A local hidden-variable model violating Bell's inequalities

A. Matzkin

Laboratoire de Spectrométrie physique (CNRS Unité 5588),

Université Joseph-Fourier Grenoble-1,

BP 87, 38402 Saint-Martin d'Hères, France

Abstract

A local hidden-variable model for two spin-1/2 particles is shown to reproduce the quantum-mechanical outcomes and expectation values, and hence to violate Bell's inequality. Contrarily to the usual preset hidden-variable (HV) distributions that have been generally considered, we relax the constraint requiring that a given HV distribution should account for the simultaneous reality of quantum-mechanical counterfactual events. We assume instead that a disturbance induced by a measurement on an eigenstate – which according to Einstein, Podolsky and Rosen hinders the existence of an element of physical reality – results in a change of the corresponding hidden-variable distribution. We first investigate the one-particle HV-distribution and then tackle in the same way the two-particle problem in the singlet state. The averages of spin measurements along different axes are obtained from the HV distributions without appealing to nonlocal effects.

PACS numbers: 03.65.Ud,03.65.Ta,03.65.Sq

In their celebrated paper [1], Einstein, Podolsky and Rosen (EPR) argued that quantum mechanics was incomplete on the ground that the wavefunction of a 2-particle entangled state allows to predict with certainty the measurement outcomes of noncommuting observables that, according to the formalism, cannot have simultaneous reality. The alternative to incompleteness was to uphold that the measurement carried out on one particle determined which obervable was real for the second particle, despite the fact that the second particle could be arbitrarily separated from the first. EPR concluded: "*no reasonable definition of reality could permit this*" nonlocal action at-a-distance. Bell, who took seriously the possibility of supplementing quantum mechanics with additional hidden-variables [2], showed in a seminal work [3] that (i) any model based on a distribution of hidden variables (HV) preset at the source so that the quantum results are (a) reproduced in a local fashion and (b) completed in the EPR sense alluded to above must satisfy an inequality involving averages taken over the hidden variable distributions; and (ii) in certain circumstances the average value of 2-particle quantum observables violates the inequality, so no local HV model will do. Experimental setups aimed at measuring observables violating the inequality have confirmed with increasing precision the quantum mechanical predictions [4]. The current received view is that Bell-type inequalities 'prove' that local realism is incompatible with quantum mechanics [5], though there is some disagreement [6] as to whether this also means that quantum mechanics itself is non-local. However recent works have been questioning the relevance of 'nonlocality proofs' to hidden-variable theories, based on data ordering dependencies [7] or on logical grounds [8].

In this work, we go further by putting forward an explicit local hidden variable model for the prototypical spin-1/2 problem that is compatible with quantum mechanical predictions (and hence violates Bell's inequalities). The basis of the argument consists in returning to the EPR criterion of physical reality, that depends on the prediction with certainty of a physical quantity "*without in any way disturbing a system*" [1]. We shall follow EPR in assuming a perturbation of the system whenever the state of the system is not in an eigenstate of the measured observable. If we introduce HV distributions supposed to underlie the quantum mechanical eigenstates, by fulfilling the EPR criterion we are led to envisage a model in which the HV distribution is dynamically and locally affected by the measurement. Bell's inequalities can then be violated, because the main assumption employed to derive them – postulating a counterfactual term accounting for the existence of quantum-mechanically incompatible events that are nevertheless taken to be meaningful from the hidden-variables point of view – does not hold.

In his original derivation of the inequalities, Bell [3] investigated the fragmentation of two spin-1/2 particles formed in the singlet state. Before getting to the 2 particle problem proper, we will first consider the hidden variable distribution for a single spin-1/2 particle. The measurement of the spin projection along an axis u making an angle θ_u with an arbitrarily chosen z axis is represented by the operator (we put $\hbar = 1$)

$$S_u(\theta) = \frac{1}{2}(\rho_+(\theta_u) - \rho_-(\theta_u)) \tag{1}$$

where $\rho_\pm(\theta_u) = |\pm(\theta_u)\rangle \langle\pm(\theta_u)|$ are the density matrices corresponding to the eigenstates with eigenvalues $\pm 1/2$ that represent the possible outcomes of a spin measurement along the axis. We assume an HV distribution such that (i) the eigenvalues depend on the hidden variables. (ii) An eigenstate is characterized by an HV distribution that is invariant relative to the transformations generated by the quantity that is being measured [9]. (iii) When a measurement disturbs (in the EPR sense) the quantum system by modifying the eigenstate, we assume that the HV distribution is modified, the post-measurement distribution obeying the invariance condition in the final eigenstate. (iv) Individual outcomes depend on the initial HV value and on the perturbation but average values over the initial HV distribution are not modified by the perturbation (this condition is necessary both for consistency of the measurement process – assumed to depend on the HV distribution – and to ensure that conservation laws are obeyed on average).

Here we shall take as the hidden variable a unit vector $\boldsymbol{\lambda}$ pointing along the direction of the angular momentum. The distribution of λ when the system is in the state described by $\rho_\pm(\theta_u)$ is set to be

$$R_{\pm u}(\boldsymbol{\lambda}) = \frac{\lambda}{2\pi}\delta(\boldsymbol{\lambda} \cdot \hat{\boldsymbol{u}} \mp \frac{1}{2}), \tag{2}$$

which is normalized on the unit-sphere ($\hat{\boldsymbol{u}}$ is a unit-vector). The rationale for this choice is that any vector $\boldsymbol{\lambda} \in R_\pm(\boldsymbol{\lambda})$ has the same projection λ_u on the u axis, whose value is precisely the outcome of the measurement (see Fig. 1(a)). If the system is in the state $\rho_\pm(\theta_u)$, measuring S_u does not perturb the system, nor the distribution $R_{\pm u}$. Then the measured value $\mathcal{M}$ is directly given in terms of the hidden variable:

$$\mathcal{M}_u(\boldsymbol{\lambda}) = \boldsymbol{\lambda} \cdot \hat{\boldsymbol{u}} = \pm\frac{1}{2}, \tag{3}$$

and the mean value of λ_u is obviously

$$\langle \lambda_u \rangle_{R_{\pm u}} = \pm \frac{1}{2}. \tag{4}$$

Suppose now that the system is initially in the state $\rho_+(0)$ (spin pointing in the positive z direction), characterized by the HV distribution $R_{+z}(\boldsymbol{\lambda})$. Measuring S_u perturbs the system, which ends up in either of the states $\rho_\pm(\theta_u)$ with probabilities obtained from elementary quantum mechanics as

$$P_{+z\to+u} = \mathrm{Tr}\,(\rho_+(0)\rho_+(\theta_u)) = \cos^2\frac{\theta_u}{2} \tag{5}$$

$$P_{+z\to-u} = \mathrm{Tr}\,(\rho_+(0)\rho_-(\theta_u)) = \sin^2\frac{\theta_u}{2}. \tag{6}$$

Following (iii), we postulate that the perturbation induced by the measurement leads the HV distribution to change from the premeasurement $R_{+z}(\boldsymbol{\lambda})$ to $R_{\pm u}(\boldsymbol{\lambda})$ with the probabilities (5)-(6). Because of the perturbation, the measured value $\mathcal{M}_u$ will not be given by Eq. (3) in terms of the premeasurement distribution; indeed, for $\lambda \in R_{+z}$ the projection λ_u depends on ϕ_λ, i.e. on the specific position of λ on the sphere (see Fig. 1). The unknown function $\mathcal{M}_u(\boldsymbol{\lambda}) = \pm 1/2$ can be introduced to specify how the perturbation turns a given λ into one of the measured outcomes. In term of this function, the average value

$$\langle \mathcal{M}_u \rangle_{R_{+z}} = \int \mathcal{M}_u(\boldsymbol{\lambda}) R_{+z}(\boldsymbol{\lambda}) \sin\theta_\lambda d\theta_\lambda d\phi_\lambda \tag{7}$$

is equal to the average over the measurements, known to be given by the quantum-mechanical expectation value $\langle S_u \rangle_{\rho_+(0)} = \mathrm{Tr}\,(\rho_+(0)S_u) = \cos\theta_u/2$. The probabilities $P_{+z\to\pm u}$ can thus be interpreted as the fraction of the original HV distribution that undergoes a transition towards the post-measurement distributions $R_{\pm u}$:

$$\langle \mathcal{M}_u \rangle_{R_{+z}} = \sum_{k_u=-1/2}^{1/2} k_u \int \tilde{R}_{+z\to k_u}(\boldsymbol{\lambda}) \sin\theta_\lambda d\theta_\lambda d\phi_\lambda = \sum_{k_u=-1/2}^{1/2} k_u P_{+z\to k_u}, \tag{8}$$

where the tilde signifies that $\tilde{R}$ is not itself a distribution but the result of cutting a given distribution R into different pieces given the function $\mathcal{M}_u(\boldsymbol{\lambda})$. According to (iv), the expectation $\langle \mathcal{M}_u \rangle$ should be given by the mean value of $\boldsymbol{\lambda} \cdot \hat{\boldsymbol{u}}$ previous to the perturbation. $\langle \lambda_u \rangle$ can be directly computed, yielding the result

$$\langle \lambda_u \rangle_{R_{+z}} = \int \boldsymbol{\lambda} \cdot \hat{\boldsymbol{u}} R_{+z}(\boldsymbol{\lambda}) d\Omega_\lambda = \frac{1}{2}\cos\theta_u, \tag{9}$$

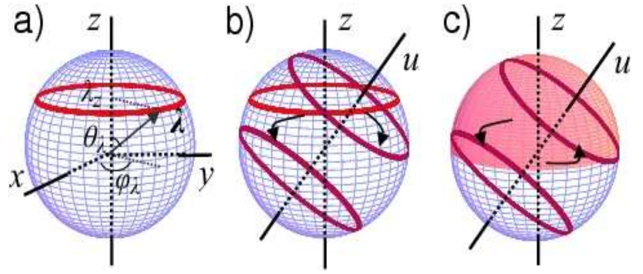

FIG. 1: (a) A vector λ on the unit sphere belonging to the distribution R_{+z} corresponding to the quantum eigenstate $\rho_+(0)$. (b) Pictorial representation of the measurement of S_u in the state $\rho_+(0)$: the initial HV distribution R_{+z} around the z axis (in red) changes as a function of the disturbance and of the HV position to one of the distributions around the u axis. (c) Same as (b) for the initial HV distribution Σ_{+z}, uniform in the upper hemisphere (see text).

which is seen to be equal to the expectation value $\langle S_u \rangle_{\rho_+(0)}$. These results are valid for a measurement of the spin projection along any axis b given an initial state $\rho_+(\theta_a)$ along any axis a, and we rewrite Eqs. (7)-(9) as

$$\langle \lambda_b \rangle_{R_{+a}} = \langle \mathcal{M}_b \rangle_{R_{+a}} = \sum_{k_b=-1/2}^{1/2} k_b P_{+a \to k_b} = \frac{1}{2} \cos(\theta_b - \theta_a). \tag{10}$$

We thus see the main consequence of postulating a correspondence between an eigenstate and an HV distribution: when the system is in an eigenstate of the measured observable, the hidden-variables have all the same value of the property being measured, that is directly determined by the HV value. When the measurement induces a perturbation (in the EPR sense), the HV distribution changes: the measured outcome depends both on the initial HV distribution and on the perturbation, although the average over the measurements only depends on the mean value of the hidden variables over the initial distribution.

An important consequence of our choice (2) concerning the correspondence between the eigenstates and the hidden variables is that different HV distributions can yield the same quantum results. In particular, to anticipate on the 2-particle problem, if the hidden variables are uniformly distributed on the positive surface of the hemisphere Σ_{+a} centered on the a axis, we have

$$\langle \lambda_b \rangle_{\Sigma_{+a}} = \int \boldsymbol{\lambda} \cdot \hat{\boldsymbol{b}} R_{\Sigma_{+a}}(\boldsymbol{\lambda}) \sin\theta_\lambda d\theta_\lambda d\phi_\lambda = \frac{1}{2} \cos(\theta_b - \theta_a), \tag{11}$$

where $R_{\Sigma_{+a}}(\boldsymbol{\lambda}) = 1/2\pi$. Hence $\langle \lambda_b \rangle_{\Sigma_{+a}} = \langle \lambda_b \rangle_{R_{+a}}$. In the case $\theta_b = \theta_a$, Eq. (11) gives (the positive version of) Eq. (4): if S_a is measured, one will find the result $+1/2$ with probability one irrespective of whether the initial HV distribution was Σ_{+a} or R_{+a}. Indeed, both Σ_{+a} and R_{+a} are invariant around the a axis and by modifying accordingly Eqs. (2) and (3), we could have chosen to let Σ_{+a} correspond to an eigenstate rather than R_{+a}. Consequently there is no effective disturbance (in the EPR sense) in the passage $\Sigma_{+a} \to R_{+a}$ [1].

We proceed to the 2 particle case. Suppose that the spin of each particle is determined by a hidden variable attached to the particle. To be specific, let us postulate the existence of two hidden variables $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ on the unit-sphere pointing in the direction of the spin angular momentum of each of the particles. Following Bell [3] we assume the particles were originally formed in the singlet state and are spatially separated so that individual spin measurements along arbitrary axes can be made. The singlet state, whose density matrix is given by

$$\rho = \sum_{k,k'=\pm 1} \frac{(-1)^{\frac{k-k'}{2}}}{2} \left| k(\theta_u) \right\rangle \left| -k(\theta_u) \right\rangle \left\langle k'(\theta_u) \right| \left\langle -k'(\theta_u) \right| \tag{12}$$

is rotationally invariant (and does not depend on the choice of θ_u). The total spin and its projection along any quantization axis vanish: this imposes a correlation between $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ that for a given ρ is fixed once and for all. Given the HV model developed above for the spin measurements of a single particle, the correlation takes the form

$$\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 = 0, \tag{13}$$

that can be seen as a consequence of the conservation of the angular momentum on the hidden variables. Eq. (13) means that in the singlet state, the hidden variables are uniformly distributed on the sphere Σ, but point in opposite directions. This is exactly what one would expect for the angular momenta of two classical particles with a uniform statistical distribution and zero total angular momentum [10]. The condition (13) imposes a distribution $R_\Sigma(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$ proportional to the delta function $\delta(\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2)$. This delta function encapsulates the premeasurement correlation between $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$: given the disturbance Eq. (13) is not expected to hold after the measurements, but as we now show, this initial correlation suffices in accounting for the quantum expectation values in terms of the conditional expectation of the HV positions.

[1] The fact that Σ_{+a} and R_{+a} have different topologies is particular to the spin-1/2 case. For the general spin-S case, the choice would be between a delta function of the type (2) and a spherical zone of width $1/(2S+1)$ around the axis, where $S+1/2$ is the radius of the sphere. The extension of the present study to an arbitary angular momentum is straightforward and will be given elsewhere.

Consider a measurement on particle 1 of the spin projection S_{1a} along an axis a. We know from Eq. (11) that if λ_1 is distributed on the positive half-sphere Σ_{+a}, we will obtain the result $+1/2$ with certainty (and $-1/2$ if λ_1 belongs to the negative half-sphere Σ_{-a}). Therefore given the uniform distribution of λ_1 on the sphere, we can obtain $S_{1a} = \pm 1/2$ with probability 1/2 which is of course the quantum result in the state (12). Now if, say, one obtained $+1/2$, then, from Eq. (13) we infer that λ_2 is on the half-sphere Σ_{-a}; hence the measurement of S_{2a} will yield with certainty $S_{2a} = -1/2$ (see Fig. 2). If instead the spin projection on an axis b is measured, the outcomes $S_{2b} = \pm 1/2$ are possible with respective probabilities $\cos^2 \theta_b/2$ and $\sin^2 \theta_b/2$ [Eqs. (5)-(6)]; the average $\langle \lambda_{2b} \rangle_{\Sigma_{-a}}$ – the mean projection of $\boldsymbol{\lambda}_2$ on the b axis if $\boldsymbol{\lambda}_1$ belonged to the positive hemisphere – is obtained from Eq. (11). By repeating this line of reasoning for each possible outcome, we can determine the total expectation

$$\langle \mathcal{M}_{1a} \mathcal{M}_{2b} \rangle_{R_\Sigma} = \sum_{k_{1a}, k_{2b}=-1/2}^{1/2} k_{1a} k_{2b} \int \tilde{R}_{\Sigma \to k_{1a}}(\boldsymbol{\lambda}_1) \tilde{R}_{\Sigma_{\sigma(k_{1a})a} \to k_{2b}}(\boldsymbol{\lambda}_2) d\Omega_{\lambda_1} d\Omega_{\lambda_2}, \tag{14}$$

where σ gives the sign of its argument. The integrals over $\tilde{R}(\boldsymbol{\lambda}_1)$ and $\tilde{R}(\boldsymbol{\lambda}_2)$ give the probabilities 1/2 and $P_{\pm a \to k_{2b}}$ [Eqs. (5)-(6)] respectively, and Eq. (14) can be seen to be equal to the quantum-mechanical expectation value

$$\langle S_{1a} S_{2b} \rangle_\rho = \mathrm{Tr}(S_{1a} S_{2b} \rho) = -\frac{1}{4} \cos(\theta_b - \theta_a). \tag{15}$$

In terms of the pre-measurement HV distribution, the average is not given by $\langle \lambda_{1a} \lambda_{2b} \rangle_{R_\Sigma}$ (since Eq. (13) is not preserved by the disturbance due to the measurement). As in the one-particle case, we require however that conservation laws hold on average[2] (see point (iv)): a given measurement on particle 1 allows to infer the zone occupied by $\boldsymbol{\lambda}_2$ on the sphere (which according to Eq. (13) is the opposite of the one occupied by $\boldsymbol{\lambda}_1$ before the measurement). This does not allow to predict the outcome of a measurement on particle 2, but only its average (exactly like in the one particle case), leading to

$$\langle \mathcal{M}_{1a} \mathcal{M}_{2b} \rangle_{R_\Sigma} = \frac{1}{2} \sum_{k=\pm 1} \langle \lambda_{1a} \rangle_{\Sigma_{ka}} \langle \lambda_{2b} \rangle_{\Sigma_{-ka}}. \tag{16}$$

[2] We see that this property only makes sense for a single particle, since the perturbation induced on the second hidden variable depends conditionally on the result of the first measurement.

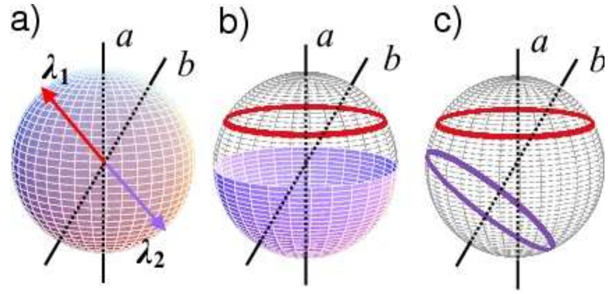

FIG. 2: Example of correlated HV distributions for two spin-1/2 particles in the singlet state. (a) Initial distribution: the HV are uniformly distributed on the sphere, correlated through Eq. (13). (b) The measurement of S_{1a} gives $+1/2$, meaning that λ_1 belonged to the upper hemisphere before the measurement, and hence that λ_2 is to be found in the lower hemisphere. (c) The measurement of S_{2b}, which can give one of the 2 outcomes as pictured in Fig. 1(c), gives $-1/2$, and the corresponding final distribution of λ_2 around the b axis is shown.

To compute Eq. (16), we remark that from Eq. (11) we have $\langle \lambda_{1a} \rangle_{\Sigma_{ka}} = k/2$ and

$$\langle \lambda_{2b} \rangle_{\Sigma_{-ka}} = - \langle \lambda_{2b} \rangle_{\Sigma_{ka}} = (-1)^{\frac{k+1}{2}} \frac{\cos(\theta_b - \theta_a)}{2}, \tag{17}$$

thereby yielding again the standard quantum result Eq. (15). Note that we can interchange $1 \leftrightarrow 2$ or $a \leftrightarrow b$, i.e. it does not matter which axis or particle is measured first.

We therefore obtain the quantum expectation value (15), known to violate Bell's inequality, by postulating HV distributions perturbed by the measurement process. Nonlocality does not play any role. The fact that in Eq. (16) $\langle \lambda_{2b} \rangle$ depends on the measurement outcome of particle 1 is not the result of action at a distance but a conditional statement, the inference being grounded on the correlation (13). As remarked by Jaynes [11], Bell's HV model does not allow to make this type of inference. Indeed, because of the completeness requirement, the Bell hidden variable distributions are required to account for the quantum-mechanically incompatible elements of reality. Here instead, the role of the measurement disturbance is twofold: obtain discrete outcomes for individual events and ensure that the spin projection of one of the correlated particles along more than one axis cannot be an element of reality. Ironically, this is the substance of Bohr's reply [12] to EPR (if one ignores the positivist and obscure tone of his reply): "*the finite interaction between object and measuring agencies conditioned by the very existence of the quantum of action*" implies that measuring the spin projection along a given axis brings in "*an essentially uncontrollable*" action that "*cuts ourselves off from any possibility of applying the law of conservation*" of rotation to the system so as to make predictions on the second particle [3].

We have shown that by assuming that hidden variable distributions are disturbed by measurements – in the same sense that Einstein, Podolsky and Rosen stated that a quantum state is disturbed when a measurement involves observables that do not commute with the corresponding density matrix – we are able to recover the quantum expectation values that violate Bell inequalities. Indeed, Bell-CHSH [14, 15] type inequalities assume preset HV distributions that would ascribe simultaneous reality to quantum-mechanical counterfactual events. This condition forces HV models to obey a Bell inequality, but as it has recently been pointed out [7, 8, 10] this assumption should not be conflated with local realism. In this respect, the violation of Bell inequalities does not appear to be a valid ground to support nonlocality, since a measurement perturbed distribution of *local* hidden variables achieves the violation that is generally thought to be possible only by considering a preset distribution of *nonlocal* hidden variables.

---

[1] A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. 47, 777 (1935).

[2] J. S. Bell, Rev. Mod. Phys. 38, 447 (1966).

[3] J. S. Bell, Physics 1, 195 (1964).

[4] Aspect, J. Dalibard and G. Roger, Phys. Rev. Lett. 49, 1804 (1982); Y. Hasegawa, R. Loidl, G. Badurek, M. Baron and H. Rauch, Nature 425, 45 (2003).

[5] A. Zeilinger, Rev. Mod. Phys. 71, S288 (1999); P. Grangier, Nature 409, 774 (2001).

[6] W. Unruh, Int. Jour. Qu. Inf. 4, 209 (2006).

[7] K. Hess and W. Philipp, PNAS 101, 1799 (2004); K. de Raedt, K. Keimpema, H. de Raedt, K. Michielsen and S. Miyashita, Eur. Phys. J B 53, 139 (2006).

[8] A. F. Kracklauer, AIP Conf. Proc. 750, 219 (2005); A. Broadbent, H. A. Carteret, A. A. Méthot and J. Walgate, N. J. Phys. 8, 302 (2006).

[9] A. Matzkin, in prep.

[10] A. Matzkin, E-Print quant-ph/0703251 (2007).

[3] Bohr referred to the position and momentum of the particles, employed by EPR [1], and not to spin projections, which were introduced in an EPR context later by Bohm [13].

[11] E. T. Jaynes, in J. Skilling (Ed.), *Maximum-Entropy and Bayesian Methods*, Kluwer, Dordrecht (1989).

[12] N. Bohr, Phys. Rev. 48, 696 (1935).

[13] D. Bohm, *Quantum Theory*, Prentice-Hall, Englewood (NJ, USA, 1951).

[14] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. 28, 880 (1969).

[15] J. S. Bell, *Speakable and unspeakable in quantum mechanics*, Cambridge Univ. Press (2004), ch. 4.