EARLY MATHEMATICS EDUCATION:
A CASE OF THE BLIND LEADING THE BLIND?*

Stephen R. Campbell

Department of Education
University of California, Irvine
Irvine, CA, 92697-5500

Telephone: 1-949-824-7736
Fax: 1-949-824-2965
Email: SENCAEL@UCI.EDU

A manuscript accepted for publication and forthcoming in Volume 15, 2002, of the *Philosophy of Mathematics Education Journal.*

Abstract:
The study of mathematics exemplifies the intellectual quest for comprehensive and consistent thinking. Today this quest, as bequeathed to us by the Greeks, has been abandoned in mathematics education in favor of relatively mundane tasks of calculating with formal algorithms and applying those calculations in supposedly more meaningful "real world" contexts. An excuse in forsaking this intellectual quest, especially in the early grades, is that young children are incapable of understanding the conceptual nature of mathematics. This "incapability thesis," which absolves teachers in the early grades themselves from developing a conceptual understanding of mathematics, is called into question. An alternative philosophy of mathematics education for teaching the conceptual nature of mathematics dating back to Parmenides and Plato is presented and illustrated in ways that may well be within the reach of young children. Some problems and potential implications for mathematics teacher education are considered.

Word Count (including endnotes): 5323

Running Head: <u>MATHEMATICS EDUCATION…</u>

Mathematics Education:
A Case of the Blind Leading the Blind?

How well can one teach a young elementary or middle school level child to manipulate symbols? How well can that child learn to apply those symbols for solving problems? If these are the sole objectives and values for mathematics education in the early and middle grades, as traditionalists and progressivists to one extent or another would have it, then even the most exemplary practices may not be addressing the main challenge of mathematics education: teaching for deeply meaningful and well-grounded understandings of mathematical concepts.[1]

The study of mathematics has long been recognized as an effective way of leading students out of the cave of human ignorance.[2] In its finest sense, mathematics is a proven key to unlocking the unlimited potential of the human mind to create new conceptual structures and to discover existing ones. To the extent that teaching and learning for conceptual understandings of mathematics is not being successfully addressed by contemporary theories and practices in mathematics education, and there are reasons to suggest that it is not, the study of mathematics may be serving more as an instrument for the oppression of the human spirit than as a means for its emancipation.

Most mathematics educators are of the mind that young children in the early grades are incapable of developing conceptual understandings of mathematics. My first task in this paper is to call this "incapability thesis" into question, suggesting traditionalists and progressivists have failed to effectively address the problem of conceptual understanding in mathematics education, and have essentially ignored it altogether in the early grades. To the extent that the incapability thesis is wrong, the challenge for mathematics teacher educators will be to develop ways of guiding elementary and middle school teachers into the light of conceptual understandings of mathematics, and to provide them with methods for helping young children along that path as well. Otherwise, the mathematics education of children in the early grades may well remain a case of the blind leading the blind.

In an attempt to redress this problem, I consider the emergence of conceptual understanding in Greek thought. This line of inquiry provides an historically based "existence proof" that the development of conceptual understanding of mathematics is possible by examining the kinds of thinking that originally brought it into being. This in turn provides a philosophical alternative for mathematics education as to what conceptual understanding of mathematics might be, illustrated with two examples for teaching mathematics accordingly in ways that are more accessible to young children.

Have both "traditionalists" and "progressivists" gotten it wrong?

Traditionalists who emphasize teaching the basics seem to believe that conceptual understanding is something that follows upon the memorization of various number facts and operations, depending on "how good a student is" at mathematics. Progressivists who emphasize the learning of mathematics through "real world" applications, often seem to equate conceptual understanding with those very applications.

In Plato's famous allegory of the cave, Socrates can be seen as admonishing the traditionalist in concluding that "education is not what it is said to be by some, who profess to put knowledge into a soul which does not possess it [i.e., algorithms], as if they could put sight into blind eyes." He can also be seen as admonishing the progressivist when he goes on to say "… the entire soul must be turned away from this changing world [i.e., applications], until its eye can bear to contemplate reality." Mathematics, as an art suitable to turning "the power of learning the truth" in every soul to "the way it ought to be," is presented as a cornerstone in Plato's curriculum for the expressed purpose of awakening intellect to the quest for knowledge.[3]

To the extent that developing conceptual understandings of mathematics is a concern at all to traditionalists and progressivists today, it is typically viewed as something that *follows upon* the teaching and learning of calculations and applications. Mathematics as we have come to know it from the Greeks, however, is by nature *conceptual*. Considering the above Socratic admonitions from a developmental perspective, if an understanding of mathematical concepts is something that can *precede* calculations and applications then traditionalists and progressivists alike have gotten it wrong.

Independently of the question as to whether developing conceptual understandings of mathematics is possible prior to, or only results from, teaching and learning calculations and applications, mathematics educators appear to be falling short of the challenge of helping most students to achieve any conceptual understanding of mathematics at all.[4] The question arises as to what conceptual understanding is and why is it so important? I address the second part of this question now, deferring the first part to the next section.

The importance of developing a conceptual understanding of mathematics, in a word, pertains to transfer: the problem of how knowledge and skills learned in one context can be generalized and applied in other contexts. Curiously, however, mathematical concepts are characteristically very general concepts, and mathematical understanding enables general problem solving skills to be applied in novel contexts in ways that simply transferring expertise from one specific domain to another does not.

A sardonic adage regarding the traditionalist approach to teaching mathematics is that mathematics is what you do in math class. A natural reaction to these "decontextualized," symbol-pushing practices is to "recontextualize" them. Common progressivist praxis is to render mathematics meaningful by applying those calculations in "real world" contexts.[5] But these instantiations of general mathematical concepts fall short of the concepts themselves, which ideally are above/beyond/beneath/within their applications.

Traditionalists seem to think that if students learn "the basics" well enough, some of them will eventually just "get it," whatever "it" happens to be. Even if most students do not, at least they will be able to balance their checkbook without a calculator, or even refund change without having to count out whatever amount a cash register tells them to.

Progressivists who equate conceptual understanding with the "meaning-making" resulting from applying mathematical calculations in specific contexts are left to wonder why their students have so much difficulty transferring their knowledge to other contexts, on the one hand. On the other, they must fend off criticism that their efforts undermine students' abilities to calculate without the aid of machines.

Let us suppose, in the best possible way, that traditionalist and progressivist approaches to mathematics education through calculations and applications, or some balance between the two, actually *can* produce exemplary practices that result in the reliable development of conceptual understanding for most learners, say as early as secondary school. In that case, one could expect some improvement in prospective elementary and middle school teachers' conceptual understanding of mathematics. So what?

If young children are not capable of developing conceptual understandings of mathematics, then it may make little if any difference whether teachers in the early grades have any conceptual understanding whatsoever of the mathematics they teach. This "incapability thesis" basically provides the rationale, if not an excuse, for ignoring the problem of teaching mathematical concepts, not just in the early grades, but in the respective teacher credential programs for those levels as well. Fortunately, contemporary educational theorists and researchers are beginning to question and openly reject this thesis, inviting the question as to whether the potential young children may have for conceptual understandings of mathematics has been given short shrift.[6]

If the incapability thesis is wrong—and it is likely not so much a matter of *if*, but a question of just *how* wrong it happens to be—a pedagogical lacuna of corresponding significance will have been revealed in the teaching of mathematics to young children. On the bright side, to the extent that the incapability thesis *is* wrong, a new realm of possibilities to a corresponding extent will emerge for improving mathematics education, along with a pressing need to reform, or better, perhaps even to *transform* mathematics teacher education accordingly. New practices would be required, not only for improving teachers' *conceptual* subject matter content knowledge, but their *conceptual* pedagogical content knowledge as well,[7] especially in the early grades.

Computer-based learning environments and other educational technologies may have a substantive role to play. Indeed, these technologies are providing educational researchers with unprecedented new opportunities to reconsider our understanding of the nature of cognition itself, especially in relation to technology.[8] At the very least, computer models and simulations of mathematical functions and structures are offering unprecedented new opportunities for augmenting imagination in bridging the gap between intuition and understanding. It remains to be determined how effective such practices will be.

A complementary realm of possibilities is opened by a return to the past. A history of the *conceptual* origins of elementary mathematics is latent in the extraordinary philosophical developments of ancient and classical Greek thought. Although many important advances in mathematical computations and applications were made in other cultures prior to that time, there were some distinctively conceptual advances of note made by the Greeks.[9]

This cognitive history can potentially inform the psychological development of young learners, and thus lead to significantly new pedagogical practices in mathematics education. This is the guiding hypothesis of the following section.

How can history inform psychology and pedagogy in mathematics education?

The main ways in which the history of mathematics has served to inform mathematics education has been in terms of historical “sidebars” providing biographical information on mathematicians or mathematical accomplishments of the past, or in terms of specific problems perceived to be of contemporary import or relevance. If the problems are deemed important enough, they may be rewarded with a cameo role in the text proper. Otherwise they may serve as interesting exotica in exercises at chapters’ end.

A more substantive way in which historical developments in mathematics can inform mathematics education is to consider if crucial qualitative shifts in the early cognitive history of mathematics can inform the psychological development of mathematical understanding in young children. In this case, historical facts and problems are much less important than identifying potential cognitive obstacles and shifts in understanding that gave rise to them. Can associated or prerequisite methods of thinking be identified which allowed for significant historical shifts in mathematical understanding to occur? Might significant cognitive developments in the history of mathematical thinking have a psychological basis of relevance to contemporary mathematics education?

It remains to demonstrate that reasonable answers to the affirmative can be found to such questions. I will do so by postulating some major developments which may be implicated in the historical emergence of conceptual understanding in arithmetic and geometry, and illustrate their potential pedagogical relevance for the early grades. The most essential question pertaining to conceptual understanding is to provide some indication as to what conceptual understanding is, and how such a thing could ever have emerged in the first place. Given the scope of this question, a brief synoptic account must suffice here.[10]

The “universal” distinction between the earth and sky in creation myths provides a good clue as to the experiential “ground” upon which the distinction between perception and understanding was originally based. Making distinctions is a natural capacity of perception in discerning patterns. Becoming aware, quite likely through language, that patterns are being recognized and that distinctions are being made indicates the sun is beginning to rise in the dawning early morning light of human understanding.

Perception, be it imagined or real, concerns the immediacy of things in the moment in which things are experienced. Reflecting on perception allows for different percepts to be compared, contrasted, and quantified.[11] The mathematical mediation of multiple percepts is a defining feature of rationality. Moreover, constancy denotes similarity and change denotes difference. The starry heavens above have long provided human perception with the very best and most reliable indicator of constancy. A lack of perceptual constancy, however, even in the heavens as exemplified by the planets, would eventually give way to a logical demand for intellectual consistency in Greek thought.[12]

Stories about how mathematical concepts have emerged have been absent in the education of young children. Yet it is well known that children love good stories and are very competent with metaphors and binary opposites.[13] These kinds of stories implicate fundamental mathematical distinctions such as change and constancy, difference and similarity, many and one, part and whole, and so many more. A central distinction around which many of these distinctions are clustered hark back to the distinction between earth and heaven, or more in terms bequeathed to us by the Greeks, the distinction between perception, the realm of sense, and understanding, or the realm of intellect.

Another story that must be told in understanding the nature and origins of modern mathematics concerns Parmenides' vision, wherein the daughters of Helios guided him into the heavens to be instructed by a Goddess in the "way of truth" and the "way of seeming."[14] The way of seeming concerns the realm of sense. It is a realm wherein things can both be and not be. In the realm of sense, objects are characterized by ambivalence and contradiction as things come and they go. The way of truth concerns the realm of intellect, wherein things either are or are not. In this realm, there is no ambivalence, things are either true or they are not—they can never be both. In the Pythagorean tradition, mathematics has come to exemplify the way of truth.[15]

Parmenides' vision in more modern terms tells us that for any mathematical proposition, that proposition must be either true or not true. This is the law of the excluded middle. The requirement that a proposition cannot be *both* true *and* false is the law of non-contradiction. Accordingly, for instance, in the way of truth which characterizes the realm of intellect, the number two is an even number—it is always and forever either even or not even, it simply cannot be thought of as both even and not even. These two laws constitute the logical foundations of mathematical proof and of Western thought in general.[16] In identifying and resolving ambivalence and ambiguity in the realm of intellect, we strive for consistency of thought.

At what point can young children, or even adults for that matter, begin to grasp the difference between the way of seeming, the realm of sense, that state of pre-reflective beliefs and practices, and the way of truth, the realm of intellect, where one attempts to hold things constant, clearly and distinctly reflecting upon them in the mind? Specific answers to these kinds of questions will ultimately be determined by empirical research, but it is important to consider first whether it may be much earlier and easier, drawing for instance on children's natural propensities for metaphor and interests in stories, than is typically thought. The way of mathematics education today, in accord with the incapability thesis, however, has largely been to ignore this possibility.

Furthermore, the pedagogical efficacy these kinds of stories need to be determined. At the very least, teachers of mathematics should know about them and at least be capable of distinguishing whether—and determining the extent to which—children's ideas follow the way of truth, and thus constitute knowledge, or the way of seeming. The goal in teaching mathematics to young children, in so far as it is possible to achieve, should be to guide them from the latter to the former. But how might this be done?

Socratic questioning as to "what *is* such and such?" is concerned with conceiving, giving birth, and nurturing clear and comprehensive definitions of concepts. The main tool in so doing is by comparing and contrasting similarities and differences between the definition itself with instances of the concept that is being defined. Ideally, that definition should not exclude instances that should be included or vice versa.

The purpose of a definition, of course, is to make things definite. The more definite things are, the more certain we can be that we know what it is that we are talking about. Like Socrates, we should resist finality regarding definitions, even mathematical ones.[17] The important thing is that we strive for clarity, comprehensiveness, and consistency. These are the hallmarks of conceptual understanding. Every teacher, especially mathematics teachers, should know this.

Every teacher should also know that imagination is a most active component in concept formation,[18] and yet it is hardly ever called upon in mathematics education. This is tragic, if not a travesty, given that children also have such fertile imaginations. The pedagogical challenge is in how children's imaginations might best be nurtured in mathematical concept formation. Let us consider how this might be done with two of the most important concepts in mathematics: the unit and the line.

Mathematical concepts are typically characterized as being abstract rather than concrete. It is easy to rationalize, again in accord with the incapability thesis, *not* teaching abstract concepts on the basis that anything abstract *at all* is considered to be *too* abstract, or *too* decontextualized for children to make sense out of. Thus, numbers and figures become reduced to things to tinker with on tables, or drawn and pointed at on blackboards or computers. This is *not* the way of truth.

A child's mind cannot be guided toward a meaningful conceptual understanding of an arithmetic unit just through the use of marbles or colored cubes used with base 10 blocks, as commonly practiced. Such practices forsake the general nature of the concept. Simply having children memorize definitions renders them meaningless. The key thing for young learners to *consciously* realize about an arithmetic unit is that it can stand for *any* thing. Pedagogically, this would involve inviting children to consider what it is that *all* things, whatever things they can possibly imagine, have in common. It is a huge leap for children to realize that the *only* thing that *all* things hold in common is that they are all objects or events of one kind or another. Even when children are still thinking of objects and events as having extension in space and time, important steps remain.[19] It is only when these last two perceptual predicates are separated from objects and events that one is left with a pure intellectual concept of an arithmetic unit.

The foregoing provides some indication as to how children's imaginations can lead them from all possible things in the realm of sense to the general concept of an arithmetic unit. I will now illustrate a way in which imagination can lead from a familiar sensible object to the pure concept of a geometrical line. Imagine if you will, a yardstick. Imagine making that yardstick twice as long and half as thick. *Mutatis mutundi*--what the mind can do once, the mind can do twice. Make the yardstick you have in mind twice as long and half as thick, again, again, and again. Imagine having done that forever. Now what do you have in mind? Is it anything you can ever perceive with your senses?

Through unit and line, I have briefly attempted to demonstrate that mathematical concepts can only be grasped and understood intellectually, they cannot be directly perceived by the senses, and that children's imaginations can likely be applied to meeting this end. The story of Parmenides' vision can be invoked to help reinforce these kinds of results. In so far as mathematics teachers are not guiding students, be they young or old, child or adult, towards and into the realm of intellect, and in the way of truth, the essence of what mathematics is about is neither being taught nor is it being learned.

"I don't understand these things, how can I expect my students to?"

It is evident by the current state of mathematics education that prioritizing the teaching and learning of calculation and applications has fallen short of providing most learners with a deep understanding of mathematical concepts and the reasoning involved. This is evident not only by an unfulfilled demand for qualified individuals for technical positions requiring a solid understanding of mathematics, it is also reflected in the poor caliber of mathematical understanding and concomitant poor attitudes towards mathematics of preservice teachers in the elementary and middle grades.[20]

The opening quotation that constitutes the heading of this concluding section is from an exasperated preservice teacher in an elementary and middle school or "multi-subject" credential program class I taught not too long ago.[21] To my mind, and in my experience, this attitude exemplifies the challenge of teaching mathematics for conceptual understanding that currently faces mathematics teacher educators.

A common and most prevalent profile for a multi-subject credential candidate at the University of California at Irvine is a Caucasian female between 22 and 24 years of age who has recently graduated with a degree in the social sciences. She enjoys working with children and is looking to gain experience with them prior to getting married and having children of her own. Mathematics is typically the main subject that she either knows the least about, or simply wishes to avoid as much as possible. This latter observation applies across the board, to one extent or another, to most multi-subject candidates. Those who do have an aptitude for mathematics typically pursue a secondary school credential.[22]

With an increasing demand for more teachers these days, there is increasing pressure being placed on teacher credential programs to increase enrollments. A considerable challenge for teacher educators is to increase quantity while also increasing quality. Indeed, it is not an inconsiderable challenge to simply maintain the status quo. Many teacher credential programs have no room for "content" courses at all, and must make due with "methods" courses that often consist of nothing more than showing anxious students rightly concerned with classroom management what to do. The mere mention of words such as "concept," "general," or "abstract," especially in context with the subject of mathematics, can make already high levels of anxiety unbearable for them.

Preservice teachers' anxieties and attitudinal difficulties regarding the teaching of mathematics can often be correlated with a concomitant lack of mathematical understanding. It can be a traumatic thing for many teacher candidates to realize that they do not understand at a very basic conceptual level what they have ostensibly and purportedly been studying for years, or even why it is that they do not understand it. This state of affairs is likely a direct result of these students never having been exposed to the conceptual foundations of mathematics, even at the most elementary levels.

Even if higher levels of quality students could be attracted into teacher credential programs in higher numbers, would this make much of a difference without making major changes in the teacher credentialing process as well? Of course, knowing how to calculate and apply those calculations in problem solving contexts would certainly be an improvement of sorts. Knowing how to do something, however, does not imply an understanding of why things are done in the ways that they are, and still less does it imply knowing how to teach such things to others. There remains the challenge of knowing how to teach what one knows, and more specifically of concern here, there remains the problem of understanding the conceptual foundations of mathematics from a developmental perspective and learning how to teach children accordingly.

After many years of calculations and applications, most students have evidently never been taught, nor have they learned about what mathematics *is*.[23] The problem is systemic and self-perpetuating. There are important stories to be told about the contexts, motives, and beliefs that have given rise to mathematics as a distinctively *conceptual* pursuit in Western culture. Some of these stories have only been touched upon here in an attempt to illustrate their potential relevance and value to early mathematics education. It remains to be established by research in mathematics education just how effectively these kinds of stories can be told in ways that are accessible and meaningful to young children and preservice teachers alike. In more general terms, it remains to be established to what extent mathematics educators have been underestimating children's propensities and abilities to think conceptually. Irrespectively, unless better ways can be found for teaching and learning mathematical concepts and the conceptual foundations of mathematics, and beginning as early as possible, what appears to be a perpetual cycle of the blind leading the blind in mathematics education will likely remain unbroken.

---

[*] A draft of this paper, the title of which owes itself to Plato's allegory of the cave, was submitted to the *American Association of Universities* Forum on Exemplary Practices and Challenges in Teacher Preparation (Boston, MA: September 30th – October 2nd, 2001).

[1] Traditionalists are portrayed here as those who would prioritize the teaching of algorithms and calculations over applications of mathematics, and progressivists are portrayed as those who would prioritize applications, or learning from "real-life" experience, over a focus on calculating with algorithms. Conceptual understanding, in both cases, is typically considered as something that *follows* in one way or another from these respective priorities. Furthermore, in both cases, the development of conceptual understanding is viewed as something that is *compromised* when a greater priority is

placed upon the other. These respective views can be seen as educational progeny of analytic-formalist and pragmatic-intuitionist traditions in the philosophy of mathematics. The view presented herein falls more in line with the phenomenological tradition as applied to the cognitive history of mathematics, e.g., Jacob Klein, "Phenomenology and the History of Science," *Philosophical Essays in Memory of Edmund Husserl*, ed. Marvin Farber (Cambridge, MA: Harvard University Press, 1940), 143-63. For more in this regard, see Stephen R. Campbell, "Three Philosophical Perspectives on the Relation between Logic and Psychology: Implications for Mathematics Education," *Philosophy of Mathematics Education Journal*, 14 (2001).

[2] Plato, *The Republic of Plato*, trans. Francis M. Cornford (London, UK: Oxford University Press, 1945/~380B.C.E.), 175-263, *passim*.

[3] Ibid., 232ff, see also 181-9.

[4] Implicit testimony to this effect is offered by those legions of learners readily professing not to be good at math, not to mention an increasingly unmet demand for workers who actually do understand mathematics and how to apply it in novel situations.

[5] Independently of how "real" these applications may be, constructivists in the progressivist tradition have come under fire for not adequately distinguishing between subjective belief and objective knowledge. See Christine McCarthy and Evelyn Sears, "Science Education: Constructing a True View of the Real World?" in *Philosophy of Education*, ed. Lynda Stone (Urbana, IL: Philosophy of Education Society, 2000), 369-377. The alternative view presented below provides historical context regarding the emergence and relevance of this important distinction.

[6] E.g., Kieran Egan, *The Educated Mind: How Cognitive Tools Shape Our Understanding* (Chicago, IL: University of Chicago Press, 1995); David Carraher, Analúcia D. Schliemann, and Bárbara M. Brizuela, "Can Children Operate on Unknowns?" in *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education*, ed. Marja van den Heuvel-Panhuizen, (Utrecht, NL: Utrecht University, 2001), vol. 1, 130-140.

[7] Knowing something does not imply knowing how to teach it to others. This distinction between "subject-matter content knowledge" and "pedagogical content knowledge" has become a mainstay in teacher education. L. S. Shulman, "Those Who Understand: Knowledge Growth in Teaching," *Educational Researcher*, 15, no. 2 (1986): 4-14.

[8] E.g., Stephen R. Campbell, "Computer-Assisted Synthesis of Visual and Symbolic Meaning in Mathematics Education," in *Proceedings of the International Conference on Mathematics/Science and Educational Technology*, R. Robson, ed., (San Diego, CA: Association for the Advancement of Computing in Education, 2000), 101-105.

[9] See, for instance, Bruno Snell, *The Greek Discovery of the Mind in Greek Philosophy and Literature*, (New York, NY: Dover Publications Ltd., 1982/1953). Note that I am *not* suggesting psychological development *must* recapitulate historical development.

[10] I have written more comprehensively about this elsewhere. See, Stephen R. Campbell, "Number Theory and the Transition Between Arithmetic and Algebra: Connecting History and Psychology," in *Proceedings of the 12th International Commission on Mathematical Instruction Study Conference on The Future of the Teaching and Learning of Algebra*, (Melbourne, Australia: ICMI, 2001), 147-154; "Zeno's Paradox of Plurality and Proof by Contradiction," *Mathematical Connections*, Series II, No. 1 (2002), 3-16; "The Problem of Unity and the Emergence of Physics, Mathematics, and Logic in Ancient Greek Thought. *Proceedings of the 4th International History and Philosophy of Science and Science Teaching Conference* (Calgary, Canada: University of Calgary, 1999), 143-152.

[11] According to Plato, "… reflection is provoked when perception yields a contradictory impression… [w]hen there is no such contradiction, we are not encouraged to reflect," *The Republic of Plato*, 239ff. Plato's three-finger example basically illustrates that the ring finger can both be large relative to the pinky, and small relative to the middle finger, thus providing a contradictory impression that it is both large and small. Quantifying provides an intellectual resolution to the contradiction.

[12] "No one, I should say, can ever gain knowledge of any sensible object by gaping upwards any more than by shutting his eyes and searching for it on the ground, because there can be no knowledge of sensible things." Plato, *The Republic of Plato*, 247. It remains a contemporary problem in the foundations of mathematics as to the extent to which complete intellectual consistency is an attainable ideal.

[13] E.g., Egan, *The Educated Mind*, 33ff.

[14] Parmenides, "On Nature," in *Source Book in Ancient Philosophy*, Charles M. Bakewell, ed. (New York, NY: Charles Scribner's Sons, 1907): 11-20.

[15] Both Parmenides and Plato can be situated squarely within the Pythagorean tradition.

[16] Stephen R. Campbell, "Zeno's Paradox of Plurality and Proof by Contradiction."

[17] In distinguishing between common belief and true knowledge and expressing humility regarding his own ignorance of the latter, Socrates notes "…we shall be better and braver and less helpless if we think we ought to inquire into what we don't know than if we give way to the idle notion that there is no knowledge, and no point in trying to discover what we do not yet know." From Plato's *Meno* (84), quoted in Zhang Loshan, "Plato's Counsel on Education," in *Philosophers on Education: New Historical Perspectives*, ed. A. O. Rorty (London and New York: Routledge, 1998), p. 36.

[18] Immanual Kant, *Critique of Pure Reason*, (New York: St. Martin's Press, 1965)

[19] This shift in the cognitive history of the arithmetic unit can be identified with the philosophies of the early Pythagoreans (a proto-atomic unit with spatial extension) and Plato (a pure indivisible unit of quantity) respectively. A further shift (to a pure divisible unit of measure) motivated in large part by the thinking of Aristotle is identified in Jacob Klein, *Greek Mathematical Thought and the Origin of Algebra*, trans. Eva Bran (New York, NY: Dover Publications, Ltd).

[20] E.g., Liping Ma, *Knowing and Teaching Elementary Mathematics* (Mahwah, NJ: Lawrence Erlbaum Associates, 1999); Janice Rech, Judykay Hartzell, and Larry Stephens, "Comparisons of Mathematical Competencies and Attitudes of Elementary Education Majors with Established Norms of a General College Population," *School Science and Mathematics* 93, no. 3 (1993):141-4. The positive side is that preservice teachers in the early grades appear to have the most to gain from mathematics teacher education, e.g., Robert J. Quinn, "Effects of Mathematics Methods Courses on the Mathematical Attitudes and Content Knowledge of Preservice Teachers," *Journal of Educational Research* 91, no. 2 (1997):108-13.

[21] In this course, I was teaching to some extent, at least with respect to conceptual content, in accord with the kinds of ideas outlined in the previous two sections.

[22] Let me be very clear about this. I believe that next to parents, and in many cases more so, elementary and middle school teachers are the main contributors to the moral fabric of society. The effects and influence of the vast majority of the bright and able young women and men, that contribute to the character development of young children, day in and day out, simply cannot be overestimated, and should never be taken for granted. Any critiques or aspirations for improvement in mathematics teacher education, in my opinion, must be kept in perspective within this much greater context.

[23] The philosophical foundations of mathematics remains a vibrant, open, and unresolved field of study, and no suggestion to the contrary is intended or implied (see, for instance, Hart, W. D. (ed.), *The Philosophy of Mathematics* (New York, NY: Oxford University Press, 1997). Rather, the point here is simply to acknowledge that there is a fundamental conceptual component to learning mathematics that involves developing well-grounded intuitions and clear reasoning, and that there is more to teaching and learning mathematics than just calculations and applications.