in these settings there might have been much duplication of work already researched, especially by Kieran (1992), the process would have been more demanding in terms of the researcher's time, and also there may have been little new information gathered about the extent of error-making and the relative frequency of these errors, although interviews might have pointed more clearly to possible causes of errors.

The pilot study was designed with these issues in mind, and, as will be discussed later, the scientific research paradigm was found to be the most appropriate. An examination of the three main research paradigms (the scientific, interpretative, and critical theoretic) discussed in Ernest (1994) suggests two reasons why most features of the scientific paradigm might be best suited for this experiment.

First, if one of the main goals of this research is to be fulfilled, it should seek general patterns: the relative frequency of each identified error should hopefully serve as a guide to other teachers so that they may see where some of the difficulties might lie even before starting to teach the topic. The scientific research paradigm, as defined by Ernest (1994, page 22), 'is concerned with objectivity, prediction, replicability, and the discovery of scientific generalisations or laws describing the phenomena in question....What is central to the scientific research paradigm is the search for general laws predicting future educational outcomes''.

Second, in order to achieve this, there must be data collected from many individuals, and not one or a small number, as within an 'interpretative' paradigm using interview methods, given the resources available to the researcher. However, it must be noted that by choosing to work within the scientific paradigm, perfectly sound elements of the other paradigms are not being criticised or dismissed. For example, researchers will always be interested in how individuals construct knowledge. But in the context of studying the relative frequency of errors made by a large sample of pupils, such impressions of how the knowledge is being constructed by each individual student, based solely on his or her written answers, are probably beyond the scope of the investigation. Even if a study is undertaken within the scientific paradigm, it can make recommendations consistent with a constructivist view of learning, i.e. those normally associated with the interpretative paradigm.

3.3 Experimental Design.

The pilot study was designed to be carried out without the pupils' knowledge that a research project was being undertaken in order to minimize the effect of the affective domain. This was done by making the linear equation question compulsory at the beginning of the end-of-year examination which was written simultaneously by all 87 First Year pupils. By placing the test item at the beginning, it was hoped that every pupil would at least attempt it. There were about five steps needed to solve the item, and the errors produced could be expected to be manageable in number, facilitating a focused analysis.

A non-integer solution was chosen to try to avoid a situation in which the pupil might get the right answer by trial and improvement. Such guessing could have led to working being omitted and unsystematic skipping of procedures, making error analysis more difficult. It should be noted that trial and improvement is not being treated as an inferior method. Indeed, as will be discussed later, use of this method demonstrates an appreciation of the meaning of the problem and, moreover, an understanding of the significance of the solution.

Contained within the solution methods of linear equations are many concepts and procedures, some of which have been mentioned already: the meaning and gathering of 'like' terms (see the deletion error in Carry et.al. (1980)); viewing the equal sign in perhaps a new way (Matz (1981), page 140); and applying, one or more times, a simple algorithmic structure, described by Matz (1979) as "deductions and reductions" (Kieran, 1989, page 48). As there is already much research on this topic, a major aim of the study was to compare the findings with other experiments from the literature. Also, on the basis of looking at pupils' work in great detail, perhaps it might be possible to specify teaching objectives within the topic more accurately i.e. plan diagnostic teaching to accommodate and rectify the errors and misconceptions.

Twenty-five of the eighty-seven papers met the criteria of both having an incorrect answer to the linear equation question, and showing some working, i.e. there were twenty five identifiable errors, several of which corresponded with known errors from the literature. Of the other 62 papers, 15 wrote no working at all, (writing only the wrong answer or leaving the question blank), and 47 obtained the correct answer.

The equation to be solved was $5x + x + 2 = 3x + 12$. In this equation there is more than one line of working. This leads immediately to a difficulty in error classification: how do we classify as an error something which has been done correctly at the beginning of the problem and incorrectly at the end? Do we view this as inconsistency, the product of distraction, or do we simply accept that if a pupil commits an error once, that does not mean that he or she will always commit it. It should be noted that this is a different kind of error from all the others mentioned here because of this committal/non-committal aspect within the same problem.

3.4 The Exhaustion Error (New).

A working definition of the exhaustion error is any error which is made towards the end of a particular problem despite the fact that this same error was not made at the beginning of the problem, even though the opportunity existed. It was felt by the researcher that this error might occur frequently enough to deserve to be an error category in its own right. An example may serve to clarify this error type:

$$\frac{x + 3}{x + 2} + \frac{x + 4}{x + 1} = \frac{(x + 3)(x + 1)}{(x + 2)(x + 1)} + \frac{(x + 4)(x + 2)}{(x + 2)(x + 1)}$$

$$= \frac{x^2 + 4x + 3 + x^2 + 6x + 8}{(x + 2)(x + 1)} = \frac{2x^2 + 10x + 11}{x^2 + 3x + 2}$$

$$= \frac{10 + 11}{3} \quad \text{(errors)} \quad = 7$$

Here the error was probably made by "cancelling" the x^2's, the x's, and the 2's. It is important to note that the error was not made at the beginning of the problem, where an opportunity for its commission existed. At a certain point the pupil makes the error for what may be one or more of the following reasons. From her experience she knows that the problem cannot be solved in one or two steps (pupils have often given me this "reasoning" verbally) or she has remembered the correct first step because she has often seen that that particular process is always used in this type of example. When the pupil gets to the middle of the problem she often simply uses her own construction (simple cancelling, as with a single value in both the numerator and denominator, and not cancelling of complete factors within a completely factorized expression) to "end" the question. This misapplication may be a good example of the individuality of construction of knowledge: at this point in the middle of the problem there are few alternatives for the pupil, misapply previous knowledge or find a correct method. This particular misapplication of the cancelling rule is tempting, especially in light of the fact that they have been told to simplify, that not only (in my experience) do about half of some entire classes do it, but that having done it, they appear to see absolutely nothing wrong with this reasoning. These pupils are applying algebraic rules in an inconsistent fashion. Of course, this is the algebra teacher's main problem: to have the pupils understand that the entire subject hangs together as a cohesive whole. For too many pupils it remains a phenomenological construct devoid of meaning, in which rules are applied at random in an attempt to please the teacher. This error, of carrying out a process correctly then later incorrectly, will be called the Exhaustion error in this paper and will join the other errors: Deletion, Switching Addends, Redistribution, and Transposing as suggested in studies by Carry, Lewis & Bernard (1980) and Kieran (1982, 1984) as mentioned in Grouws (1992, pages 398, 400, and 402).

3.5 The Lack of Transposing Errors (Lit).

It is interesting to note that there were no Transposing errors in the pilot study. This may be because of the emphasis placed during teaching on the structural approach and the need for symmetry at each step. Another possible explanation could be that the pupils have not been dealing with the topic for long enough to start constructing their own (erroneous) transposing approach as is commonly seen among older pupils. However, the most likely explanations of the absence of transposing errors are that the format of the question does not lend itself to this kind of error because it has no denominator, and that the pilot study has a small sample size.

3.6 Categorising the Errors.

The papers were sorted according to the three error types already mentioned in the literature: Deletion, Redistribution, and Switching Addends, plus the newly defined Exhaustion error. There were also a number of papers left over which did not fall into any of these four categories. These papers were examined to see if they had any errors in common. This then led the researcher to identify five new types of error: Omissions (i.e. where one term in a long string has been dropped for no apparent reason), Misuse of Additives Inverse, Inability to Isolate Variable, Division Error, and Absence of Structure. In this way every paper was accounted for, except one which defied analysis. (For completeness, it is important to note that 15 questions were left blank and so there was no way of knowing if any errors would have been made). These types of error are exemplified in Table I.

Table I

Error Types in the Pilot Study

Sample Question: 5x + x + 2 = 3x + 12

Deletion error (Lit)	Redistribution error (Lit)	Switching Addends error (Lit)
3x - 3 + 2 = 12 - 3	5x + 2 - 2x = 2x + 12 - 2x	6x + 2 = 3x +12
x + 2 = 9	3x + 2 + 2 = 12 -2	9x = 14

Exhaustion Error (New)	Omissions error (New)	Misuse of Additive Inverse error (New)
5x + x + 2 = 3x +12	5x + x + 2 = 3x + 12	5x + x + 2 = 3x + 12
6x + 2 - 2 = 3x + 12 - 2	6x + 2 - 2 = 3x + 12	6x + 2 + 2 = 3x + 12 + 2
6x = 3x + 10		
6x + 3x = 10		

Inability to Isolate Variable error (New)	Division error (New)	Absence of Structure error (New)
3x = 10	3x = 10	5x + x + 2 = 3x + 12
blank	x = 3.1	3 + 2 = 3x - 8

Note that (Lit) indicates an error found in the literature specified above.

Table II

Frequency of Errors in the Pilot Study

Deletion (Lit)	3	
Redistribution (Lit)		1
Switching Addends (Lit)		2
Exhaustion		3
Omissions		3
Misuse of Additive Inverse	2	
Inability to Isolate Variable	3	
Division Error		5
Absence of Structure	3	

As can be seen from the Table II, the first three types of error as described in the literature occurred a total of six times.

3.7 The Misuse of Additive Inverse Error.

The Misuse of Additive Inverse error, as illustrated below, is very common when the pupils are first introduced to the topic, but has persisted here only twice through to the final examination.

$$x + 6 = 9$$
$$x + 6 + 6 = 9 + 6$$
$$x = 15$$

This error may show some understanding of the balance analogy, in that the pupil has done the same to both sides. However, it is evident that the pupil has not grasped the real structural aspect of the problem - she may see the problem as one in which terms are 'eliminated' from one side and 'turn up' on the other. It could be argued that the error may persist as it may not be producing the discomfort associated with the exposure to new material which conflicts with existing 'knowledge' - a condition deemed necessary for learning to occur according Piagetian theory. In order for this pupil to recognize that there is a conflict, she must see that her methods are not producing solutions which will satisfy the original equation. In other words, she must improve her structural understanding of linear equations before she can see her errors on her own.

On examination of the Misuse of Additive Inverse error itself, it was found to be almost synonymous with the Switching Addends error since the outcome is the same. The only slight difference is that in the Misuse of Additive Inverse error it is shown explicitly that the pupil may think that the 'opposite' of +6 is +6 (referring to the example above). In the Switching Addends error any reasoning is hidden. This similarity between the two errors led to their amalgamation in the large-scale study.

3.8 The Division Error.

The Division error may seem a relatively unimportant one in the context of solving linear equations. However, until such division is mastered, pupils without calculators may often be unable to find non-integer solutions to linear equations.

3.9 The Inability to Isolate the Variable Error.

The Inability to Isolate the Variable error appeared at the same point in all three cases. The pupils correctly reached the $3x = 10$ step but stopped there. This error may arise because the pupil does not realise what must be done i.e. even towards the end of the question, we must do the same to both sides. Perhaps this is an indication that the concrete analogy with the balance has not persisted to the end of the question and the case could be made to put this error in the same category as Exhaustion, or that the pupil does not see that not only may one add and subtract on both sides but that one may also divide. A second explanation could be that the pupil does not appreciate that 3x means '3 times x'. This inability to even identify the operation of multiplication could explain why the pupil cannot do the opposite operation (division) to both sides. A third explanation is that the pupil does know that 3x means ‘3 times x’, but does not realise that it may be appropriate to divide both sides by 3 at this point. A fourth explanation could be that the pupil fears making a Division error, thus leaving the question unfinished. A fifth explanation is that the pupil expects an integral solution from previous experience and hence is confused. In view of these diverse explanations, the Inability to Isolate the Variable error and the Division error are amalgamated in the large-scale study.

3.10 The Absence of Structure Error.

The cases of Absence of Structure error were those in which the pupil demonstrated a lack of understanding of 'doing the same to both sides'.

Chapter 4 - The Large-Scale Study Methodology

4.1 Experimental Design.

In order to test more fully the hypotheses that (a) errors can be grouped into this set of types and (b) these error types can be put in order of relative frequency, the sample size was greatly expanded for the main study. Eighty-seven pupils who formed one year group each answered one question in the pilot study. However, the whole school, with the exception of the fifth years (who write external exams instead of school exams at the end of the school year), participated in the large-scale study. The pilot study design and rationale were evaluated after analysis of the data and several improvements were suggested e.g. some error types were amalgamated to facilitate analysis.

The subjects of the large-scale study were a group of 246 pupils. They comprised the entire school from first to fourth year, with approximately equal numbers in each of the four year groups. Therefore there was no selective sampling, in that no one in the school who was taking exams was omitted. This was felt to be an important feature of the study because of the limitations of selective sampling, given the relatively small school size. Since the pupils all took the examination at the end of the school year, they had been solving such equations for between one and four school years. The subjects' ages ranged from twelve to sixteen years old and they attend a selective, academic, co-educational secondary school in Bermuda. The school's population is comprised of approximately the 'top' 40% of the general population of Bermuda as identified by two hours of Maths and English examinations given at age eleven to approximately 80% of the population (the other 20% of the school population being in private schools). This method of selection combines the Maths and English scores and so allows for a wide range of abilities in Maths (and English) i.e. a pupil my be accepted to the school with an outstanding English score and a Mathematics score which places her in the third quartile of those who sat the exam for eleven year olds. The children are from all socio-economic backgrounds and nearly all have lived in Bermuda since birth. Nevertheless the sample is skewed upwards in its representation of the range of mathematics achievement levels.

The process of collecting data was not complicated: it was to be pure survey research, based on pupils' answers in an examination, with no special teaching input or emphasis. The possible influences of the affective domain were thus intentionally minimized. Clearly there may have been some element of stress, especially when one considers that the final exams, counting as they do for 60% of the year mark in each subject, are taken seriously in the school by both teachers and students.

4.2 The Questions and their Design.

The pilot study question $5x + x + 2 = 3x + 12$ was improved upon, because $5x + x$ should be collected at some point. This collection adds, for the pupil, a small amount of work which is irrelevant to this particular study. The questions in the expanded study are:

Question 1	$7x = 28$	First year pupils only
Question 2.	$x - 1 = -3$	First year pupils only
Question 3	$\frac{10}{x} = 5$	First year pupils only
Question 4.	$5x - 1 = 38$	All pupils
Question 5.	$4x - 2 = x - 1$	All pupils
Question 6	$5 + \frac{x}{2} = 2$	All pupils

It was felt appropriate that the first year pupils have more questions on this topic in the final exam for two reasons. The topic is introduced and given a large block of time in first year, and the first year pupils are given some less intimidating questions, allowing them more opportunity to get some credit for their understanding of equations.

Question 1. $7x = 28$

This is a lead-in question designed to make pupils feel comfortable and confident. The only obvious anticipated errors are Division errors (not being able to apply division correctly) and the Other Inverse error i.e. $x = 28 - 7$.

Question 2. $x - 1 = -3$

This is designed to see how many pupils can make a single application of the 'deduction - reduction' algorithm. Note that guessing the correct solution is difficult here because of the location of the solution in the negative segment of the number line. As will be seen, it may be possible to attribute specific errors in the case of the incorrect solution $x = -4$ with the help of other answers on the exam, but it must be noted that this is an ambiguous situation in which the commission of an error could be viewed as stemming from one of several mechanisms.

Question 3. $\frac{10}{x} = 5$

This is a multistep question if solved using the 'deduction – reduction' algorithm. Trial and improvement using substitution of values for 'x' probably takes less time, especially in view of the fact that the solution is a positive integer.

Questions 1 to 3 were not originally designed to be part of an error study - they are solely test items of moderate difficulty for first years to ascertain their achievement in solving a range of simple linear equations.

Question 4. $5x - 1 = 38$ (the first item for years 2 to 4)

This is an example of an 'arithmetical' equation mentioned by several authors, including Filloy and Rojano cited in Kieran (1992, page 402). The 'deduction – reduction' algorithm may be used twice or this question may be solved with difficulty (because of the non-integer correct solution) by trial and improvement using substitution of values for 'x' from the beginning. It is therefore designed to steer the pupil into taking a structural direction (e.g. using the algorithm), with a lot of working being shown, hence making it possible to identify the type of any error. However, on the right hand side there is a number, which is often viewed as an answer while numbers are successively substituted in place of x on the left hand side. As will be seen in the analysis, Question 4 did indeed provide its share of errors.

Question 5. $4x - 2 = x - 1$

This is an example of an ‘algebraic’ equation (see Filloy and Rojano in Kieran, 1992, page 402). Guessing would be tedious in the extreme. Many scripts showed little working and much confusion. This is probably due in part to the fact that, whereas in Question 4 the number 38 can be used in every attempt to solve it by trial and improvement, in this "algebraic" type of equation, the right hand side changes with each new numerical instantiation. The analysis of errors and performance in this question confirms how difficult pupils may find "algebraic" equations.

Question 6. $5 + \frac{x}{2} = 2$

Since the Transposing error was not seen in the pilot study, despite its being mentioned in the literature by Kieran (1992, page 400), this test item was designed, in part, to elicit the error from pupils who were unsure of a correct method. Deliberate adjacent positioning of the 2's here on the part of the researcher may have served to prompt the "change side - change sign" rule oversimplification more than if the problem had read

$$\frac{x}{2} + 5 = 2$$

leading to (perhaps) $x + 10 = 2$

which may be more readily felt by the pupil to be an obvious error, leading to a re-examination of the question.

In summary, these six questions were designed both to test pupils in an examination setting and to make possible an identification and analysis of errors.

4.3 The Determination of Error Types in the Large-scale Study.

Both the literature and the analysis of exam scripts during the gathering of data in the pilot study were used to define the error categories in the large-scale study in the following ways. The Deletion, Redistribution, Switching Addends and Transposing error types were all retained because of their presence in the literature. The Exhaustion Error did not survive as an error type because there were only two of them found in the large-scale study. In both cases the error made was judged to be of the Switching Addends error type. The Omissions error type was retained because of its occurrence in the pilot study. As has already been mentioned, was a condensing of four error types in the pilot study into two pairs: Switching Addends (including Misuse of Additive Inverse), and Division (including Inability to Isolate Variable). The Absence of Structure error could not be amalgamated with any other type, and it was felt to be important as it might be possible to link it with some form of structural confusion, either from the uses of the equals sign or the application of the algorithm. This leaves the Other Inverse and Number Line errors to complete the list of nine identifiable error types, as seen in Table III.

Table III

Error Types in the Large-Scale Study

Deletion error (Lit) 3x - 3 + 2 = 12 - 3 x + 2 = 9	Redistribution error (Lit) 5x + 2 - 2x = 2x + 12 - 2x 3x + 2 + 2 = 12 -2	Switching Addends error (Lit) 6x + 2 = 3x +12 9x = 14
Transposing error (Lit) $5 + \frac{x}{2} = 2$ 5 + x = 4	Omissions error (New) 5x + x + 2 = 3x + 12 6x + 2 - 2 = 3x + 12	Other Inverse error (New) 4x = 1 x = 1 – 4
Number Line Error (New) -3 + 1 = - 4	Division error (New) 3x = 10 x = 3.1	Absence of Structure error (New) 5x + x + 2 = 3x + 12 3 + 2 = 3x - 8

Note that (Lit) indicates an error found in the literature specified above.

Chapter 5 – Results

Table IV

Frequency table of errors in the large-scale study of errors made in linear equations.

Error type	First Year 66 pupils Q1 to Q6	Second Year 53 pupils Q4 to Q6	Third Year 68 pupils Q4 to Q6	Fourth year 59 pupils Q4 to Q6	Totals for each error Q4 to Q6	Totals for each error Q1 to Q6
Deletion (Lit)	--------	1(Q4) 2(Q5)	1(Q6)	1(Q5)	5	5
Redistribution (Lit)	5(Q2)	1(Q4)	---------	--------	1	6
Switching Addends (Lit)	5(Q2) 7(Q5) 1(Q6)	3(Q4) 3(Q5) 1(Q6)	2(Q4) 4(Q5) 1(Q6)	10(Q5) 3(Q6)	35	40
Transposing (Lit)	9(Q6)	9(Q6)	7(Q6)	15(Q6)	40	40
Omissions (New)	2(Q4)	1(Q5)	1(Q5) 3(Q6)	1(Q4)	8	8
Other Inverse (New)	1(Q4) 1(Q5)	3(Q4) 1(Q6)	1(Q6)	1(Q5)	8	8
Number Line (New)	7(Q2) 3(Q5)	1(Q5) 1(Q6)	1(Q5) 1(Q6)	4(Q5)	11	18
Division (New)	15(Q4) 2(Q5)	5(Q4) 3(Q5)	4(Q4) 1(Q5)	3(Q4) 1(Q5)	34	34
Absence of Structure (New)	3(Q4) 1(Q5)	-------	--------	3(Q5)	7	7
Identified errors in Q4 to Q6 by year {Q1 to Q3 in 1st yr}	45 {17}	35	27	42	149	166
Mean number of correct responses*	0.83	1.41	2.06	1.85		

* Questions 4, 5 and 6 were each allotted one mark for the purposes of analysis in this study, then these scores were totaled.

Note that 2(Q5) means that the particular error occurred twice in Question 5.

5.1 Notes on the Data.

The results of the pilot and large-scale studies are tabulated in the Table II in Chapter 3 and in Table IV above.

Two important points must be stressed about the data in Table IV. First, these are the errors as classified by the researcher according to the planned scheme developed and adopted by the researcher. Second, whenever errors appear in a pupil's working, they are put into one of the nine error categories.

To give an example, the first year pupils achieved a mean of 0.83 correct responses out of a maximum of three (there being a score of one given for each correct answer to each of the questions numbered 4, 5 and 6 on the examination). However, the first year pupils made 45 identifiable errors, compared with 42 errors made by the fourth year pupils, whose mean number of correct responses was more than double at 1.85. Since the year group sizes are similar at 66 pupils and 59 pupils, this discrepancy is accounted for by the fact that many more first than fourth year pupils wrote nothing in response, as can be seen in Table VI in Chapter 6.17.

5.2 The Errors Made, in Decreasing Order of Frequency, in Questions 4 to 6.

The Transposing error was made 40 times, out of a total of 149 errors, and was the most frequently detected. Although it occurred only 7 times in the first year, this may have been because it occurred in what was too daunting a question. In support of this, 27 of the 66 first year pupils either gave an incorrect answer without any working or left it blank. The Transposing error occurred 15 times in the fourth year i.e. approximately twice as often as in each other year group, a finding which will be examined in the next chapter.

The Switching Addends error was the second most frequent with 35 occurrences, again with the fourth year group accounting for more than any of the other years.

The Division error occurred 34 times, despite the fact that this error could not easily be made in Question 6 because of the unitary coefficient of x. The first years accounted for half of all the Division errors made.

Again, it is stressed that these are frequencies of errors actually made and observed, with no inferences being drawn from blank responses.

The above three were the most frequently made errors, accounting for 73% of all observed errors.

The other errors were, in decreasing order, Number Line, Omissions, Other Inverse, Absence of Structure, Deletion, and Redistribution errors.

5.3 Statistics Related to the Frequency Table.

The identified errors made in Questions 4 to 6 on the exam were totaled according to year group. The first years made the most with 45 errors, closely followed by the fourth years who made 42. Again it must be stressed that errors must be made before they can be identified e.g. many first years omitted working altogether, but this fact does not show up in Table IV. In an attempt to get a clearer picture of how these errors relate to overall performance the "mean number of correct responses" statistic was included. The fourth years scored more than twice as highly as the first years on average (1.85 versus 0.83), with the highest average (2.06) being achieved by the third years.

Chapter 6 - Discussion of Results

6.1 The Applicability of the Results to Other Populations.

Throughout this discussion it must be remembered that this sample of pupils is not representative of the general population, but, as has been mentioned, is a certain subset of the top three quarters of the full mathematical achievement range, as represented by the population of one school. The same applies to references to the literature: a particular study will almost invariably have been done with students of different abilities from those in this study. Thus, in general, it would be impossible to translate these results when trying to predict the totals of each type of error about to be made by another group of pupils. Rather, in searching for a crude frequency distribution in Table IV, it is hoped that there would be benefits for the classroom teacher in studying these results. Obviously, teaching styles vary, and this may also account for variations in results. For example, at the school in which this study was done, great use is made of manipulatives by all mathematics teachers. Using white dice to represent the constants 1 to 6, black dice to represent the constants -1 to -6, red plastic pawns to represent each variable 'x', and blue plastic pawns to represent each variable 'y', the mathematics teachers showed, for instance, $2x - 5$ as 2 red discs and a '5' on a black die. This was done in an attempt to show that constants and variables are 'unlike' terms and therefore cannot be added or subtracted. This may help to account for the scarcity of errors of the Deletion type e.g. $2x - 2 = x$, made by this particular population. Given that the deletion error is prevalent in the literature, this might have suggested that the error would appear more often than it did in this large-scale study. Confusing 2x with 2 does not appear to be a problem for this particular sample of pupils. A joint effect may be at work. Higher-achieving pupils who are given access to manipulatives might be likely to see, and remember, the difference between 2x and 2 because 2x has been represented by two pawns, and 2 by the number 2 on a die. However, as has been noted by Daniels & Anghileri (1995, page 42) when referring to manipulatives, at the other end of the ability scale the pupils "found the blocks.....as abstract, as disconnected from reality, mysterious, arbitrary, and capricious as the numbers these blocks were supposed to bring to life", which might account for the 12 blank responses by first years in Question 5.

6.2 The Transposing Error (Lit) Frequency.

This was the most frequent error observed, this despite the fact that, because of its structure, this error could be made in only one of the questions. It has been observed over many years by the researcher that the change side - change sign 'rule' has an immediate and powerful appeal to less successful algebra students because of its simplicity.

One explanation for why pupils find it easier to Transpose than ‘do the same to both sides’ may be that pupils do not have to *find* a multiplicative inverse then remember to do the same to both sides of the equation, as they would have to do in the more formal method. With Transposing, they are simply moving *existing* numbers (or letters).

However, it is interesting to note that Transposing is often used by pupils who have only had this researcher as a mathematics teacher at this school and who have never, at least in my classes, been taught this method. This makes it either a good example of Hiebert & Carpenter's "invented strategy" (1992, page 74), or quite possibly, a strategy taught by their parents or by peers from another teacher’s class. Indeed, seeing a process in which the denominator on one side of the equals sign simply appears to join the numerator on the other side e.g.

$$\text{Density} = \frac{\text{Mass}}{\text{Vol.}} \qquad \text{and} \qquad \text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

$$\Rightarrow \ \text{Density X Vol} = \text{Mass} \qquad \Rightarrow \ \text{Speed X Time} = \text{Distance}$$

may lead many pupils to think that such Transposing is always mathematically correct, irrespective of the shape of the equation. Remedying the transposing error is difficult because simple transposing, as outlined in the two examples above, often works.

6.3 The Transposing Error (Lit) and the Fourth Years.

By designing Q6 so that simple 'transposing' does not yield the correct answer, Transposing errors were revealed, especially in the papers of the fourth year pupils:

$$5 + \frac{x}{2} = 2$$

$$\Rightarrow \quad 5 + x = 4$$

It is interesting to note here that, even though this is an "arithmetical" equation (to use Filloy & Rojano's terminology in Kieran (1992)) with an integer solution, pupils appear to have been frequently preferred to try a structural approach using their transposing mal-rule. This may demonstrate that using an over-simplified rule is preferred over trial and improvement, especially in this question as it is "arithmetical" in nature. In this case using the over-simplified rule certainly is less work, but this is hardly surprising since mal-rules may be invented by pupils, at least partly, to save time and energy. In other words, this situation in which some fourth year pupils are inventing a convenient mal-rule may appear to demonstrate some pupils' self-constructed approach. This may spring directly from many of the equations from Physics as noted above e.g. Resistance = Voltage / Current. Also, the fact that this is taking place more in fourth year lends weight to the idea that it takes time for a pupil to construct his own knowledge or methods. However, there are several, less complicated, alternatives to this constructivist interpretation.

First, that the fourth year group may, as a whole, be weaker at solving linear equations (or maybe weaker at mathematics) than the other year groups. The four year groups are distinct and comparable only to the extent that they attend the same school and have the same teachers.

Second, that the fourth year may not have received as much linear equation instruction (and certainly no manipulative work with the balance) during the past year as the other classes may have. This is certainly the case when comparing the first and fourth years.

Further, large numbers of the fourth year did not try to check the answer by substituting into the question and realize that something had gone wrong. This may confirm the findings of Greeno (1982) summarized in Kieran (1992, page 403), that it does not always occur to pupils that the correct solution to a simple linear equation must be able to be substituted into the original equation so that the left side equals the right side. There is, however, the possibility that many of the fourth years did know their answer to Question 6 was wrong, but could not fix it or felt they had little time to fix it. The examination setting for the experiment may have contributed to this, with pupils feeling under pressure to complete an exam in a limited amount of time.

6.4 The Division Error.

This was seen 17, 8, 5, and 4 times respectively in the first to fourth year groups. It should be reiterated that this error type was an amalgamation of the division error e.g. 5x = 39 => x = 7.4, and the inability to isolate variable error, characterised by nothing being written after 5x = 39. Twenty six of the thirty four errors were division errors (according to the original, restricted definition). The other eight were of the inability to isolate variable error, which, as has already been noted, may have at least four different possible explanations for its commission. Whatever the reason, the fact remains that the equation 5x = 39 could not be solved by one quarter of the pupils in first year. Arithmetically, this should be within the capability of this particular sample of pupils. This demonstrates that, in terms of percentage of correct answers, a low percentage may not be indicative of poor algebraic skills alone. Rather, it may be indicative of poor subordinate skills e.g. the ability to divide integers to give non-integer answers.

6.5 Subordinate Arithmetic Skills.

It may be tempting for a teacher to conclude that a pupil's score on a linear equations test is indicative of the pupil's understanding of linear equations. However, close analysis of pupils' work can sometimes reveal weak subordinate skills. There are many subordinate skills used in the solution of linear equations, and the application of these skills may have as much influence on the final success rate as does any understanding of linear equations or their structure.

This was observed in the large-scale study among the first year pupils, with division identified as one of these weak subordinate skills. Many of the first year pupils have, in going through a particular question, performed the algebraic manipulation successfully, only to commit an error, in this case division, in an operation with which they had several years' experience. Further, given the ability of this particular sample of pupils, the operation should have been mastered prior to the introduction of linear equations. On the subject of subordinate skills, the case may be made that the only errors which should be analysed in regard to linear equation questions are those made in algebraic manipulation as it relates to the structural aspects of the solution of such equations. However, the fact remains that poor subordinate skills in one area can be a contributory factor in a pupil's lack of success in another area, and so a stronger case may be made for the inclusion of all types of errors, especially those made while using subordinate skills.

As far as solving linear equations is concerned, not only should the four basic operations have been mastered, but they should have been mastered with directed numbers, otherwise the pupil has a poor prognosis for success. For example, if an 'algebraic' equation of the form $ax + b = cx + d$ is to be solved, then at some point the additive inverse of either ax or cx must be found, in order to collect 'like' terms and therefore isolate the variable. Without mastery of directed numbers, luck may be playing the largest part in finding $-ax$ or $-cx$. Such mastery takes time to achieve in any large class.

6.6 The Switching Addends Error (Lit).

As has already been mentioned, when this error is observed it may show a lack of understanding of the structure of a linear equation (or perhaps simple carelessness). A typical example of the Switching Addends error from pupils' work is

$$4x - 2 = x - 1$$

$$\Rightarrow \quad 4x + x = -1 - 2$$

This error was found to be persistent in the large-scale study, occurring 7, 7, 7, and 12 times in first to fourth year respectively. Indeed, the Switching Addends error frequency pattern is not unlike that of another error associated with a lack of understanding of structure i.e. the Transposing error. Whereas the Transposing error may spring from a pupil constructing a method in accordance with what appears to work often, there is a possibility that there may be different mechanisms at work during the production of the Switching Addends error.

In committing the Switching Addends error, the pupil may be attempting to give structure to the solution of a linear equation question in an examination (even, in an extreme case, if only in the sense that it contributes more working in accordance with the exam instruction that marks may not be gained unless full working is shown). If it is indeed, for whatever reason, an attempt to provide structure, then perhaps this reasoning

$$x + 3 = 5$$

$$\Rightarrow \quad x = 5 + 3$$

could, because of its attempt at being more structural, be regarded as a mental step in the right direction. Thus the Switch Addends error might be perceived in a positive light, as one being made by a pupil approaching understanding of linear equations.

Alternatively, it could also be argued that by writing

$$x + 3 = 5$$

$$\Rightarrow \quad x = 5 + 3$$

the pupil may be simply trying to follow the teacher's advice (in trying to prevent the pupil from making a Transposing error) of "Don't just change side-change sign", but is confused about how to do this, and in an honest attempt to follow a piece of negative advice, the pupil changes side but not sign. Such tortuous reasoning may seem improbable, but mathematics teachers are used to a wide variety of mal-rules being given back to them as their own e.g. two minuses make a plus. This should be viewed in a positive way i.e. as part of a pupil's genuine attempt to construct knowledge.

It is interesting to observe, from Table IV, that the Switching Addends error, was found 24 times in Question 5 (the 'algebraic' equation) as, typically,

$$4x - 2 = x - 1$$

$$\Rightarrow \quad 4x + x = -1 - 2.$$

This may demonstrate that a pupil is reluctant to employ the rule of 'doing the same to both sides' by taking 'x' from both sides, perhaps for the following reason: if the value of 'x' is unknown, how can it be taken from both sides? From the results of this study, there is little doubt that the fourth year pupils found the 'algebraic' equation type more difficult than the 'arithmetical' type. This may be shown by the important fact that not a single Switching Addends error was made by fourth years in Question 4, an 'arithmetical' equation, compared with 10 Switching Addends errors made by the same 59 pupils in Question 5, the 'algebraic' equation. In committing the Switching Addends error there seemed to be a genuine attempt by many pupils to solve by structural rather than procedural means. However, the 'algebraic' nature of the equation in Question 5 proved too difficult for the fourth years, one third of whom made errors in this question.

These findings, when coupled with the frequency of Transposing errors from Table IV, mean that there were 28 Switching Addends or Transposing errors made in 118 questions answered by the fourth years (118 being the 2 questions, numbers 5 and 6, multiplied by 59 pupils in fourth year). From this, it would appear that many of the fourth years are having difficulties stemming from the structural aspects of linear equations. However, reliable inference of the mechanisms behind such difficulties is not possible from the study or its data, and many of the ideas above are merely suggestions based on anecdotal evidence collected by the researcher in the classroom.

6.7 The Switching Addends Error in 'Algebraic' versus 'Arithmetical' Equations.

It is significant to note that, overall, the Switching Addends error occurred 5 times in the 'arithmetical' equation $5x - 1 = 38$, compared with 24 times in the 'algebraic' equation $4x - 2 = x - 1$. Such a difference in occurrence was not unexpected in the light of the literature on the subject. For instance, Filloy and Rojano tried to "uncover some of the obstacles experienced by students during the period of transition from arithmetical to algebraic equations" (Kieran, 1992, page 402).

One such obstacle which may contribute to a lack of success in 'algebraic' equation solving has been noticed by the researcher over the years i.e. the ability to gather 'like' terms. The transition from 'arithmetical' to 'algebraic' equations may be analogous to the transition from arithmetic to algebra. One of the features of arithmetic is that, except where different units of measurement are concerned, all the terms are numbers and therefore 'like' each other, although this point may not be made explicitly to pupils. These numbers may be confidently added, subtracted, divided, and multiplied. Not only that, but pupils have become used to doing this. At around the time pupils reach secondary school, algebra is introduced, usually in the form of expressions first. In algebra, terms may not be automatically added or divided etc.. The pupil must make a judgement about which terms are 'like' which other terms, with a view to 'collecting the like terms'. This process has no equivalent in arithmetic, except when different units are concerned (which, incidentally, might help explain why pupils have difficulties with units in primary school because they may not classify terms as 'like' or 'unlike' until secondary school). Thus it may be misleading when "school algebra is sometimes referred to as generalized arithmetic" (Booth, 1984, quoted in Kieran, 1992, page 395), when one considers that some often-used processes which are necessary in algebra occur to a more limited extent in arithmetic.

Younger pupils can find the process of collecting 'like' terms so difficult that they cannot confidently simplify an expression such as $3x + 2x$. Difficulties are not restricted to younger pupils. Older pupils can be so confused by second and third year algebra that, by fourth year, they find it difficult to explain why 3x can be multiplied by 2y, but cannot be added to 2y. Indeed, "Wenger (1987) has described some of the poor strategic decisions made by students with extensive algebra experience – decisions that result in their 'going round in circles' while carrying out simplification transformations" (Kieran, 1992, page 397). Also, some pupils have difficulty collecting the 'like' terms separated by an 'unlike' term within an expression, such as in

$$3x + 2y + 4x.$$

It is therefore likely that such pupils who experience any of these difficulties with 'like' terms within an expression, when faced with an 'algebraic' equation in which the 'like' terms are now *separated* by an equals sign, will have even more difficulty in successfully arriving at the correct strategy of, in this case in Question 5, subtracting 'x' from both sides to arrive at a point where it will be possible to collect 'like' terms. Further, there may even be some reluctance to do this since they do not want to put themselves in the position of having to perform a process in which they lack confidence.

As has been discussed in Chapter 2.14, there is room for confusion between expressions and equations. Confusion may be further compounded because 'like' terms can lie adjacent within an expression, separated by an 'unlike' term within an expression, separated by an equals sign within an 'arithmetical' equation, or separated by an equals sign within an 'algebraic' equation.

6.8 The Number Line Error.

This is characterised by such mistakes as $-3 + 1 = -4$. There may be many explanations for this error. The pupil may be confusing $-3 + 1$ with $-(3 + 1)$, or she might be misapplying the rules for the order of operations (BODMAS, to name one acronym for this order) by thinking that the addition of $3 + 1$ must be done first in $-3 + 1$ and then the negative sign is dealt with by simply placing it in front of the 4 obtained. Indeed, it is the researcher's experience that pupils find many different ways of attempting to perform $-3 + 1$, especially if this is embedded within an equation. One reason for this may be that the manipulation of negative numbers, a subordinate skill necessary in the solution of equations, is a source of difficulty and is taught only a short time before the topic of linear equations is introduced to the pupils in this study. Practice is needed in using such devices as a number line to simplify expressions such as $- 3 + 1$. However, too much reliance on such devices as the number line may not be advisable in that its usefulness may break down in the face of fractional integers and, as the literature suggests, "such devices……had to be rejected when it came to multiplication and division" (Orton, 1992, page 69).

However, without mastery of directed number manipulations, occurring, as they tend to, in the 'reduction' step of the deduction-reduction algorithm, the prognosis of success in solving linear equations may be poor. Confirmation of this assertion may be found in the large-scale study by the fact that the number line error was made 10 times by first year pupils.

6.9 The Omissions Error.

e.g. $5x + x + 2 = 3x + 12$

$\Rightarrow \quad 6x + 2 - 2 = 3x + 12$

Since this usually occurs in the middle of the solution of the problem, and occurs while a pupil is using a structural method, the error itself should not give too much cause for concern. The complexity of the problem and the pressure of the exam setting may both cause the pupil to omit letters or numbers.

6.10 The Other Inverse Error.

As has already been stated, this error, where

$$4x = 1$$

$$\Rightarrow \quad x = 1 - 4$$

does not appear to be present in the literature, and yet occurs 8 times in the large-scale study. This may be a more serious error than the previous one for several reasons.

First, it may demonstrate a lack of understanding of the meaning of the statement $4x = 1$.

Second, it may point to a confusion of the additive and multiplicative inverses. The pupil may be familiar with the concept of inverses, but not sure when each is appropriate.

Third, it may confirm Greeno's (1982) assertion that a certain number of pupils cannot, or do not know that it is possible to, check the truth of the solution either in the original equation or in any one of the equalities in the chain of equations i.e. the solution of $x = 1 - 4$ will not satisfy the original equation.

Fourth, it may be another example of $4x$ being misinterpreted: 3¾ means 3 + ¾ so why does $4x$ not mean $4 + x$? (Matz in Sleeman, 1982, page 392) This leads directly to the use of the additive rather than the multiplicative inverse. This confusion, of '4 added to x' with '4 multiplied by x' when pupils are faced with finding a meaning for $4x$, has implications beyond simple linear equations.

6.11 The Absence of Structure Error (New).

As can be seen from the example in Table 1, this response typically defies analysis. Constants and variables are confused more than they are in the Deletion error. Change side-change sign manipulation may be attempted, but the Transposing error is impossible to detect with certainty, as is the Switching Addends error. A pupil operating at this level is probably aware of his difficulties, but because of their number and/or complexity, may not be able to articulate them to the teacher.

Even among the fourth years, who have had several years of exposure to simple linear equations per se as well as equations at the end of other algebra and geometry problems, there were Absence of Structure errors made by 3 of the 59 pupils.

It should be noted that the Absence of Structure error is not an identifiable single error, but a category into which many uncategoriseable errors are placed.

6.12 An Analysis of an Answer Showing "Absence of Structure".

In solving the equation $4x - 2 = x - 1$

one of these three fourth year pupils wrote

line 2 $2x - 4 = -1$

line 3 $-12x - 4 = -1 - 1$

line 4 $x = 2x - 3$

Much more could be learned by investigating this pupil's thought processes in an interview situation. However, following the design of this study, inferences may be drawn from the written work. Line 2 could be some attempt at transposing internally in which 2x - 4 is deemed to be equivalent to 4x - 2. Line 3 shows the pupil might be ‘adding’ -1 to both sides to arrive at -12x i.e. concatenating -1 and 2x to make -12x.

Line 4 defies analysis. It is noteworthy that every line does contain an equals sign, so perhaps the term 'Absence' in this error type may be too strong to accurately describe this particular pupil's error.

6.13 The Deletion Error (Lit) Revisited.

The Deletion error, in which 3x - 3 is incorrectly simplified to x, has not been found frequently in this study. However, it is treated as an important algebraic error by Carry et.al. (1980) and Matz (1979), and, as has already been noted, this may be partly because of its *persistence* through to college. There may be several explanations of the apparent difference in importance attached to the Deletion error by this study and by the literature.

In the study by Carry et. al. (1980), there were 60 Deletion errors observed out of a total of 227 general algebraic errors i.e. errors made both while solving equations and simplifying expressions. In this study there were 5 Deletion errors observed out of a total of 149 errors made while solving equations.

However, it must be reiterated that the sample of pupils in this study is not representative of the general population, by virtue of the selective nature of the school. Also, the questions in the large scale study may not have given much opportunity for the commission of Deletion errors.

In discussing the results of this large-scale study, there has been some divergence noted from studies from the literature, especially with regard to the Deletion error. This is only to be expected since parameters vary from one experiment to another. Also, question types and formats may affect the frequencies of different error types. However, the small number of Deletion errors in the large-scale study compared with that of Carry et. al. (1980) may beg questions concerning both the type of analysis done and the element of bias in the large-scale study.

6.14 The Exhaustion Error.

This error occurred less frequently in the large-scale study than in the pilot study, despite a greatly increased sample size. It would therefore appear to be an error of little importance in the context of simple linear equations. However, the error does correspond closely with an observation of Greeno (1982) who found that "beginning algebra students are consistent neither in their approach to the testing of conditions before performing some operation nor in the process of performing the operations" (Kieran, 1992, page 397). In other words, the commission of the Exhaustion error is in agreement with Greeno's (1982) finding that a pupil can be inconsistent within the solution of a particular problem. Despite this, it was removed from the list of error types because of its low frequency in this study.

6.15 The Redistribution Error (Lit).

In the literature this error is observed by Kieran (1984) and further noted by Grouws (1992).

Judging from its very low observed frequency of commission in both the pilot and large-scale studies, this error would appear to be either undetected (but as it was an error reported by other researchers (Grouws, 1992) it was looked for) or unimportant. This highlights one of the features of the quantification of error-making: it does facilitate the ranking of errors for the purpose of determining their relative importance, which in turn may help the teacher.

Contrast the study by Erlwanger (1973) observing "Benny", in which the errors derived their importance from the interesting way in which the mal-rules have been invented, rather than from their frequency. There is little doubt that there is much to be learned from Erlwanger's (1973) landmark study. His exploration of an individual's thought processes and coping strategies offer valuable insight to every teacher of an analytical subject.

Therefore, analyzing errors by frequency alone, as has been done in this large-scale study, is just one way of systematically classifying and organising errors. Thus, just because an error, in this case the Redistribution Error, seldom appears does not automatically mean that error is unimportant.

6.16 Subordinate Skill Errors in Different Education Systems.

In Bermuda pupils are promoted from year to year based on their performance on a wide range of subjects. In the United States the courses Middle School Math (often called pre-Algebra), Algebra 1, Geometry, and Algebra 2 must be passed, and in that order, to qualify the student for graduation from high school. In the U.S. system it is common, for instance, to have 'advanced' 14 year olds and repeating 17 year olds in the same Geometry classroom. This focus on mastering (or at least being minimally successful on tests on) material before promotion to the next Mathematics course has many merits, among them being that there should be less opportunity for subordinate skill errors to be made: there being a greater likelihood that previous material should, overall, have been mastered. In the Bermuda system, all pupils gain social promotion from one year level to the next. Therefore there can be no such assumption of even partial mastery of previously-learned material can be made, particularly in the lower achieving streams. This difference in the two systems has implications for error analysis, the main one being that there may be a greater scope for the commission of subordinate skill errors in the Bermuda system. However, while teaching in the U.S. system, this researcher noted that memory came into play, especially in the year between Algebras I and II, when the pupils were studying Geometry. Skills mastered in Algebra I were often forgotten by the time the pupils reached Algebra II.

6.18 A Comparison of Total Errors Made in Questions 4,5 and 6.

Table V

Errors Made by Question.

	First year n = 66	Second year n = 53	Third year n = 68	Fourth year n = 59	Total
Question 4	21	13	6	4	44
Question 5	14	10	7	20	51
Question 6	10	12	14	18	54

Total	45	35	27	42	149

From Table V, compiled from the data in Table IV, it is possible to find several patterns of error-making. Before analysing the results it must be restated that these are frequencies of observed errors, and do not include non-attempts to answer a particular question.

Fewer errors are made in Question 4 ($5x - 1 = 38$) as pupils get older. Given the procedural nature of the question with its constant right hand side, it is possible to both trial-and-improve quickly and even check the solution in the original question. It is therefore not unexpected that some improvement in success rates has occurred with practice and increased maturity.

In Question 5 ($4x - 2 = x - 1$) the fourth years made more errors than the third years, which may be consistent with the perceived complexity of the questions by Filloy and Rojano (in Kieran, 1992, page 402), as outlined previously in the discussion of "arithmetical" and "algebraic" equations i.e less direct instruction on how to solve an "algebraic" equation, or simply forgetting that variables may be added to or subtracted from both sides, may have contributed to the fourth years' reversal of the downward trend in error rates for this question as pupils get older. A longitudinal study could help determine whether memory is a factor in the fourth years' ability to solve "algebraic" equations, or whether this particular fourth year finds "algebraic" equations more difficult than does this particular third year. Perhaps a case could be made for making the point explicitly in textbooks that there does exist a difference in difficulty between "arithmetical" and "algebraic" equations, perhaps to the extent that they be introduced in different chapters.

In Question 6 ($5 + \frac{x}{2} = 2$) the fourth years are committing a larger number of errors than those in other years. One possible interpretation of this fact is that the fourth years are trying harder than those in lower years to arrive at the correct solution. If this is the case, then this may be a positive sign i.e. perhaps the fourth years are not daunted by the task and are willing to take risks. In order to test this conjecture it is necessary to tabulate the blank responses.

Table VI

Blank Responses by Question.

	First year n = 66	Second year n = 53	Third year n = 68	Fourth year n = 59	Total
Question 4	7	1	5	3	16
Question 5	12	2	3	3	20
Question 6	15	3	1	8	27
Total	34	6	9	14	63

From this table it appears that the fourth years are indeed having difficulty with Question 6 as 8 fourth year pupils did not write anything in response to this question, as opposed to one third year pupil.

Perhaps this confirms the assertion that this may result from confusion with simpler equations such as Speed = Distance/Time from Arithmetic, and D = M/V from Physics. Transposing errors may be masked while solving these simpler equations – equations which are seen more frequently, and in more guises (e.g. R = V/I), as pupils progress through the school - until they are revealed in more complicated equations such as Question 6. Such masking may be an important feature of the Transposing error.

The first year pupils are making the most omissions. This appears to reinforce Kieran's (1989, page 52) warning that "an often documented finding concerns the inability of beginning algebra students to 'see' the surface structure of algebraic expressions containing various combinations of operations and literal terms".

The topic of linear equations is introduced in first year, strongly reinforced in second year, and taught very little in its own right in third or fourth year, but reinforced in such topics as quadratic equations, changing the subject of the formula, and general problem-solving. It would appear from the data in this study that many of these particular fourth years would benefit from a review of the solution of linear equations. However, the nature of this study does not permit an extrapolation to fourth years in general.

Chapter 7 – Conclusion and Evaluation

7.1 The Transposing Error.

"Formal methods of equation solving include transposing and performing the same operation on both sides of an equation" according to Kieran (1992, page 400). The researcher's own opinion, before the study was done, was that it may be inappropriate to label transposing, certainly as done by many of his own pupils over twenty years, as a 'formal' method, because it often leads pupils towards making errors. Indeed, over the years the researcher has increasingly dissuaded pupils from using a transposing method for two reasons.

First, there seemed to be little understanding on the part of the transposers of the structure or meaning of equations. Second, this lack of understanding, coupled with many pupils' natural desire to make things simpler whenever possible, very often led to an over-simplification of the 'change side – change sign' rule. (Incidentally, analogous over-simplification has often been seen by the researcher elsewhere e.g. in the pupil-generated "two minuses make a plus" mal-rule).

To remove any doubts that the researcher may have had about the dangers of transposing, despite the literature's implication that it is a 'formal' method, the decision was taken analyse Transposing errors within the framework of a strictly controlled study. Doubts also sprang from the fact that colleagues did not appear to attach much importance to the exact way in which a pupil solved an equation. The researcher's overriding concern has always been that not only should pupils be successful in solving equations, but that the process should be *meaningful* to as many pupils as possible. If transposers were using a device or trick to obtain the solution to an equation, then they may not have fully appreciated exactly what they had accomplished in solving that equation. It was felt by the researcher that without such an appreciation, the whole process of teaching and learning how to solve linear equations becomes merely mechanical.

Fortunately for the Mathematics Education researcher looking at equations work done by a pupil, it is often possible to find out if that pupil has indeed fully appreciated the meaning of what he has written in response to a simple linear equation question i.e. the correct solution must be able to be inserted into the original equation to yield a true statement. Evidence of this can often be seen.

Just as importantly the teacher can tell if it has not been done, either verbally or on a paper, if there appears to be no attempt to rectify what is obviously an incorrect solution. This process (of trying to determine if a pupil appreciates the significance of what he has been engaged in; in this case solving an equation) cannot be done as easily in some other types of question e.g. in simplifying an expression a pupil cannot tell if the simplified answer is correct without performing the same operations all over again.

As has been noted, the test question $5 + \frac{x}{2} = 2$

was designed to elicit the Transposing error by those pupils who may be prone to using an over-simplified transposing method. The large-scale study showed that this error did appear, especially among the fourth year pupils (a quarter of whom made this error). This finding may confirm that the fourth year pupils have over-simplified transposing to the point that if a number appears on the denominator on one side of the equals sign, then on the next line it can always be written on the numerator of the other side of the equals sign. Indeed, the fourth year pupils who have committed this error may have invented a device or trick to solve *all* linear equations in which the variable to be found is divided by a number, based on the fact that this is *sometimes* seen to work e.g. while solving for 'D' in $S = D/T$ when 'S' and 'T' are known constants. Further, those fourth years who committed the Transposing error and found that $x = -1$ either did not test this answer in the question, or if they did, were unable to change their thinking in view of the fact that if $x = -1$, the left side of the question does not equal the right side. Either way, this does demonstrate a lack of understanding of the structural aspects of equation solving.

Thus the findings of this study have both removed some doubts about the dangers of using transposing and served to reinforce the researcher's view that transposing can lead to many errors.

Kieran (1992, page 400) appears to have labelled the transposing method as 'formal' because of its widespread use by algebra teachers. However, the word 'formal' does imply a certain validation of the method and maybe, for instance, even a recommendation of it to the novice teacher. This study would appear to refute such a validation, by virtue of the fact that transposing can lead to the taking of short-cuts and the commission of errors. Therefore it is the opinion of this researcher, based on the findings of this study, that transposing would be better described as a 'widely-used' method of solving simple linear equations, with the warning that the 'change side – change sign' method is open to over-simplification and abuse.

7.2 Subordinate Arithmetic and Algebraic Skill Errors.

Another finding in this study is that there is a greater variety of errors made in simple linear equations than may have been envisaged by mathematics teachers or than might have been gleaned by referring to the current literature. In addition to the previously documented, and therefore expected, types of errors, this particular experiment yielded five new identifiable errors in the context of simple linear equations, two of which, the Number Line error and the Division error, involve subordinate arithmetic skills.

The Division error, typically found in some younger pupils' inability to proceed beyond $5x = 39$, appears worthy of further investigation. In an improved study, it would appear to be desirable for two reasons to somehow ensure that as many pupils as possible attempt to solve every equation. First, there would be less need to speculate as to why the pupil could not progress beyond a certain point. Second, those pupils not finishing a question or leaving the whole question blank may be the same pupils prone to making errors. In other words, there might be a larger number of errors to categorise from a study using a given sample size.

The Other Inverse error could be classed as a subordinate algebraic skill error. Although this error occurred only eight times in this study, mastery of multiplicative and additive inverses remains important in the structural understanding of the solution of linear equations. Also, it must be remembered that it is impossible to ascribe reasons for blank responses, some of which could also be due to the fact that some pupils are uncomfortable with inverses.

Thus this study points to a lack of sufficient subordinate skills, both arithmetic and algebraic, in many of the younger pupils, demonstrating their apparent lack of readiness to be able to solve equations of this type. If, in the words of Gagne (quoted in Orton, 1992, page 54), "developmental readiness for learning any new intellectual skill is conceived as the presence of certain relevant subordinate skills", then perhaps the school in this study should consider introducing linear equations only when it is satisfied that a good level of understanding has been reached in such subordinate skill topics such as integer division and adding and subtracting integers i.e. in the second year. As there are many instances of the use of simple linear equations in first and second year in other subjects e.g. the sciences, other departments would be affected by such a decision to delay the teaching of simple linear equations.

7.3 The Switching Addends Error.

By far the most frequent errors in the study as a whole were the Transposing, Division, and Switching Addends errors. While the Division error is of the subordinate skill type, the Transposing and Switching Addends errors may be produced because of a lack of understanding of the structural aspects of solving linear equations.

Whatever the causes of the Switching Addends error are, constructivist theory may help suggest, in a positive way, some mechanisms for its commission. Again, a subordinate algebraic skill, the collecting of 'like' terms, may be partly responsible for a pupil's lack of success in equation work. This would appear to strengthen the case for delaying the teaching of simple linear equations until pupils have enough confidence with the necessary subordinate skills to make smoother the process of understanding the structural aspects of solving equations.

7.4 The Effectiveness of Manipulatives.

Some researchers, such as Davis (quoted in Chapter 2.10), have encouraged the use of manipulatives while others, such as Daniels & Anghileri (quoted in Chapter 6.1), have urged caution in their use. In addition to seeing ambivalence in the attitudes of researchers to the use of manipulatives in general, it may be discouraging for algebra teachers to further learn that Filloy and Rojano's "interviews revealed that the use of (balance and area) concrete models did not significantly increase most students' ability to operate at the symbolic level with equations having two occurrences of the unknown" (Kieran, 1992, page 402), especially since, at least according to this large-scale study, pupils appear to have difficulty solving such 'algebraic' equations. Orton (1992, page 92) summarized it well in saying that "certain research has suggested that there are gains (in using apparatus), but there has usually been little evidence of long-term benefits. There does not, however, seem to be any evidence that the use of structural apparatus is in any way harmful or detrimental to learning."

7.5 Standardisation of Teaching Methods.

A detailed examination of pupils' work, such as was done in this study, may make effective changes in teaching methods possible. It is possible that a mixture of methods taught or employed could be a source of confusion, especially, for instance, if the teacher of second year pupils favours transposing and the teacher of third year pupils stresses ‘doing the same to both sides of the equation’. Thus there may be lessons here for heads of Mathematics departments who might consider recommending that all teachers within the department use one approach.

It would make an interesting study to see whether such a standardisation of methods alone, irrespective of which method was adopted and especially if done in concert with other departments such as Science, would lead to an improvement in overall proficiency in solving linear equations.

7.5 Suggestions for Improvements to the Large-Scale Study.

This study could have benefited from a larger sample size, given the fact that it was done in the scientific research paradigm. However, there remains a failing of the study which cannot be rectified by enlarging the sample size alone: the test items were placed at the beginning of the compulsory section of a school examination in order to maximize the number of responses, but there were 63 blank responses compared with 149 documented errors out of a total of 738 possible responses to questions i.e. 8.5 % of all the questions were left blank compared with 20 % which showed errors. The researcher feels that there may have been too many blank answers, as compared with the number of answers with identified errors, for the identified errors alone to paint an accurate picture of the difficulties encountered by this set of pupils in these questions. Therefore, in an improved study, it would be desirable to raise the response rate. This could also be achieved by simply increasing the number of diagnostic questions.

In this study there was an element of bias, referred to in Chapter 6.13, in the design of Questions 4,5, and 6 in order to investigate errors made in ‘arithmetical’ and ‘algebraic’ equations as well as to find any Transposing errors, perhaps due to over-simplification of the transposing method. In trying to investigate errors, one of the methods used was to try to minimise guessing by procedural means such as ‘trial and improvement’ and therefore maximise the use of more ‘formal’ methods in order that pupils' working be shown as often as possible. For this reason, positive non-integer solutions were chosen for Questions 4 and 5, and a negative integer solution was chosen for Question 6 (a non-integer solution for Question 6 was not chosen because of the unnecessary degree of difficulty in checking the answer i.e. the need to divide a fraction by a whole number). Thus, this may have led inadvertently to an investigation of the slightly different topic - "An analysis of errors made in solving simple linear equations *not easily soluble by procedural means.*" Also, this approach may appear to downplay the importance of such methods as 'trial and improvement', which was not the intention of the researcher for two main reasons. First, the use of some procedural methods, such as trial and improvement, can show a solid understanding of the meaning of what one is engaged in while solving an equation. Second, 'trial and improvement' becomes a formal method, for instance at 'A' level in Numerical Analysis, when other methods do not yield results.

This avoidance of the use of positive integer solutions may appear to have led to a greater degree of difficulty in the examination than may have been desirable, and also to a discrimination against pupils who prefer procedural methods. This begs the question "Are formal methods superior?" In her extensive review of the literature on solving linear equations, Kieran (1992, page 400) lists seven methods, two of which "transposing and performing the same operation on both sides…..are often referred to as formal methods". The other five methods, use of number facts, use of counting techniques, cover-up, undoing (or working backwards), and trial and error substitution, are not collectively given a name. Since the first two methods are referred to as formal, there may be an implication that these other five methods are not fully formal in the same way. When solving simple linear equations, the 'trial and improvement' method can be efficient for solving equations with positive integer solutions, and this method may be important for the two reasons outlined in the previous paragraph, but there remain obvious limitations associated with teaching a method which works efficiently only on certain occasions. Perhaps teaching two different methods of solving linear equations might be an effective way of promoting both success and understanding. Such an approach would appear to be in agreement with Whitman (1976), whose "findings suggest that the students who had been taught to solve equations by the formal methods alone were not conceptually prepared to operate on equations as mathematical objects with formal, structural operations." (Kieran, 1992, page 400)

As far as the degree of difficulty is concerned, it should be mentioned that in the examination papers of all four years there were many other questions containing simple linear equations whose solutions were positive single-digit integers e.g. in a third year set theory question, which reduced to $4x - 3 = 5$ towards the end of the problem, methods such as 'trial and improvement' could be used efficiently. It could be argued that the mixture of integer and non-integer answers in an examination should be reflective of real life, and in this regard it is the researcher's opinion that none of the four examinations was unfair in this respect. In an improved study, those questions with embedded linear equations could also be analysed in an attempt to find out the relative use of formal and informal methods. Thus with more data from each individual, it might even be possible to discover how many pupils use both formal and informal methods, and which kinds of equations or solutions trigger the use of each method.

In an improved study it might be possible to elicit a different error from those described above, namely an error of ordering. This was not attempted in this study because the specific type of linear equation, which has been seen by the researcher to elicit this error, may not be universally regarded as 'simple', especially for first year pupils. This type of question involves a negative coefficient in an 'arithmetical' equation, and the observed faulty reasoning is included as follows:

Line 1		$4 - 3x = 10$
Line 2	=>	$4 - 3x + 4 = 10 + 4$
Line 3	=>	$3x = 14$

In Line 1 there is an error of ordering i.e. the negative sign in $-3x$ is regarded as 'belonging' to the 4. The rest follows logically. To reach Line 2, the additive inverse of the now-negative 4, being positive 4, is added to both sides of Line 1. Line 3 may be reached in either of two ways: make the ordering error again by cancelling $4-$ and $+4$, or ignore the $3x$ and simplify $4- + 4$ to zero. All of these pieces of reasoning have been given to the researcher during class, and the appeal of these constructs may lie in the fact that, by Line 3, x no longer has a negative coefficient, eliminating the need to divide throughout by a negative number. It would be interesting to observe the frequency of such an ordering error among upper secondary pupils.

7.6 Conclusion.

Familiarity with these nine errors should enable me, and perhaps other teachers, to be better equipped to forestall the most common mistakes. Indeed, such errors could be discussed at the appropriate point, both in lessons and in textbooks. This may be especially important at the introductory level because it prevents the formation of bad habits as well as the development of inaccurate constructions on the part of the learner.

It may also be useful for the teacher, when recognising a specific error, to point it out to the pupil for, as Borasi (1994, page 166) observed, "although teachers and researchers have long recognised the value of analysing student errors for diagnosis and remediation, students have not been encouraged to take advantage of errors as learning opportunities in mathematics instruction."

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