A Partial Elucidation of the Gauge Principle To appear in *Studies in History and Philosophy of Modern Physics* ⋆

Alexandre Guay

CIRST, UQÀM, Case postale 8888, Succ. Centre-ville, Montréal (Qc) H3C 3P8, Canada

Abstract

The elucidation of the gauge principle "is the most pressing problem in current philosophy of physics" Michael Redhead in 2003. This paper argues for two points that contribute to this elucidation in the context of Yang-Mills theories. 1) Yang-Mills theories, including quantum electrodynamics, form a class. They should be interpreted together. To focus on electrodynamics is potentially misleading. 2) The essential role of gauge and BRST symmetries is to provide a local field theory that can be quantized and would be equivalent to the quantization of the non-local reduced theory. If this is correct, the gauge symmetry is significant, not so much because it implies ontological consequences, but because it allows us to quantize theories that we would not be able to quantize otherwise. Thus, in the context of Yang-Mills theories, it is essentially a pragmatic principle. This does not seem to be the case for the gauge symmetry in general relativity.

Key words: gauge principle, gauge symmetry, Yang-Mills theory, BRST symmetry

1 Introduction

Three of the four best theories for modelling fundamental interactions are of the Yang-Mills type (YM): quantum electrodynamics, quantum electroweak theory and quantum chromodynamics. At the core of each of these theories lies a local gauge symmetry of the same kind. Gauge symmetries in YM theories are peculiar. They do not have direct empirical significance (Brading & Brown, 2004). They imply indeterminism at a classical level (Earman, 2003). They are postulated as a principle

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Email address: `alexandre.guay@gmail.com` (Alexandre Guay).

and are parts of a priori gauge arguments.[1] Nevertheless, these arguments are at best heuristic and/or incomplete (Martin, 2002). In fact, gauge symmetry, which is necessary to keep track of the ambiguity of mathematical representation implicit to these theories, seems to be a useful trick, nothing more. Nevertheless, is there something philosophically valuable to learn from such formal symmetry? Is gauge arbitrariness actually "a deep and far reaching principle" (Itzykson & Zuber, 1980, p. 10)?

Before going further, let us make clear some elements of vocabulary. In this paper, a theory is a *gauge theory* if it exhibits *gauge freedom*. Gauge freedom is an ambiguity in the mathematical representation of physical states. In particular, it implies that the phase space of such theory is structured by orbits of equivalent descriptions (*gauge orbits*). This equivalence is what we define as the *gauge symmetry*. Transformations between elements in the same orbit are called *gauge transformations*. These transformations form the *gauge group* $\mathcal{G}$. A gauge symmetry is said to be *local* if the gauge group is a Lie group depending on a finite number of arbitrary functions. For more details, see (Belot, 2003).

In this paper, two main points will be argued. The first one is methodological. Since all YM theories share the same structure, they should be studied together as a class. Concentrating only on quantum electrodynamics can be misleading. In the next section, we will show what exactly is the common gauge structure possessed by all YM theories and what essential aspect quantum electrodynamics does not share with the others. It is our understanding that the substantial literature concerned with such notions as true gauge and connection substantivalism is the result of focusing on electrodynamics. The second point is more substantial. If the gauge structure of YM theories plays any significant role, it must do so in the quantization process. In this paper, we show that we have good reasons to believe gauge and BRST surplus structure are essentially tools to obtain a field theory[2] that can be easily quantized and *would be equivalent* to the quantization of the reduced theory.[3] So the gauge principle is important, not so much because it "dictates" interactions (Ryder, 1985, p. 81), but rather because it facilitates quantization.

Based on this analysis, we propose an elucidation of the gauge principle in the con-

[1] A gauge argument is an argument justifying and showing the consequences of the *gauge principle*, which asserts that a local gauge symmetry is a requirement for physical theories including interaction. In other words, the imposition of the gauge symmetry is understood to be something like an axiom restricting the possible forms of Lagrangian. A paradigmatic example of gauge argument can be found in (Peskin & Schroeder, 1995, ch. 15).

[2] In this paper, a field is defined as a dynamical variable (scalar, vector...) for a continuous system whose points are indexed by spacetime points. In a physical field theory, fields are freestanding and undecomposable unities by themselves. More details may be found in (Auyang, 1995, ch. 2).

[3] In this context, the reduced theory is a theory where the gauge degrees of freedom have been eliminated. This theory is not necessarily a field theory.

text of YM theories. According to Redhead (2003), the question is how could "the ambiguity of mathematical representation", which is the consequence of the gauge symmetry, have any "physical" significance? Our answer is that these "surpluses of structure", when their role is well understood, do not have ontological significance. Nevertheless, they possess a deep pragmatic significance, because they are just what we must add to quantize certain theories for which we do not have practical alternative quantization methods. In other words, these surpluses are the result of cognitive constraints imposed by available mathematical techniques. If this is correct, the gauge principle plays a different role in YM theories and in general relativity. In general relativity, it seems that the gauge symmetry is involved in a genuine ontological question, namely the problem of time (Earman, 2002).

Note that one originality of the current paper is that all discussions about quantization are made using Feynman functional formalism. Thus, this paper is a complement to (Earman, 2003) who advocates for the constrained Hamiltonian formalism. Also in this paper we use Dewitt's condensed notation. Since this notation is not well known, an introduction to it is presented in Appendix A.

It should be noted that this paper is implicitly arguing for a shift of perspective in the philosophical analysis of YM theories. Usually, the focus of the analysis revolves around classical formulations of YM theories. [4] We argue that, since only quantized versions of YM theories have been successfully applied, quantum YM theories should be the main object of philosophical inquiry. [5] This paper is an example of this position. The fact that we are able to propose a new interpretation of gauge symmetry in this context seems to vindicate this choice.

2 The choice of the typical YM theory

In the study of the YM theories that have been successfully applied in experimental contexts, we face two main difficulties: 1) most of them exhibit a non-Abelian gauge group, and 2) all of them are quantum theories. In the light of these difficulties, it seems reasonable to begin the analysis with the simplest case, the classical $U(1)$ YM theory (the classical Abelian YM theory associated with quantum electrodynamics), hoping that the study of non-Abelian YM theories is unnecessary to interpret the role of the gauge principle. In the next section, we will show how important quantization is for our task. In the present section we will challenge the choice of the classical $U(1)$ theory. This theory possesses specific features that can mislead us in understanding the gauge structure of classical YM theories. However, we are not claiming that the $U(1)$ YM theory does not belong in the same class as

[4] A good review of such analyses is (Martin, 2003).

[5] Note that we do not consider the classical $U(1)$ YM theory to be equivalent to classical electromagnetism. More details in Section 3.

the other YM theories. It decomposes in gauge orbits the space of field histories Φ in a similar way. See Appendix B for details about the specific gauge structure of YM theories. As we can see in Appendix B, the gauge constraint on what can be a YM theory is strong, but not strong to the point of uniquely defining interacting terms of a Lagrangian. It is why gauge arguments require more assumptions than the imposition of gauge symmetry. The gauge constraint does suggest, however, that YM theories form a "natural" class and, therefore, should be analysed together.

Let us now show the main difference between the Abelian and non-Abelian YM theories. In their seminal paper, Yang and Mills, while discussing the possibility of a $SU(2)$ YM theory, assert one of the fundamental aspects of YM theories:

> [W]e wish to explore the possibility of requiring all interactions to be gauge invariant under *independent* rotations of the isotopic spin at all space-time points, so that the relative orientation of the isotopic spin at two space-time points becomes a physically meaningless quantity. (Yang & Mills, 1954, p. 192, original italics)

According to Yang and Mills, the local "phase" (in this case isospin) at a space-time point *is not* a meaningful physical quantity. This passage is often interpreted in a passive way but the active version follows. For example, in the principal fibre formalism, for every change of local section (passive viewpoint) there is a corresponding global automorphism of the principal bundle (active viewpoint) (Choquet-Bruhat et al., 1982, ch. Vbis). So in the context of YM theories, even the value at a space-time point of a free matter field is, strictly speaking, not a physical quantity. Consequently, the notion of a true gauge seems difficult to ground. Furthermore, in YM theories a potential local interaction, when understood as the local value of an interacting field, is not physical. This can be deduced from a paper of Wu and Yang. When discussing $U(1)$ YM theory (quantum electrodynamics) they write: "[Quantum] electromagnetism is thus the gauge-invariant manifestation of nonintegrable phase factor" (Wu & Yang, 1975, p. 3846). This is the signature of a YM interaction and can be generalized to any YM theories. A literal interpretation of this point is the source of gauge potential substantivalism. An example of such a defence can be found in (Leeds, 1999). However, nonintegrable phase factors, when calculated using the gauge potential, are not generally gauge invariant. In fact only phase factors associated with the electromagnetic interaction and calculated for a closed loop are gauge invariant. This is equivalent to noting that, contrary to the case of non-Abelian YM theories, only the field tensor of electrodynamics is gauge invariant. So the electromagnetic interaction is not the norm but the exception, even if it is clearly in the YM class.

The elements already mentioned should make us suspicious of philosophical interpretations of the gauge principle based only on the $U(1)$ theory. Let us clarify in greater detail the difference between Abelian and non-Abelian theories in the geometrical framework of the principal fibre bundle construction (a philosopher's favorite). This representation comes very naturally from a simple model: a charged quantized particle interacting with a *classical* electromagnetic gauge potential. In this case the wave function $\psi(x)$ plays the role of the matter field in a $U(1)$ YM theory. The gauge potential $A_\mu(x)$ acts like a connection and modifies the derivative in such a way that the Schrdinger equation is gauge invariant (Aitchison & Hey, 1989, ch. 2). Taking into serious consideration the notion of A_μ as a connection, we can propose a principal bundle representation. Using this framework, it is easy to generalize so as to include all classical YM theories. [6] So in a YM principal fibre bundle $P(M, G, \pi)$, M corresponds to the spacetime manifold, G to $SU(N)$ (called the structure group). The gauge potential corresponds to the connection in P and the matter field to a section of the associated vector bundle. Active gauge transformations (elements of the gauge group $\mathcal{G}$) correspond to vertical automorphisms of P. This geometrical construction induces a two-step research strategy: 1) concentrate on $P(M, U(1), \pi)$ and identify the relevant features. 2) Propose an interpretation that can be generalized to all YM theories. In a remarkable paper, Healey (2001) invalidates many such positions founded on the $U(1)$ principal bundle, such as the possibility of a true gauge or connexion substantivalism.

It is our belief that these positions are the result of a misunderstanding of the correspondence between a YM theory and a principal fibre bundle. If we return to Yang and Mills, they assert that the local values of fields have no physical meaning, only relations between values at different spacetime points are meaningful. The principal bundle with its rich fibre structure hides this fact. What is needed is a formalism that insists on relations between different spacetime points rather than a formalism like the principal bundle where there is apparently attribution of phase property to each point of the base manifold. This alternative formalism exists. It was developed by Charles Ehresmann in the 1950's following his work on the principal fibre bundle. [7] It is called an Ehresmann or gauge groupoid. [8] The gauge groupoid is a geometrical construction close to the principal fibre bundle. The main difference is that information about particular points in fibres is lost in the groupoid; only relative relations between fibres are kept. Thus the gauge groupoid is more in accordance with Yang and Mills' conception. A definition of the gauge groupoid and its relation to the principal bundle can be found in Appendix C. What is important to remember is that the way in which to relate principal bundle to gauge groupoid is by means of one or more reference points. This dependence made Kirill Mackenzie assert that if a phenomenon on a principal fibre bundle is formulated in groupoid terms then it will be an intrinsic concept (MacKenzie, 1987, p. 30). Thus the gauge groupoid is describing the intrinsic relations between properties at different spacetime points. The introduction of a charge space (fibre in a principal bundle) was inspired by the way we represent degrees of freedom in spacetime, but it is not

[6] For more details on this geometrical construction, see (Daniel & Viallet, 1980).
[7] For an example of his early work on principal bundle see (Ehresmann & Felbau, 1941).
[8] For a general introduction to groupoids in differential geometry, see (Weinstein, 1996).

appropriate here.

Now if we study different YM gauge groupoids, we note that there are many intrinsic structures in Abelian theory that are not intrinsic in non-Abelian theories. For example, holonomy groups are intrinsic only in $U(1)$ theory. However in the general case, the various holonomy groups and isomorphic holonomy bundle arising from a connexion refer to the holonomy groupoid only through the choice of a reference point (MacKenzie, 1987, p. 80).[9] This suggests that philosophical interpretations driven by the $U(1)$ theory should, at best, be made with caution.

YM theories are not the only theories where the associated space of field histories Φ is structured in a fibered way. General relativity is another example. But the manner in which the fibre bundles are structured through the action of a specific set of vector fields $\mathbf{Q}_\alpha$ on history space Φ that do not modify the action S is the signature of YM theories (see Appendix B).[10] Quantum electodynamics, quantum electroweak theory, and quantum chromodynamics admit proper gauge groups $\mathcal{G}$ isomorphic to $U(1)$, $SU(2) \times U(1)$ and $SU(3)$ respectively. Only electrodynamics has an abelian group. Its distinctness comes from this fact. By focusing on the tree of electrodynamics, we are missing the forest.

3 Quantization of Yang-Mills theories

It is useful to compare and contrast the gauge structure between classical YM theories and other classical gauge theories such as General Relativity. But to limit analyses to this kind of work is missing an essential point: only quantized versions of YM theories have been successfully applied in experimental contexts. The model described below is of course the exception. In a strict sense, electromagnetism is not the classical version of quantum electrodynamics. Electromagnetism is a theory of localized charges interacting through the electromagnetic field. The gauge structure is generated from the fact that the field-strength tensor can be expressed using an infinite class of distinct gauge potentials. In quantum electrodynamics, when understood as a YM theory, matter is described by a field. The gauge structure is the result of the particular *coupling* between matter and gauge potential. Contrary to electromagnetism, both fields are changed in a gauge transformation. Why then are these two different theories considered as describing the same interaction? First, they seem to describe the same kind of phenomena. Second, their gauge groups are isomorphic. And third, there is an intermediate theory lying between them: a quantized particle interacting with an unquantized gauge potential. As mentioned before, this intermediate case corresponds roughly to a classical $U(1)$ YM theory.

[9] A similar point is argued in (Healey, 2001).

[10] Note that contrary to i, α is a group index, not a field index. Unless said otherwise, in the rest of the paper this convention between Latin and Greek indices will hold.

In the case of non-abelian YM theories, no such semi-classical version has any experimental application. Here again electrodynamics is the exception.

This said, many difficulties arise. Whatever quantization technique is chosen, the gauge surplus will have to be dealt with. There are four standard ways for doing so: [11] 1) Choose a gauge and then quantize. 2) Reduce the phase space and quantize; in other words, quantize the reduced theory formulated in gauge invariant variable I's that label gauge orbits $\Phi/\mathcal{G}$. 3) Reformulate the YM theory as a constrained Hamiltonian system and then quantize (Dirac, 2001). 4) Impose BRST symmetry and then quantize the resultant theory that is described by a gauge dependent action and that contains unphysical ghosts fields (Becchi et al., 1976). [12]

Let us briefly discuss the first method. Fixing the gauge can be achieved by adding a covariant gauge breaking term to the Lagrangian density, for example $(\partial^\mu A^a_\mu)^2$. Note that to be coherent with the familiar notation, the already mentioned convention between Greek and Latin indices will not be followed for the gauge potential A^a_μ and the field tensor $F^a_{\mu\nu}$. It is well known that in the case of a non-abelian YM theory, the quantum theory so obtained is not unitary. It is believed the gauge condition fails to force the system to be on a hypersurface in Φ that meets each of the gauge orbits exactly once. Equivalently, it is possible to rigorously prove that it is impossible in these cases to define coordinates K^α serving as global coordinates, where K's label points of Φ within each gauge orbit, orbits labeled themselves by coordinates I's. This means that the differential operator $\hat{\mathfrak{F}}$, described in Appendix B, cannot be non-singular globally. Since the vector fields $\mathbf{Q}_\beta$ on Φ are defined as tangent to gauge orbits, the non-singularity of $\hat{\mathfrak{F}}^\alpha_\beta = \mathbf{Q}_\beta K^\alpha$ would mean that $\delta/\delta K^\alpha$ (defined in Appendix B), which are vector fields that commute with each other, would generate Abelian gauge orbits. This result is called the Gribov ambiguity (Singer, 1978). This seems to exclude the possibility of defining a true global gauge. At best we can define arbitrary local gauges (local coordinate systems), an approach aligned with the original (Yang & Mills, 1954). For these reasons we will not discuss further this method.

Once one of the remaining three methods is chosen, a quantization technique has to be chosen. Most of these techniques come in two families: A) canonical methods and B) Feynman functional method. The combination of a method to deal with gauge degrees of freedom plus a quantization technique constitute a quantization path. In the case of the constrained Hamiltonian method, the canonical technique is

[11] A reference on the subject is (Henneaux & Teitelboim, 1992).

[12] Note that previous quantization of YM theories using ghost fields based on Feynman formalism, like (Faddeev & Popov, 1967) is now considered to be using implicitly the BRST method. As for Mandelstam (1968), he starts with a gauge independent formulation of YM theory. However, because of technical difficulties, he does not directly quantize this theory. Rather he adds auxiliary variables, just as one does in the classical gauge dependent formulation. When this construction is quantized using canonical techniques, he obtains the same Feynman rules, with ghost loops, as Faddeev and Popov.

the Dirac constraint quantization. Roughly it corresponds to the canonical quantization, but with the first-class constraints promoted to operators on a Hilbert space which includes unphysical states. These operators are required to annihilate physical states (Henneaux & Teitelboim, 1992, ch. 13), thereby decomposing the Hilbert space appropropriatly. There is no Feynman technique corresponding to this method of reduction of the gauge surplus.

Thus we are left with five quantization paths for YM theories. There is no general proof that these paths are equivalent. In this paper the Feynman sum over histories method will be discussed in more detail. This quantization method has been chosen for three main reasons: 1) It is more attuned with the lessons of special relativity than canonical techniques since it does not depend on the (3+1) dimensional baggage of conjugate momenta and constraints. 2) This approach provides an elegant transition between classical and quantum systems. 3) More importantly, in many relativistic experiments, it is the quantized versions of YM theories obtained by using this method that have been tested. Therefore, at least in the context of YM theories, we can be very confident in this method of quantization. However, during the following subsections other quantization paths are briefly discussed.

3.1 Non-relativistic quantum mechanics

Most philosophical discussions about quantized gauge theory are in the framework of nonrelativistic quantum mechanics and are about the simplest case: quantized particles interacting with an unquantized electromagnetic potential. In particular, this is the framework used to predict the Aharonov-Bohm effect. A local gauge transformation of the potential and of the wave function (represented in the Feynman formalism by the propagator) is a symmetry of the system (Aitchison & Hey, 1989, p. 50). In this case the Feynman propagator [13] takes the form of an integral over all possible trajectories from x to y, $K(y,x) = \int D(\vec{q}(t))e^{iS[\vec{q}(t)]}$, where S is the classical action associated with path q. With the free particle case as a reference, the action of the electromagnetic interaction is to multiply the contribution of each path q (of each possible history) by a nonintegrable phase factor $U(y,x) = e^{-ie\int_q A_\mu dx^\mu}$ (called a Wilson line). Recall that this is, according to Wu and Yang, characteristic of a YM type of interaction. Note that a Wilson line is not in general gauge independent *but the relative change of phase between paths, caused only by electromagnetic interaction, is gauge invariant*. In other words, Wilson loops $U(x,x)$ are gauge independent. Since in the Feynman formalism the phenomena are the result of the interference between contributions of different his-

[13] In this context, the Feynman propagator is the transition amplitude that a charged particle be present at a space-time point y if it was at x. In this paper, we adopt the convention $c = \hbar = 1$.

tories, it is tempting to attribute a physical significance to Wilson loops. [14] In this context, the Aharonov-Bohm effect is just a straightforward illustration of the non-local character of Wilson loops. The non-locality of Wilson loops can be understood in two ways: 1) a Wilson loop, even if it is defined at a certain point $x \in M$, is not a free-standing field variable since it depends on properties at other points. 2) The value of a Wilson loop depends on the flux of the electromagnetic field $F_{\mu\nu}$ inside the loop. A phenomenon that depends strictly on the value of the Wilson loop (like the Aharonov-Bohm effect) could occur in a region where $F_{\mu\nu} = 0$, but still be dependent of the value of $F_{\mu\nu}$ in other regions.

From this we can propose an explanation of the apparent necessity for the gauge principle: the gauge structure is the result of the freedom to produce local descriptions compatible with non-local entities representing interaction, namely Wilson loops. Let us reformulate this point in more detail. A) To quantize this system, we need to be able to compute the action associated with the possible classical trajectories of a particle. B) Using Feynman formalism, this task does not seem possible using directly gauge invariant variable I's, labeling gauge orbits, like Wilson loops. C) Consequently, we describe the system in terms of gauge dependent fields φ^i, like the gauge potential. In this new description, actions of particular paths are computable. D) Finally, we impose gauge invariance in order to keep track of the added unphysical variables. Retrospectively, the gauge principle is pragmatically necessary in order to get a usable quantum theory. Could this explanation be generalized to relativistic cases? We will see in the next section that it is basically possible, but things are not so simple.

3.2 Relativistic quantization

There is a limit to what can be deduced from non-relativistic mechanics. If the focus of one's study is theories modelling fundamental interactions, to do so without special relativity is not an option. Let us now see how we can generalize the propagator $K(y,x) = \int D(\vec{q}(t)) e^{iS[\vec{q}(t)]}$ already discussed. This transition amplitude is a sum over possible paths (or histories). This sum takes the form of a functional integral over the compact Hausdorff space of functions q. To rigorously define the measure of this integral is in general problematic. Nevertheless, by understanding that this measure is a generalization of the Wiener measure (Nelson 1964) and by imposing unitarity, one can argue that $Dq = \mu[q][dq] = const \times [dq]$, where $[dq] := \prod_t d\vec{q}(t)$. Thus the *measure functional* $\mu[q]$ plays the role of a volume density in the space of trajectories (or histories). Now to get a relativistic quantum theory we have to quantize a covariant field theory. The straightforward generalization of the propa-

[14] Note that, in the non-Abelian case, the Wilson line is defined as $U(y,x) = P\left\{ e^{ig\int_0^1 ds \frac{dx^\mu}{ds} A^a_\mu(x(s)) t^a} \right\}$ where P is a prescription of path-ordering. The associated gauge independent Wilson loop is the *trace* of $U(x,x)$.

gator (the transition amplitude between "in" and "out" states) takes on the form in the condensed notation:

$$\langle out|in\rangle = \int e^{iS[\varphi]}\mu[\varphi][d\varphi], \quad [d\varphi] := \prod_i d\varphi^i, \tag{1}$$

where S is the field action. [15] In Equation 1, the sum is not over possible paths but over possible field histories. Analogously to the nonrelativistic case, $\mu[\varphi]$ plays the role of a volume density in the space of field histories Φ. The general evaluation of the measure functional is an unsolved problem. However, by imposing hermiticity one can approximate μ in certain cases (DeWitt-Morette, 1969), namely when $\mu[\varphi]^{-1}\mu_{,i}[\varphi]$ depends only on the properties of φ in the immediate vicinity of the spacetime point associated with i.

As already seen, even after having chosen Feynman quantization, more than one path can bring us from a classical YM theory to its quantum version: 1) first reduce the gauge surplus then quantize in order to obtain a reduced Hilbert space, presumably the "physical" Hilbert space. The reduction is necessary because we want to sum over the distinct (i.e. gauge independent) field histories. 2) Get the quantum theory by quantizing an extended theory built by imposing BRST symmetry. At least in the perturbative regime, the second strategy has been successfully applied. Nonetheless, even if it has not been fully carried out, the first strategy seems more philosophically satisfying since the physical variables are identified. Both strategies will be discussed in this paper.

3.2.1 Reduction of phase space method

In the Feynman approach, the functional integral associated with any transition amplitude between "in" and "out" states takes on the general form

$$\langle out|in\rangle = \int e^{i\dot{S}[I]}\dot{\mu}_I[I][dI], \quad [dI] := \prod_A dI^A \tag{2}$$

where $\dot{S}$ is the classical action S plus all counter terms that will be needed to render the amplitude finite, and $\dot{\mu}_I$ is the functional measure necessary for the normalization of the integral. The index A refers to points of the space of gauge orbits $\Phi/\mathcal{G}$. The integral runs over distinct field histories, thus over I. In this form the integral is too abstract to be very useful. A particular set of variable I's has to be identified in order to proceed further. Since YM theories have no classical application, no classical experiments can help us to choose particular I's. However we notice that all readily available I's depend non-locally on the φ^i (these coordinates depend on the value of φ at more than one point of M). This observation is important

[15] Note that to make the equation more transparent, we omitted the sum over homotopy equivalence classes and sums over parameters compatible with boundary conditions. These are not essential for the remainder of the discussion.

because it suggests that the gauge independent structure of *classical* YM theories is incompatible with a straightforward interpretation of the Humean supervenience defended by David Lewis. Humean supervenience is “[T]he doctrine that all there is to the world is a vast mosaic of local matters of particular fact” (Lewis, 1986, p. ix). Some facts, associated with the gauge independent structure of classical YM theories, are not local. They cannot be expressed as properties strictly defined at spacetime points.

Among the possible choices of specific I’s, Wilson loops are a possibility. But this is not the only choice. For example, if the asymptotic boundary conditions are empty Minkowskian, then one can introduce gauge invariant line integrals coming from infinity. For other conditions other choices can be proposed. How can we choose among these gauge invariant variables? Do we have to choose? An enlightening lesson can be drawn from a simpler case. In classical electromagnetism, the interaction field can be described locally by the electromagnetic field tensor $F_{\mu\nu}$ or non-locally by gauge invariant factors $\oint A_\mu dx^\mu$. Why choose the local description over the non-local one when they are equivalent? There are at least two reasons: 1) experimental applications are more easily explained by a local bearer of force. 2) If energy is conserved locally, $F_{\mu\nu}$ seems better suited as bearer of electromagnetic energy. This example illustrates that, in order to choose among equivalent descriptions, we need external constraints, either experimental or theoretical. For YM theories we lack these resources. No classical applications are known and it is not clear what ontological commitment is suggested by relativistic quantum experiments. Since all available reductions are non-local, a locality commitment cannot be invoked as a constraint. All we can say is that the nonrelativistic case discussed above pushes us towards Wilson loops, but this is not much.

After choosing particular I’s, another serious difficulty is the evaluation of the measure functional $\dot{\mu}_I[I]$. As we have seen, in a system without gauge surplus, $\mu[\varphi]$ plays the role of a volume density in the space of field histories. To give an explicit expression for $\mu[\varphi]$ the usual techniques rely heavily on locality conditions. For example, $\mu[\varphi]^{-1}\mu_{,i}[\varphi]$ should depend only on the properties of φ in the immediate vicinity of the spacetime point associated with i. Because of this, when we deal with non-local variables, quantization is much more complicated, even at the level of approximation. Thus we have a compelling reason to adopt a local description (a field theory), even if it implies adding unphysical variables. But it should be noted that compelling reasons are not necessities. Nothing in equation 2 forbids future developments of the method.

It is important to note that the difficulty in quantizing a reduced YM theory is not just an artefact of the Feynman method. Canonical techniques also encounter serious difficulties when applied to gauge reduced theories. For example, A) the gauge independent formulation of the theory is not always manifestly covariant. B) Since the reduced theory is not a local theory, operators are more difficult to define. C) The Hamiltonian, when expressed in terms of gauge independent degrees of freedom, may turn out to be impossible to give a quantum definition. D) The resultant quantum theory could be very difficult to renormalize. For details on these problems and others see (Gambini & Pullin, 1996).

In this subsection, we discussed motivations to work with theories formulated in terms of fields. Of course, these reasons were not the ones that motivated Yang and Mills. They probably proposed local field theories as a matter of mathematical convenience and by analogy with electromagnetism. It is our inability to appreciate the usefullness of a local description for quantization that created the gauge mystery.

3.2.2 BRST symmetry method

As we said, the BRST method has been successfully applied. This success alone should have been sufficient to incite philosophers to study this method. Why this was not the case we are not sure.

In short, the BRST method consists in imposing a new *global* symmetry on Yang-Mills theories and then quantizing. In practice this consists of adding a gauge breaking term in the action and other dynamical terms that involve new unphysical fields: ghosts, antighosts, and an auxiliary field. In the interest of conciseness, for the remainder of the paper all these unphysical fields will generically be called ghosts. If the new theory is quantized, the quantum theory obtained is unitary and renormalizable (Becchi et al., 1976). The transition amplitude for this formulation of YM theory takes the form

$$\langle out|in\rangle = \int e^{i(\dot{S}[\varphi]+i\chi_\alpha\hat{\mathfrak{F}}^\alpha_\beta[\varphi]\psi^\beta+\omega_\alpha K^\alpha[\varphi]-\frac{1}{2}\omega_\alpha(\kappa^{\alpha\beta})^{-1}\omega_\beta)}\dot{\mu}[\varphi][d\varphi][d\chi][d\psi][d\omega], \quad (3)$$

where κ is a symmetric ultralocal invertible continuous matrix [16] , ψ, χ, ω are different ghost fields, and φ represents the gauge potential and the matter fields. These new fields are called ghost fields because they do not have the same status as the other fields φ. In computation, they essentially compensate for non-physical modes carried by the transition amplitude calculated with the gauge dependent description using fields φ^i. Furthermore in perturbative theory, the ghost fields generate close loops contributions that guarantee renormalizability. Of course what is really puzzling is that, to get rid of a gauge surplus, we *add* another surplus that is apparently even more bizarre than the first one since the added fields are apprently fictive. Thus BRST construction has no simple classical interpretation such as the one based on local/non-local descriptions that we proposed earlier for gauge symmetry. Presented this way, the BRST method looks mysterious. It apparently gives a "Platonist-Pythagorean role for purely mathematical considerations in theoretical physics" (Redhead, 2003, p. 138). This would be plausible if one could prove that

[16] A continuous matrix is ultralocal if it is of the form $\gamma_{\alpha\beta}\delta(x,x')$ and does not contain a differentiated δ function.

the BRST method is incompatible with the reduced space method. If Bryce Dewitt's proof is correct, this conclusion is not justified. [17] According to Dewitt the BRST surplus arises *entirely* from the fibre bundle structure of Φ. To obtain the ghosts, it is not even necessary to integrate over the gauge group as done in the usual presentation. DeWitt's proof makes two important points in the context of *Feynman quantization*:

- In the Feynman formalism, the reduced phase space and the BRST methods give equivalent quantum theories. This is a non-trivial point since an explicit Feynman quantization of a reduced YM theory has not yet been produced.
- The BRST surplus is just what we need to add to a YM theory to make its corresponding quantum version equivalent to an eventual quantum reduced theory. Does this mean that we should have included ghosts in the definition of Φ? We do not think so. Equation D.7 suggests that the two kinds of unphysical variables, gauge and ghost, play a different role in the quantization process. Ghosts are the result of the use of gauge variables. The reverse does not seem true. Gauge variables are doing the localizing job. Ghosts are assuring equivalence between quantum theories.

To quantize a BRST extended YM theory using canonical techniques is also instructive. Since BRST is a global symmetry it implies, by Noether's first theorem, the existence of a conserved charge. The operator associated with this fictitious charge is nilpotent. Its action divides neatly the extended Hilbert space into physical and unphysical subspaces of states (Kugo & Ojima, 1978). It is because of this property that the ghost degrees of freedom are considered metaphorically to be negative degrees of freedom which cancel the positive gauge degrees of freedom. Note that the equivalence between Feynman and canonical BRST quantization has been perturbatively verified. This verification is implicit in (Mandelstam, 1968).

Now we have a richer story. A) To quantize a YM theory we should work with nonlocal variable I's, in a gauge reduced formulation of the theory. B) Since we do not have the mathematical tools to do so, we are forced to quantize an alternative theory in terms of local fields φ^i. In other words, we implicitly apply the gauge principle. C) The discussion above shows that in order to obtain, by integrating on φ^i, a quantum theory equivalent to the hypothetical quantum reduced theory, we need to add ghosts, in other words to impose BRST symmetry. In conclusion, gauge and BRST symmetries are *pragmatic necessities*. They are needed to get what we want: computable quantum YM theories.

[17] The proof is presented in Appendix D.

3.3 *A few words about the Dirac constraint quantization*

For YM theories, the main alternative to BRST quantization (using Feynman or canonical methods) is the Dirac constraint quantization. While we do not have a rigorous proof, we have good reasons to believe that in most cases these quantization paths are formally equivalent; they seem to identify the same physical states in their respective Hilbert space. Observables in each formalism seem also equivalent (Henneaux, 1985). However they could, for practical reasons, be inequivalent. For example, Marc Henneaux suggested that there could be operator ordering problems and other difficulties when using the Dirac method. Faced with a discrepancy between quantization paths, Henneaux proposed the adoption for the BRST as more fundamental. At least in the case of YM theories, the good results obtained with BRST incline us to agree. It would not necessarily be the case for other gauge theories, however.

3.4 *A few words about renormalization*

At the end of (Martin, 2002), Christopher Martin suggested that the requirement of renormalizability is a better principle than the gauge symmetry as the basis for a fundamental theory. Clearly he is right. Renormalizability is a much more fundamental criterion than the gauge symmetry for selecting what kind of interacting terms we should keep in a Lagrangian. Thus renormalizability is a better postulate than gauge symmetry as “a logic of nature”. But in what measure does renormalizability clarify the role of gauge symmetry? Not all gauge theories are renormalizable, for example general relativity. And not all renormalizable theories are gauge theories, for example the ϕ^4 theory used in the study of critical phenomena is not a gauge theory (LeBellac, 1988). [18] Nevertheless all YM theories are renormalizable and the fact that a YM theory can be formulated in terms of fields is important to prove this (Becchi et al., 1976). In fact this property is one of the reasons for the development of these theories. If the requirement of renormalizability is physically fundamental, the analysis of previous sections shows how the gauge symmetry is pragmatically essential to obtain a field theory that is more or less easy to quantize.

[18] The ϕ^4 theory could be considered by some as not a “fundamental” theory. Since we do not really know how to define such terms in the light of research on effective theories, we will not dicuss this further.

4 Conclusion

We have argued that YM theories should be discussed together. It is possible to conceptually define quantization of a YM theory expressed only in the language of its physical non-local variables, but for pragmatic reasons a gauge structure is preferable. The role of the added unphysical variables is precisely to give a field version of the theory. Still, to achieve equivalence between quantization paths using Feynman formalism more unphysical variables (ghosts) are needed.

In conclusion gauge symmetries in YM theories are not as mysterious as is usually thought. The gauge principle is an important step in the quantization process. The real mystery lies behind the gauge structure. How should we characterize physical non-local variables? Already some interesting discussions can be found in the literature, for example (Belot, 1998) or (Healey, 2001), but the intuition behind these is built on electrodynamics. This is problematic since this theory is the exception rather than the rule in the class of YM theories.

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A Some aspects of the condensed notation

In this appendix we present Bryce DeWitt's condensed notation (DeWitt, 1984, part 1). This notation is abstract. Nevertheless, it allows to discuss the quantization of YM theories without getting lost in technical details. To simplify further the discussion we will consider only theories where matter is represented by bosonic fields. Thus let us imagine that we are starting with a collection of boson fields. We will denote $\{\varphi^i(x)\}$ the collection of components of these fields at a point x in an chart $(\mathcal{U}, \phi)$, where a chart is a subset $\mathcal{U}$ of the spacetime manifold M together with a bijective mapping $\phi : \mathcal{U} \to \mathbb{R}^n$ from $\mathcal{U}$ onto an open subset of Euclidean n-space. A chart is also called a coordinate patch. [19] In typical contexts, it is necessary to

[19] Note that in this paper we will put aside the fact that, in general, x is also defined relatively to a field of local Lorentz frames. This simplification does not affect the arguments defended here.

consider several points of spacetime in the same expression. In these cases different points will be distinguished by associating primes to x. For example, we may write:

$$\frac{\vec{\delta}}{\delta\varphi^i(x)}\varphi^j(x') = {}_i\delta^j\delta(x,x'), \quad \varphi^i(x)\frac{\overleftarrow{\delta}}{\delta\varphi^j(x')} = \delta^i{}_j\delta(x,x') \tag{A.1}$$

where $\delta(x,x')$ is a Dirac function on M and ${}_i\delta^j$ and $\delta^i{}_j$ are Knonecker deltas. To simplify these expressions, we can introduce an abbreviation which consists of suppressing the symbol x and placing all primes on their associated indices. Equations A.1 become

$$\frac{\vec{\delta}}{\delta\varphi^i}\varphi^{j'} = {}_i\delta^{j'}, \quad \varphi^i\frac{\overleftarrow{\delta}}{\delta\varphi^{j'}} = \delta^i{}_{j'} \tag{A.2}$$

In the condensed notation, we take a step further. We lump the symbol x's (represented by primes) with the generic index i to make the latter do double duty as a discrete label for the field components and as a continuous label for the points of spacetime. [20] This leads to a condensed notation where primes are omitted. So Equations A.2 become

$$\frac{\vec{\delta}}{\delta\varphi^i}\varphi^j = {}_i\delta^j, \quad \varphi^i\frac{\overleftarrow{\delta}}{\delta\varphi^j} = \delta^i{}_j \tag{A.3}$$

Finally we introduce one last abbreviation. Functional differential will be indicated by a comma followed by one or more indices. Equations A.3 are then written

$${}_{i,}\varphi^j = {}_i\delta^j \quad , \quad \varphi^i{}_{,j} = \delta^i{}_j \tag{A.4}$$

To summarize, in this context, the condensed notation consists in:

(1) Represent all fields in the same category with the same symbol, for example φ.
(2) Represent all indices (contiunous and discrete labels) with generic indices, for example i.
(3) Represent functional derivation by a comma followed by appropriate indices.

B Common aspects of the gauge structure of YM theories

Let us characterize in more detail the gauge structure of YM theories. [21] Components of gauge potentials $A^a_\mu(x)$ and matter boson fields are represented, in the

[20] As (DeWitt, 2003, p. 11, original italics) underscores, the condensed notation implies the following point: "*The manifold M of spacetime, independently of any physical fields that may be imposed on it, is an index set.*"
[21] This appendix relies on (DeWitt, 2003, ch. 2 and 24).

condensed notation, generically by φ^i. The action S is a functional $S : \Phi \rightarrow \mathbb{R}$, where Φ is the space of all possible field histories over spacetime, both those that do and those that do not satisfy the dynamical equations of the system. The full description of Φ generally requires an atlas, a collection of charts of which the φ^i are the coordinates.

For YM theories, the gauge invariance implies the existence, on Φ, of a set of flows that leave the action invariant. More precisely, there is an infinite set of nowhere vanishing vector fields $\mathbf{Q}_\alpha$ on Φ such that $S\mathbf{Q}_\alpha \equiv 0$, where the vector fields are written as operators acting from the right. In coordinates $S_{,i}{}^{i}Q_\alpha \equiv 0$, where ${}^{i}Q_\alpha$ are the components of those fields. As usual in the condensed notation the index α has both a discrete and a continuous (spacetime) part. For YM theories Lie brackets of the $\mathbf{Q}$'s depend linearly on the $\mathbf{Q}$'s themselves: $[\mathbf{Q}_\alpha, \mathbf{Q}_\beta] \equiv \mathbf{Q}_\gamma c^{\gamma}_{\alpha\beta}$. We use primes to distinguish the points associated with various indices: $c^{\gamma''}_{\alpha\beta'} = f^{\gamma}_{\alpha\beta}\delta(x'', x)\delta(x'', x')$, where $f^{\gamma}_{\alpha\beta}$ are the structure constants of the typical fibre of the YM principal bundle, for example $U(1)$ for the case of quantum electrodynamics or $SU(3)$ for the case of quantum chromodynamics.

From these definitions we can deduce that the action remains invariant under infinitesimal changes of φ^i of the form $\delta\varphi^i = {}^{i}Q_\alpha \delta\xi^\alpha$, where $\delta\xi^\alpha$ are arbitrary φ-independent functions over spacetime. The group $\mathcal{G}$ of gauge transformations generated by these infinitesimal transformations is called the gauge group or more precisely proper gauge group. In this paper we will put aside the notion of a full gauge group, which is obtained by appending to the proper gauge group all other φ-independent transformations of Φ onto itself that leave S invariant and do not arise from global symmetries. [22] The closure expressed by the Lie brackets relation implies that Φ is decomposed into subspaces, called *orbits*, to which the $\mathbf{Q}_\alpha$ are tangent. Thus Φ can be viewed as a principal fibre bundle of which the orbits are the fibres (note that this construction is not the one usually discussed in the literature where the spacetime manifold is the base space).

Knowing the fibre structure of Φ, it is convenient to make, at least conceptually, the transformation $\varphi \rightarrow I^A, K^\alpha$, where I's label points in the space of gauge orbits $\Phi/\mathcal{G}$ (considered as fibres) and are gauge invariant $\mathbf{Q}_\alpha I^A \equiv 0$. The K's label points within each fibre (gauge orbit). I^A, K^α constitute a fibre adapted system of coordinates of Φ. As usual for a fibre bundle construction there is no canonical way of associating points on one fibre with those on another.

One often chooses a reference for K's. Usually this consists in singling out a base point φ_* in Φ and choosing the K's to be local functionals of the φ's such that the matrix $\hat{\mathfrak{F}}^{\alpha}_{\beta} = \mathbf{Q}_\beta K^\alpha = K^{\alpha}_{,i}{}^{i}Q_\beta$ is a non singular differential operator at and in

[22] It should be noted that there exist physical observables (invariant under the proper gauge group) that are not invariant under the full group. Therefore the name "gauge transformation" does not seem appropriate for some elements of the full group.

the neighborhood of φ_*. In the region where the operator $\hat{\mathfrak{F}}$ is not singular it can be shown that $\frac{\overleftarrow{\delta}}{\delta K^\alpha} = -\overleftarrow{\mathbf{Q}}_\beta \hat{\mathfrak{G}}^\beta_\alpha$, where $\hat{\mathfrak{G}}$ is a Green's function of $\hat{\mathfrak{F}}$. As expected vertical fields along fibres are generated by gauge transformations. In principle we can also make a specific choice for I's but in practice theses choices depend non-locally on φ^i. In other words, they depend on the components of the fields at different spacetime points.

C Groupoid and principal fibre bundle

A groupoid is a generalization of a group that is particularly handy to express local symmetries of geometrical structure. [23] The particular groupoid that we need here is called the Ehresmann or gauge groupoid. As we will see in this formalism we do not find internal spaces above the base space but arrows that denote differences of internal properties between points of space-time.

Definition 1 (The gauge groupoid) *Let $P(M, G, \pi)$ be a principal bundle. Let G act on $P \times P$ to the right by $(p_2, p_1)g = (p_2 g, p_1 g)$; denote the orbit of (p_2, p_1) by $\langle p_2, p_1 \rangle$ and the set of orbits by $\frac{P \times P}{G}$, then the gauge groupoid consists of two sets $\Omega = \frac{P \times P}{G}$ and M, called respectively the groupoid and the base, together with two maps $\alpha : \langle p_2, p_1 \rangle \mapsto \pi(p_1)$ and $\beta : \langle p_2, p_1 \rangle \mapsto \pi(p_2)$, called respectively the source and the target, a map $\varepsilon : M \to \Omega; x \mapsto \tilde{x} = \langle p, p \rangle$, called the object inclusion map, and of a partial multiplication in Ω defined on the set $\Omega \times \Omega = \{(\langle p_3, p_2' \rangle, \langle p_2, p_1 \rangle) \in \Omega \times \Omega \,|\, \alpha(\langle p_3, p_2' \rangle) = \beta(\langle p_2, p_1 \rangle)\}$*

$$\langle p_3, p_2' \rangle \langle p_2, p_1 \rangle = \langle p_3, p_1 \delta(p_2', p_2) \rangle \tag{C.1}$$

where $\delta : P \times P \to G$ is the map $(pg, p) \mapsto g$.

We can choose any representative of a groupoid element (an orbit). For example, we want to multiply $\langle p_3, p_2' \rangle$ by $\langle p_2, p_1 \rangle$. To know that $\alpha(\langle p_3, p_2' \rangle) = \beta(\langle p_2, p_1 \rangle)$ guarantees that p_2 and p_2' referred to the same spacetime point. By means of a "gauge transformation" one can always choose representatives on the orbits so that $p_2 = p_2'$. In this case, the multiplication becomes a "cancellation of the middle". This gauge freedom is clearly passive. The principal fibre bundle has more structure than the gauge groupoid. Therefore the two constructions are not bijective, but an isomorphism between them can be defined.

Definition 2 (Isomorphism between the bundle P and the gauge groupoid Ω) *Let $P(M, G, \pi)$ be a principal fibre bundle. Choose $p_0 \in P$ (a reference point) and*

[23] Groupoids are also used outside differential geometry. For example, a groupoid can be seen as a special case of category where all morphisms are reversible (Ehresmann, 1965, ch. 1).

write $x_0 = \pi(p_0)$. Then the map

$$P \to \left. \frac{P \times P}{G} \right|_{x_0}; \quad p \mapsto \langle p, p_0 \rangle \tag{C.2}$$

is a homeomorphism. The map

$$G \to \left. \frac{P \times P}{G} \right|_{x_0}^{x_0}; \quad g \mapsto \langle p_0 g, p_0 \rangle \tag{C.3}$$

is an isomorphism of topological gauge groups. Together they form an isomorphism between the principal bundle and the gauge groupoid.

D DeWitt's proof of quantization equivalence

The reader might be surprised by the following proof about quantization equivalence because it is not well known. [24] At least formally we know that the right way to quantize is equation 2. Unfortunately working directly with non-local I's is difficult. To bring the local variables φ^i into the integral one must first introduce the remaining variables K^α of the fibre adapted coordinates of Φ. Only then will one be able to transform coordinates I, K's to φ's. The strategy adopted is to introduce an unity in the integral (Equation 2) using a functional $\Delta[I]$ in which the variables K's are explicitly integrated. Let $\Omega[I, K]$ be a real scalar functional on Φ such that the integral

$$\Delta[I] := \int e^{i\Omega[I,K]} \mu_K[I, K][dK], \quad [dK] := \prod_\alpha dK^\alpha \tag{D.1}$$

exists and is non-vanishing for all I. For $\Delta[I]$ to be invariant under changes (generally I dependent) of the fibre adapted coordinates K^α, the measure must transform like $\mu_{K'}[I', K'] = \mu_K[I, K]\frac{\delta(K)}{\delta(K')}$. Using Δ we can rewrite Equation 2 (the transition amplitude defined as an integral on I's) as

$$\langle out|in \rangle = \int [dI] \int [dK] \dot{\mu}_{I,K}[I, K] \Delta[I]^{-1} e^{i(\dot{S}[I] + \Omega[I,K])}, \tag{D.2}$$

where $\dot{\mu}_{I,K}[I, K] := \dot{\mu}_I[I]\mu_K[I, K]$. To pass from I, K to the local φ one must include the Jacobian $J[\varphi] = \frac{\delta[I,K]}{\delta[\varphi]} = det \begin{pmatrix} I^A_{,i} \\ K^\alpha_{,i} \end{pmatrix}$. So equation D.2 becomes

$$\langle out|in \rangle = \int [d\varphi] \dot{\mu}_{I,K}[\varphi] \Delta[\varphi]^{-1} J[\varphi] e^{i(\dot{S}[\varphi] + \Omega[\varphi])}, \quad [d\varphi] := \prod_i d\varphi^i, \tag{D.3}$$

[24] Note that the following discussion relies greatly on (DeWitt, 2005).

In this form the measure of the integral is not well defined. To clarify this point let us study how the Jacobian behaves under an infinitesimal transformation of the fibre adapted coordinates: $K'^{\alpha} = K^{\alpha} + \delta K^{\alpha}[I, K]$. Formally this transformation will modify the Jacobian $J[\varphi]$:

$$\delta J = J \varphi^{i}_{,\alpha} \delta K^{\alpha}_{,i} \tag{D.4}$$

The factor $\varphi^{i}_{,\alpha}$ is apparently difficult to interpret because of the presence of two kinds of indices. However in Apprendix B we define the derivative $\frac{\overleftarrow{\delta}}{\delta K^{\alpha}} = -\overleftarrow{\mathbf{Q}}_{\beta} \hat{\mathfrak{G}}^{\beta}_{\alpha}$, where $\hat{\mathfrak{G}}$ is a Green function of $\hat{\mathfrak{F}}$, also defined in the same Appendix. Using this derivative we obtain

$$\varphi^{i}_{,\alpha} = -{}^{i}Q_{\beta} \hat{\mathfrak{G}}^{\beta}_{\alpha}. \tag{D.5}$$

At this point different choices of $\mathfrak{G}$ are possible, but if we choose to stay coherent with the boundary conditions appropriate to the integral then

$$\delta \ln J = -{}^{i}Q_{\beta} \hat{\mathfrak{G}}^{\beta}_{\alpha} \delta K^{\alpha}_{,i} = -\hat{\mathfrak{G}}^{\beta}_{\alpha} \delta \hat{\mathfrak{F}}^{\alpha}_{\beta} = -\delta \ln \det \hat{\mathfrak{G}}. \tag{D.6}$$

This proves that $J \det \hat{\mathfrak{G}}$ is independent of how the coordinates K^{α} are chosen. Therefore it only depends of I's and hence is gauge invariant. Moreover this product transforms as a scalar density of unit weight under transformations of the coordinates φ^{i} (DeWitt, 2003, ch. 10). This product is thus an essential element for building the functional measure.

Now if we place ourselves in the context of the loop expansion (the context in which ghost fields are usually used), we can pretend that the K^{α} can be global coordinates. In other words the K^{α} are coordinates of the tangent space. In this case an appropriate choice for Ω is $\Omega := \frac{1}{2}\kappa_{\alpha\beta} K^{\alpha} K^{\beta}$, where $\kappa_{\alpha\beta}$ is a symmetric ultralocal invertible continuous matrix. Since we are staying in a single chart we can choose $\mu_{I,K}[K] = 1$, then $\Delta = const \times (\det \kappa)^{-1/2}$. Equation D.3 takes the form

$$\langle out|in \rangle = \int [d\varphi] \dot{\mu}[\varphi] (\det \hat{\mathfrak{G}})^{-1} e^{i(\dot{S}[\varphi] + \frac{1}{2}\kappa_{\alpha\beta} K^{\alpha} K^{\beta})}, \tag{D.7}$$

where $\dot{\mu}[\varphi] = const \times \dot{\mu}_{I}[\varphi] (\det \kappa)^{1/2} J[\varphi] \det \hat{\mathfrak{G}}$. This new measure is to be used when the integration is carried out over the whole space of histories Φ rather than just over the base space $\Phi/\mathcal{G}$.

Two remarks about equation D.7. 1) There is now a gauge breaking term in the exponent of the integrand. It was put in naturally to guarantee the good behaviour of the integral when summing over K's. 2) A factor $(det\hat{\mathfrak{G}})^{-1}$ appears in the integrand. If expanded, it is this factor that gives rise to all ghost loops in the loop expansion that make the action invariant under a BRST transformation. In other words, it can be shown that equation D.7 and equation 3 are equivalent (DeWitt 2003, ch. 24).

This derivation shows that the surplus structure of the BRST construction was expected. It is the result of the fibre structure of Φ when defined with fields and from the Jacobian of the transformation from I, K's to φ's. In consequence there is no mystery related to BRST symmetry.

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