# The Duality between the Boussinesq and Non-Boussinesq Hydrostatic Equations of Motion

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ABSTRACT

The hydrostatic equations of motion for ocean circulation, written in terms of pressure as the vertical coordinate, and without making the Boussinesq approximation in the continuity equation, correspond very closely with the hydrostatic Boussinesq equations written in terms of depth as the vertical coordinate. Two mathematical equivalences between these non-Boussinesq and Boussinesq equation sets are demonstrated: first, for motions over a level bottom; second, for general motions with a rigid lid. A third non-Boussinesq equation set, for general motions with a free surface, is derived and is shown to possess a similar duality with the Boussinesq set after making due allowance for exchange of the roles of bottom pressure and sea surface height in the boundary conditions, a reversal of the direction of integration of the hydrostatic equation, and substitution of specific volume for density in the hydrostatic equation. The crucial simplification in these equations of motion comes from the hydrostatic approximation, not the Boussinesq approximation. A practical consequence is that numerical ocean circulation models that are based on the Boussinesq equations can, with very minimal rearrangement and reinterpretation, be made free of the strictures of the Boussinesq approximation, especially the ones that follow from its neglect of density dilatation in the conservation of mass.

## 1. Introduction

In the theory of ocean circulation, it is commonplace to make the Boussinesq approximation in which the full variation of density is retained only in the buoyancy force in the vertical momentum balance. Wherever else it occurs, in the horizontal momentum balance, and in the continuity equation, density is replaced by a constant (in the simplest implementation). In particular, this means that the three-dimensional velocity field is assumed to be solenoidal.

Spiegel and Veronis (1960) are generally credited with the justification of the accuracy of the Boussinesq approximation, though their focus was on nonhydrostatic, convective motion. However, the quantitative accuracy of the approximation for ocean circulation studies has been questioned. McDougall and Garrett (1992) concluded that the neglect of the dilatation term in the continuity equation may be consequential for conservation of scalars like salinity. Boussinesq circulation models, conserving volume rather than mass, may not accurately predict long-term sea level change in response to heating and cooling, whether by surface exchange or advection (Kawabe 1994). Simple global corrections have been proposed to remedy this deficiency (Greatbatch 1994; Mellor and Ezer 1995). Dukowicz (1997) considered the effects of taking the dilatation term into account in ocean circulation models. Huang et al. (2001) developed a non-Boussinesq ocean circulation model based on normalized pressure coordinates. Greatbatch et al. (2001) and McDougall et al. (2002) suggested a reinterpretation of the Eulerian velocity to accomplish a similar purpose. Lu (2001) proposed a transformation of the vertical velocity that makes the continuity equation mass- rather than volume-conserving.

We will reexamine the Boussinesq approximation in this paper and show that it is inessential to the simplified form of the equations of motion that it seems to afford. Rather, the crucial simplification is the hydrostatic assumption, and the further Boussinesq approximation is unnecessary. The key to this demonstration is the representation of the equations of motion in terms of pressure as vertical coordinate instead of depth. We will obtain a set of equations of the same form as the Boussinesq equations, but with the ''free-surface'' boundary condition imposed on the bottom pressure (instead of sea surface height). These equations conserve total ocean mass, not volume. By further transformation of these equations, the

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free-surface condition may be transposed to the ocean surface. A non-Boussinesq set of equations, which conserve mass, not volume, may be obtained with an additional approximation in the horizontal momentum balance only. This set is an exact dual of the Boussinesq equations, in the sense that the roles of bottom pressure and sea surface height are reversed.

## 2. Primitive equations in *p* coordinates

The equations of motion written in terms of pressure as an independent variable are well-known, much used in dynamical meteorology, and can be found, for example, in Haltiner and Williams (1980):

$$\frac{D\mathbf{u}}{Dt} = -\nabla_p M - f\mathbf{k}\times\mathbf{u} + \mathbf{F}, \quad (2.1)$$

$$\frac{\partial M}{\partial p} = -\alpha + \rho_0^{-1}, \quad (2.2)$$

$$\nabla_p\cdot\mathbf{u} + \frac{\partial\omega}{\partial p} = 0. \quad (2.3)$$

$$\frac{D\theta}{Dt} = Q, \quad (2.4)$$

and, for the ocean,

$$\frac{DS}{Dt} = Q_s. \quad (2.5)$$

In these equations, $\mathbf{u}$ is the horizontal velocity, $M = gz + p/\rho_0$ is the Montgomery function, $\mathbf{F}$ is the frictional force per unit mass, and $\nabla_p$ is the gradient along pressure surfaces; the substantial rate-of-change operator is

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t}\right)_p + \mathbf{u}\cdot\nabla_p + \omega\frac{\partial}{\partial p}$$
$$= \left(\frac{\partial}{\partial t}\right)_z + \mathbf{u}\cdot\nabla_z + w\frac{\partial}{\partial z}, \quad (2.6)$$

where $\omega$ is the pressure tendency, defined by

$$\omega = \frac{Dp}{Dt} \quad (2.7)$$

(see appendix A for details), $w$ is vertical velocity, and $\nabla_z$ is the horizontal gradient at constant depth $z$. Specific volume (reciprocal of density) is given by the equation of state

$$\rho^{-1} = \alpha = \alpha(S, T, p) = \tilde{\alpha}(S, \theta, p), \quad (2.8)$$

the second form of which is written in terms of potential temperature $\theta$ rather than in situ temperature $T$ (Jackett and McDougall 1995). The constant bias $\rho_0^{-1}$ applied to specific volume in (2.2) and used in defining $M$ is quite arbitrary but is conveniently chosen so that $\rho_0$ is a mean density. Temperature and salinity are controlled by the heat and salt balances, (2.4) and (2.5), in which the irreversible transfer processes are given by $Q$ and $Q_s$, respectively. Because of (2.6), scalar conservation equations such as (2.4) and (2.5) have the same form in pressure or depth coordinates.

The solenoidal form of the continuity Eq. [(2.3)] for the three-dimensional pseudovelocity ($\mathbf{u}$, $\omega$) is a consequence of the hydrostatic assumption (2.2); it does not require the neglect of dilatation $\rho^{-1}D\rho/Dt$. For when the continuity equation,

$$\frac{D\rho}{Dt} + \rho\left(\nabla_z\cdot\mathbf{u} + \frac{\partial w}{\partial z}\right) = 0 \quad (2.9)$$

is transformed to a general coordinate $p$ (not necessarily pressure) that replaces $z$, it becomes (appendix A)

$$\frac{D}{Dt}(\rho z_p) + \rho z_p\left(\nabla_p\cdot\mathbf{u} + \frac{\partial\omega}{\partial p}\right) = 0 \quad (2.10)$$

in which the Jacobian of the transformation $\partial z/\partial p$ multiplies the density (de Szoeke 2000). But if $p$ is indeed pressure, assumed hydrostatic, then $\rho z_p = -1/g$, which is constant, so that (2.3) follows. Hence, (2.3) represents full conservation, not of volume but of *mass*, including compressibility, or dilatation, effects. The same hydrostatic assumption, whereby vertical inertia is neglected, is responsible for a crucial simplification in the horizontal momentum balance of (2.1). Otherwise, (2.1) would contain an additional term on the left, $(\nabla_p z)\, Dw/Dt$, representing the effect of vertical inertia along slanting pressure surfaces.

### *Boundary conditions*

At the free surface, $z = \eta(x, y, t)$, we assume that pressure is constant (taken to be zero), and that the surface is material:

$$M = g\eta, \qquad \omega = 0 \quad \text{at } p = 0 \quad (2.11)$$

The ocean bottom at $z = -H(x, y)$, where pressure is $p_b(x, y, t)$, is likewise a material surface, so that

$$M = \frac{p_b}{\rho_0} - gH, \qquad \omega = \frac{\partial p_b}{\partial t} + \mathbf{u}\cdot\nabla p_b$$
$$\text{at } p = p_b. \quad (2.12)$$

By integrating the continuity equation (2.3) from $p = 0$ to $p = p_b$, and using (2.11) and (2.12) for $\omega$, one obtains

$$\nabla\cdot\left(\int_0^{p_b}\mathbf{u}\,dp\right) + \frac{\partial p_b}{\partial t} = 0, \quad (2.13)$$

which furnishes a prognostic equation for $p_b$.

There are a number of useful results related to (2.11)–(2.13), which provide simplification of these exact boundary conditions. By integrating the hydrostatic relation (2.2) from $p = 0$ to $p = p_b$, one obtains a relation between the free surface $\eta$, bottom topography $H$, and bottom pressure $p_b$:

$$g(\eta + H) = \int_0^{p_b} \alpha\, dp. \quad (2.14)$$

By writing $\alpha = \rho_0^{-1} + \hat{\delta}$, where $\rho_0$ is constant, so chosen that $\hat{\delta} \ll \rho_0^{-1}$, one sees that, approximately,

$$p_b = \rho_0 g(H + \eta - \hat{\eta}), \quad (2.15a)$$

where

$$\hat{\eta} = g^{-1} \int_0^{\rho_0 gH} \hat{\delta}\, dp. \quad (2.15b)$$

Free surface variations $\eta$, and contributions to bottom pressure due to internal density changes, expressed as an equivalent surface height $\hat{\eta}$, are of order 1 m. The replacement of $p_b$ by $\rho_0 gH$ in the limit of integration in (2.15b) incurs an error of millimeters or less in the deep ocean. In appendix B, we shall consider a different partition of $\alpha$ that produces a slightly more accurate approximation than (2.15).

In appendix C, we show that conditions (2.12), imposed at time-varying $p_b$, are equivalent to

$$M = \frac{p_b}{\rho_0} - gH,$$
$$\omega = \rho_0 g\mathbf{u}\cdot\nabla H + \frac{\partial}{\partial t}p_b + \nabla\cdot[\mathbf{u}(p_b - \rho_0 gH)] \quad \text{at } p = \rho_0 gH, \quad (2.16)$$

correct to second order in $(\eta - \hat{\eta})/H$. These conditions are imposed at a time-fixed pressure. Such fixed conditions are more tractable and convenient than ones at a point varying in time. Integrating (2.3) from $p = 0$ to $p = \rho_0 gH$ and using (2.11) and (2.16), rather than (2.12), we arrive at the same equation as (2.13). The neglect of the third term for $\omega$ in (2.16) is seen to be equivalent to approximating by $p_b = \rho_0 gH$ the upper limit of integration in the mass flux integral of (2.13).

Equation (2.13) is a prognostic equation for $p_b$. Once $p_b$ is advanced in time, it may be used as the bottom boundary condition for $M$ [first of conditions (2.16)]. The hydrostatic equation [(2.2)] may then be integrated from the bottom to the surface ($p = 0$), where it furnishes the sea surface elevation $\eta$ [first of (2.11)].

In summary, the set (2.1)–(2.5), with equation of state (2.8), boundary conditions (2.11) and (2.16)—the latter with variable bottom topography, $H(x, y)$—and the derived prognostic equation (2.13) for $p_b$, is a practical set of non-Boussinesq equations for hydrostatic ocean circulation. For example, this very set, further transformed into isopycnal coordinates, is the basis of the ocean circulation model developed by Bleck and Smith (1990).

In the next section we will write down the hydrostatic Boussinesq equations and exhibit the first of the promised correspondences with the set above.

## 3. Boussinesq equations in *z* coordinates

A commonly used version of the Boussinesq approximation in $z$ coordinates gives the following equations:

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho_0}\nabla_z P - f\mathbf{k}\times\mathbf{u} + \mathbf{F}, \quad (3.1)$$

$$\frac{1}{\rho_0}\frac{\partial P}{\partial z} = -g(\rho - \rho_0)/\rho_0, \quad (3.2)$$

$$\nabla_z\cdot\mathbf{u} + \frac{\partial w}{\partial z} = 0, \quad (3.3)$$

$$\frac{D\theta}{Dt} = Q, \quad (3.4)$$

$$\frac{DS}{Dt} = Q_s, \quad \text{and} \quad (3.5)$$

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t}\right)_z + \mathbf{u}\cdot\nabla_z + w\frac{\partial}{\partial z}, \quad (3.6)$$

with boundary conditions

$$P = \rho_0 g\eta, \qquad w = \frac{D\eta}{Dt} \qquad \text{at } z = \eta, \quad (3.7)$$

$$P = p_b - \rho_0 gH, \qquad w = -\mathbf{u}\cdot\nabla H \quad \text{at } z = -H, \quad (3.8)$$

where $P = p + \rho_0 gz$ is the excess pressure over standard hydrostatic pressure $-\rho_0 gz$ ($\rho_0$ is a suitably chosen reference density; see above).

Integrating (3.3) from $z = -H$ to $z = \eta$, and using (3.7), (3.8), one obtains

$$\nabla\cdot\left(\int_{-H}^{\eta}\mathbf{u}\, dz\right) + \frac{\partial\eta}{\partial t} = 0 \quad (3.9)$$

[cf. (2.13)]. By means similar to those described in appendix B, one may show that the conditions (3.7) are approximately equivalent to

$$P = \rho_0 g\eta, \qquad w = \frac{\partial\eta}{\partial t} + \nabla_z\cdot(\mathbf{u}\eta) \quad \text{at } z = 0. \quad (3.10)$$

The neglect of the $\nabla_z\cdot(\mathbf{u}\eta)$ term in (3.10) is equivalent to approximating $\eta$ in the limit of integration in the volume flux integral in (3.9) by zero.

Comparing (3.1)–(3.6) with (2.1)–(2.6), we note an exact parallel between the two sets, upon making the following identification of variables:

$$\left.\begin{aligned} p &= -\rho_0 gz \\ \omega &= -\rho_0 gw \\ M &= P/\rho_0 \\ \rho_0(\alpha - \rho_0^{-1}) &= -(\rho - \rho_0)/\rho_0. \end{aligned}\right\} \quad (3.11)$$

These are purely mathematical substitutions, certainly not exact physical identities, made to bring out the duality between the two sets of equations. Their meaning is that if one merely reinterprets $z$ and $w$, for example, from the putative Boussinesq model (3.1)–(3.6) as pressure $p$ and pressure tendency $\omega$ (due care being taken of rescaling and sign reversals), then one sees that no Boussinesq approximation was ever necessary. The last of (3.11) is perhaps prone to confusion: it does not contradict the elementary physical identity $\alpha = \rho^{-1}$. What it means is that in the Boussinesq model, one must replace scaled density anomaly, $\rho_0^{-1}[\rho(S, \theta, p) - \rho_0]$, by the negative of scaled specific volume anomaly, $-\rho_0[\alpha(S, \theta, p) - \rho_0^{-1}]$, to replicate the non-Boussinesq model. A useful bonus of this replacement is that it automatically overcomes a difficulty noted by Dewar et al. (1998) in the integration of the Boussinesq hydrostatic equation (3.2), where $p = -\rho_0 gz + P$ occurs on the right side through the equation of state $\rho(S, \theta, p)$, necessitating some iteration to solve accurately for $P$. However, with (2.2), it is merely the independent variable $p$ that occurs conveniently in the specific volume $\alpha(S, \theta, p)$ on the right side.

The boundary conditions do not preserve the exact parallels of the differential equations. Still, the contrasts and similarities are interesting. On substituting (3.11), the boundary condition (2.11) becomes

$$P = \rho_0 g\eta, \qquad w = 0 \quad \text{at } z = 0, \tag{3.12}$$

while (2.16) becomes

$$\begin{aligned} P &= p_b - \rho_0 gH, \\ w &= -\mathbf{u}\cdot\nabla H - (\rho_0 g)^{-1}\left\{\frac{\partial p_b}{\partial t} + \nabla\cdot[\mathbf{u}(p_b - \rho_0 gH)]\right\} \\ &\quad \text{at } z = -H. \end{aligned} \tag{3.13}$$

The contrasts with (3.10) and (3.8) are curious: the free-surface condition involving the unknown time-dependent variable $\eta$ at $z = 0$ in (3.10) has shifted to one involving the unknown time-dependent bottom pressure $p_b$ at $z = -H$. An exact parallel between the boundary conditions can be established if mean depth is constant, $H =$ const. For then the substitutions

$$\left.\begin{aligned} z' &= -(z + H) \\ w' &= -w \\ -(\rho - \rho_0)/\rho_0 &= \rho_0\hat{\delta} \\ \eta' + H &= p_b/\rho_0 g \\ p_b'/\rho_0 g &= \eta + H \end{aligned}\right\} \tag{3.14}$$

show that (3.12) and (3.13) become

$$P = p_b' - \rho_0 gH, \qquad w' = 0 \qquad \text{at } z' = -H, \tag{3.15}$$

$$P = \rho_0 g\eta', \qquad w' = \frac{\partial \eta'}{\partial t} + \nabla\cdot(\mathbf{u}\eta') \quad \text{at } z' = 0, \tag{3.16}$$

while the differential equations retain the form (3.1)–(3.6). Conditions (3.15), (3.16) are similar to (3.8), (3.10), respectively.

This means that a Boussinesq model such as (3.1)–(3.6), (3.8), (3.10) may be reinterpreted as a non-Boussinesq model merely by making the substitutions (3.11) and (3.14), at least for the case of a level bottom. The substitutions (3.14) show that the roles of free-surface elevation and bottom pressure variation are interchanged between the Boussinesq and non-Boussinesq models. In the non-Boussinesq model bottom pressure is a prognostic variable, predicted by (2.13), while sea surface elevation is diagnostically determined from the Montgomery function at $p = 0$; for a Boussinesq model the roles of the variables are reversed. Too much should not be made of this symmetry. Boundary conditions (3.12) at the sea surface, $z = 0$, represent only the kinematic and normal-stress dynamical conditions. In addition there will be a tangential-stress condition involving the wind stress, and analogous conditions involving heat flux and salt flux. [The kinematic conditions $w = 0$ of (3.12) can readily be modified to take into account evaporation and precipitation: $w = E - P$.] In the primed set of equations given by making the substitution (3.14), these conditions are imposed at $z' = -H$. So it appears as though the wind stress, surface heat flux, etc., are imposed on the ocean bottom in the primed system! A forced ocean circulation solution obtained from a Boussinesq model corresponds to an upside-down, forced from the bottom, non-Boussinesq solution. Nonetheless, the underlying mathematical framework is identical, and, when the non-Boussinesq view is taken, it is *not* subject to the usual strictures of the Boussinesq approximations; namely, the neglect of volume dilatation terms in the continuity equation, and density anomalies in the horizontal momentum balance.

When bottom topography is not level, the analog of the free-surface condition still appears at the bottom in the non-Boussinesq model, but is complicated by the addition of the topographic effect. An exact parallel to the Boussinesq equations can only be drawn if the conventional Boussinesq boundary conditions are modified slightly. The advantages of this approach, which is outlined below, are that mass (rather than volume) remains conserved, and that small density variations in the horizontal momentum balance are not neglected, all without any essential mathematical complication. This last point is important, for the express calculation of volume dilatation, for example, in the Boussinesq continuity equation requires extra effort and approximation, and does not even guarantee conservation of mass (Dukowicz 1997).

## 4. Rigid lid

Before proceeding to the modified mass-conserving equations in section 5, we show that an exact parallel exists in the presence of variable bottom topography if the rigid-lid approximation is made, that is, if boundary conditions (3.7) are replaced by

$$P = \rho_0 g\eta, \qquad w = 0 \quad \text{at } z = 0 \tag{4.1}$$

Boundary condition (4.1) resembles condition (2.11) from the pressure-coordinate set of equations (2.1)–(2.8), (2.11), (2.12). Bottom boundary condition (2.12) can be made to resemble (3.8) by insisting that $p_b$ be independent of time (analogous to clamping the surface):

$$M = p_b/\rho_0 - gH, \qquad \omega = \mathbf{u}\cdot\nabla p_b \quad \text{at } p = p_b(x, y). \tag{4.2}$$

The first of these relations prescribes $H$ as a function of time! The variable part of this can be interpreted as equivalent bottom pressure variation, due to free-surface and steric height variations, given by (2.15). This is analogous to the interpretation of surface pressure $P$ in the rigid-lid $z$-coordinate equations as surface height $\eta$.

The pressure-coordinate model, (2.1)–(2.6) with boundary conditions (2.11) and (4.2), is an exact mathematical analogue of the rigid-lid $z$-coordinate Boussinesq model, (3.1)–(3.6) with boundary conditions (4.1) and (3.8). In the former, no Boussinesq approximation is invoked. Although both models share the limitations of the rigid-lid approximation as a representation of the ocean, this analysis illustrates that the reinterpretation of rigid-lid, Boussinesq, depth-coordinate, ocean circulation models as fixed bottom-pressure, non-Boussinesq, pressure-coordinate models is straightforward. This suggests that the concerns raised by McDougall and Garrett (1992) regarding non-Boussinesq effects on tracer mixing may be at least partially addressed by an a posteriori change of variables and thus that these effects evidently lead only to a mild distortion of the non-Boussinesq solution rather than to fundamental changes in it.

## 5. A mass-conserving form of the equations

We return now to the general case of a compliant free-surface and variable-depth ocean, and we consider a modification of the equations of motion (2.1)–(2.6), and the boundary conditions (2.11), (2.12), that will arrive at an approximate set of equations that conserve mass rather than volume, but nonetheless display a duality with the Boussinesq equations. We transform from $p$ as independent coordinate to modified pressure, defined by

$$p' = p - \rho_0 g[\eta(x, y, t) - \hat{\eta}(x, y, t)], \tag{5.1}$$

where $\eta$ is the free-surface displacement and $\hat{\eta}$ is given by (2.15b). This entails a corresponding shift in $\omega$,

$$\omega' = \omega - \rho_0 g\frac{D(\eta - \hat{\eta})}{Dt}. \tag{5.2}$$

Hence boundary conditions (2.11), (2.12) become

$$M = g\eta \qquad \omega' = -\rho_0 g\frac{D(\eta - \hat{\eta})}{Dt} \quad \text{at } p' = -\rho_0 g(\eta - \hat{\eta}), \tag{5.3}$$

$$M = g(\eta - \hat{\eta}), \qquad \omega' = \rho_0 g\mathbf{u}\cdot\nabla H \quad \text{at } p' = \rho_0 gH \tag{5.4}$$

where, as before, $M = gz + p/\rho_0 = gz + p'/\rho_0 + g(\eta - \hat{\eta})$. The boundary condition (5.3) may be approximated by

$$M = g\eta - \rho_0\hat{\delta}g(\eta - \hat{\eta}),$$
$$\omega' = -\rho_0 g\left\{\frac{\partial(\eta - \hat{\eta})}{\partial t} + \nabla_{p'}\cdot[(\eta - \hat{\eta})\mathbf{u}]\right\} \quad \text{at } p' = 0, \tag{5.5}$$

following a procedure similar to the approximation of the free-surface condition (3.7) by (3.10).

Equations (2.1)–(2.6) become

$$\frac{D\mathbf{u}}{Dt} = -\nabla_{p'}M - g\rho_0\hat{\delta}\nabla_{p'}(\eta - \hat{\eta}) - f\mathbf{k}\times\mathbf{u} + \mathbf{F}, \tag{5.6}$$

$$\frac{\partial M}{\partial p'} = -\alpha + \rho_0^{-1}, \tag{5.7}$$

$$\nabla_{p'}\cdot\mathbf{u} + \frac{\partial\omega'}{\partial p'} = 0, \tag{5.8}$$

$$\frac{D\theta}{Dt} = Q, \tag{5.9}$$

$$\frac{DS}{Dt} = Q_s, \tag{5.10}$$

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t}\right)_{p'} + \mathbf{u}\cdot\nabla_{p'} + \omega'\frac{\partial}{\partial p'}. \tag{5.11}$$

The integrated continuity equation (2.13) becomes

$$\nabla\cdot\left(\int_{-\rho_0 g(\eta-\hat{\eta})}^{\rho_0 gH}\mathbf{u}\,dp'\right) + \rho_0 g\frac{\partial}{\partial t}(\eta - \hat{\eta}) = 0. \tag{5.12}$$

By making the formal substitutions

$$\left.\begin{aligned} p' &= -\rho_0 gz,\\ \omega' &= -\rho_0 gw,\\ M &= P/\rho_0,\\ \rho_0(\alpha - \rho_0^{-1}) &= -(\rho - \rho_0)/\rho_0,\end{aligned}\right\} \tag{5.13}$$

into (5.6)–(5.11), we see that they are identical to (3.1)–(3.6), upon neglect of $O(\rho_0\hat{\delta}) \sim 10^{-3}$ terms in (5.6). The boundary conditions (5.3) [or (5.5)] and (5.4) become [neglecting an $O(\rho_0\hat{\delta})$ term in (5.5)]

$$P = \rho_0 g \eta, \qquad w = \frac{D}{Dt}(\eta - \hat{\eta}) \qquad \text{at } z = \eta - \hat{\eta} \tag{5.14}$$

$$\left[\text{or: } P = \rho_0 g \eta, \qquad w = \frac{\partial}{\partial t}(\eta - \hat{\eta}) + \nabla_z \cdot [(\eta - \hat{\eta})\mathbf{u}] \quad \text{at } z = 0\right], \tag{5.14$'$}$$

$$P = \rho_0 g(\eta - \hat{\eta}), \qquad w = -\mathbf{u} \cdot \nabla H \qquad \text{at } z = -H. \tag{5.15}$$

The integrated continuity equation (5.12) becomes

$$\nabla \cdot \left( \int_{-H}^{\eta - \hat{\eta}} \mathbf{u}\, dz \right) + \frac{\partial}{\partial t}(\eta - \hat{\eta}) = 0. \tag{5.16}$$

Comparison with (3.7) [or (3.10)] and (3.8) reveals the replacement in the $w$ condition (5.14) [or (5.14)$'$] of $\eta$ by $\eta - \hat{\eta}$ $(= p_b/\rho_0 g - H)$. Similarly, (5.16) [cf. (3.9)] contains $\eta - \hat{\eta}$ rather than $\eta$. Thus the prognostic integration of (5.16) yields the time-variable part of bottom pressure $\rho_0 g(\eta - \hat{\eta})$, rather than the sea surface elevation $\eta$. This forms the boundary condition at the bottom [first of (5.15)] for the vertical integration of (3.2). On the right of (3.2) one must use $-\rho_0 \hat{\delta} = -\rho_0(\alpha - \rho_0^{-1})$ [see the last equation of (5.13)] in place of $(\rho - \rho_0)/\rho_0$. This quantity is calculated from the equation of state as $\hat{\delta}[\theta, S, -\rho_0 g(z - \eta + \hat{\eta})]$, where $\theta$, $S$ have been obtained prognostically from (3.4), (3.5). The $\rho_0 g(\eta - \hat{\eta}) \sim 1$ dbar bias in the pressure is negligible for this purpose. From the surface pressure, so calculated, one diagnostically obtains the sea surface height $\eta = P(0)/g\rho_0$, from the first of (5.14).

This demonstrates again that, but for the exchange of the roles of bottom pressure and sea surface height, and the reversed direction of integration of the hydrostatic equation (3.2) (bottom to top), the mass-conserving, non-Boussinesq set (5.6)–(5.11) with (5.4), (5.5), is mathematically equivalent to the standard Boussinesq set. The neglect of $O(\rho_0 \hat{\delta})$ terms in (5.6) and (5.5) is no worse than the neglect of terms of similar order in the horizontal momentum equations (3.1). [Indeed, it would be quite easy to restore these terms in (5.6) and (5.5).] Most important, however, it is not necessary to compromise the dilatation terms in the continuity equation to obtain (5.8), and hence, after the formal substitution (5.13), to obtain (3.3).

There are circumstances in which the neglected dilatation term in the conventional equation (3.3), $\rho^{-1}D\rho/Dt$, may be consequential. For example, McDougall and Garrett (1992) argue that this neglect can lead to an error in the Reynolds-averaged forms of (3.4) and (3.5) that is an appreciable fraction of the turbulent diffusion terms, while Dukowicz (1997) finds measurable effects on modeled sea level. The present result shows that if (i) the vertical coordinate in a ''Boussinesq'' circulation model is reinterpreted as modified pressure; (ii) the roles of bottom pressure and sea surface height are reversed, along with the direction of the integration of the hydrostatic relation; and (iii) density anomaly is replaced by specific volume anomaly [last of (5.13)], the neglect of the dilatation term is thereby entirely evaded.

This seems a neater and less problematic way of dealing with the dilatation than calculating it by brute force and inserting it into the continuity equation (Dukowicz 1997). It answers comprehensively criticisms of sea level changes calculated from Boussinesq models (Greatbatch 1994; Mellor and Ezer 1995), and improves on those authors' suggested ex post facto corrections for erroneous global volumes. Lu (2001) recently proposed a redefined vertical velocity variable that appears to swallow up the dilatation term. But the resulting vertical velocity anomaly (the difference between redefined and true vertical velocity) must still be calculated from the dilatation (so that this rather resembles Dukowicz's 1997 measures) and acts as an additional advective term in the momentum and any scalar equations. No such extra terms appear in the rearrangement proposed here. Greatbatch et al. (2001) and McDougall et al. (2002) proposed redefining the mean Eulerian velocity by weighting with density to obtain a solenoidal continuity equation, although this holds strictly only in the steady case. No such restriction applies to the present measures. The order-$\rho_0 \hat{\delta}$ terms neglected in the momentum balance and surface dynamical boundary condition in order to establish the correspondence with the Boussinesq equations (which have neglected terms of similar order) could be readily restored, with no effect on the mass conservation.

## 6. Summary

It has long been known that, when the hydrostatic equations of motion are written in terms of pressure as independent variable, the mass conservation equation takes a solenoidal form without neglecting volume dilatation (Haltiner and Williams 1980). This suggests a simple way of reorienting analytical theories or numerical models to make them immune to the usual strictures inherent in the Boussinesq equations that are consequent to their neglect of dilatation. That is, in essence, to reread such theories or models with level $z$ (in meters) substituted by pressure $p$ (in decibars) as independent variable. However, a complicating factor is raised by the consideration of the boundary conditions on normal stress (pressure) and normal velocity at the sea surface and bottom. The boundary conditions are not perfectly symmetric between the non-Boussinesq-pressure equations and the Boussinesq-depth equations. Yet there is a clear duality between the two sets of conditions. This can be best stated as follows. A solution from a Boussinesq-depth theory or model can be reinterpreted as a non-Boussinesq-pressure solution where the sea surface elevation in the Boussinesq theory is reinterpreted as (equivalent) bottom pressure variation in the non-Boussinesq theory. Just as bottom pressure is calculated in the Boussinesq theory by integrating the hydrostatic equation from the surface down to the bottom, so sea surface height in the non-Boussinesq theory is calculated by integrating the hydrostatic equation up from the bottom. Where scaled density anomaly, $(\rho - \rho_0)/\rho_0$, appears in the Boussinesq-depth hydrostatic equation, negative of scaled specific volume anomaly, $-(\alpha - \rho_0^{-1})\rho_0$,—almost the same thing—appears in the non-Boussinesq-pressure equation.

A further complication is posed by variable bottom topography, which affects one boundary and not the other. However, for a level bottom, the duality is perfect: a Boussinesq solution as a function of depth can be regarded as a non-Boussinesq solution as a function of pressure, with the sea surface height in the Boussinesq solution reinterpreted as bottom pressure in the non-Boussinesq solution. In a rigid-lid solution the duality is also perfect, even with variable bottom topography. The non-Boussinesq dual solution so obtained has neglected neither volume dilatation in the mass conservation equation, nor the small variations in density in the horizontal momentum balance.

In the case of variable bottom topography, we considered a slightly different independent variable, modified pressure, which is in situ pressure corrected by the variable part of bottom pressure. We established a near-perfect duality between Boussinesq-depth solutions and non-Boussinesq-modified pressure solutions. The sole imperfection in the duality is the neglect of terms of the order of density variations in the non-Boussinesq horizontal momentum balance. However, volume dilation is not neglected in the mass conservation. Again, the roles of sea surface height and bottom pressure variation, and of density anomaly and specific volume anomaly, are exchanged between the Boussinesq and non-Boussinesq solutions.

One important practical consequence of this duality is that a numerical Boussinesq model may be readily converted into a non-Boussinesq model by minimal or even no changes. (None are required for converting a rigid-lid model.) To convert a Boussinesq free-surface depth-coordinate model with variable bottom topography to become a non-Boussinesq modified-pressure model requires as the only substantive changes that 1) ''pressure'' at each ''level,'' as already obtained in such a model from the hydrostatic relation, be calculated by integrating upward from the bottom, with bottom pressure as boundary condition, rather than down from the top; and that 2) where density occurs, calculated from the equation of state, its reciprocal, specific volume, be substituted. [The first change merely requires the reversing of a do-loop in the computer code. This is so even if the model is implemented with sigma coordinates (height normalized by bottom depth) or density coordinates, and if the barotropic time integration is split from the baroclinic (Higdon and de Szoeke 1997).] Alternatively, the hydrostatic pressure, as presently calculated by integrating from the top down, may be corrected by the difference between bottom pressure and sea surface height (which is the steric height). The rest consists of considering horizontal velocity, temperature, salinity, etc. as functions of modified pressure rather than depth, and reinterpreting vertical velocity as modified pressure tendency. (The pressure-like dependent variable is Montgomery function $gz + p/\rho_0$.) The dilatation term in the continuity equation, $\rho^{-1}(D\rho/Dt)$, is not neglected but transformed into invisibility.

The crucial simplification that permits the development of the set of equations (2.1)–(2.6), or (5.6)–(5.11), is the hydrostatic approximation. While it is often said that the Boussinesq approximation eliminates or filters acoustic waves, the hydrostatic approximation has almost the same effect. [There is one notable exception, the Lamb mode (Gill 1982). In the atmosphere, where scale height (based on compressibility) is smaller than total height, the Lamb mode exists as a vertically evanescent hydrostatic–acoustic wave trapped on the lower boundary. Thus, a pressure-coordinate model of the atmosphere will support the Lamb mode. In the ocean, however, the scale height is far larger than total depth (200 vs 5 km), and the Lamb mode is subsumed into the external gravity mode.] For nonhydrostatic motions, the Boussinesq approximation is critical for mathematical tractability (Lighthill 1978). Yet once the hydrostatic approximation is made, the Boussinesq approximation offers no essential further mathematical simplification to the equations for ocean circulation.

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## APPENDIX A

### Coordinate Transformation

Consider replacing the depth coordinate $z$ by a general coordinate $p(x, y, z, t)$. Then partial derivatives become, in terms of the new variable,

$$\frac{\partial}{\partial z} = \frac{\partial p}{\partial z}\frac{\partial}{\partial p}, \qquad \left(\frac{\partial}{\partial x_i}\right)_z = \left(\frac{\partial}{\partial x_i}\right)_p + \left(\frac{\partial p}{\partial x_i}\right)_z \frac{\partial}{\partial p}, \qquad \text{(A1)}$$

where $x_i$ stands for $x$, $y$, or $t$. Applying these to $z$ itself, one obtains

$$1 = \frac{\partial p}{\partial z}\frac{\partial z}{\partial p}, \qquad 0 = \left(\frac{\partial z}{\partial x_i}\right)_p + \left(\frac{\partial p}{\partial x_i}\right)_z \frac{\partial z}{\partial p}, \tag{A2}$$

whence (A1) become

$$\frac{\partial}{\partial z} = \frac{1}{z_p}\frac{\partial}{\partial p}, \qquad \left(\frac{\partial}{\partial x_i}\right)_z = \left(\frac{\partial}{\partial x_i}\right)_p - \frac{1}{z_p}\left(\frac{\partial z}{\partial x_i}\right)_p \frac{\partial}{\partial p}, \tag{A3}$$

provided $z_p \equiv \partial z/\partial p \neq 0$.

Using (A1),

$$\frac{D}{Dt} \equiv \left(\frac{\partial}{\partial t}\right)_z + \mathbf{u}\cdot\nabla_z + w\frac{\partial}{\partial z}$$
$$= \left(\frac{\partial}{\partial t}\right)_p + \mathbf{u}\cdot\nabla_p + \left[\left(\frac{\partial p}{\partial t}\right)_z + \mathbf{u}\cdot\nabla_z p + w\frac{\partial p}{\partial z}\right]\frac{\partial}{\partial p} \tag{A4}$$

so that, if one defines

$$\omega = \frac{Dp}{Dt}, \tag{A5}$$

it follows that

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t}\right)_p + \mathbf{u}\cdot\nabla_p + \omega\frac{\partial}{\partial p}. \tag{A6}$$

An alternative way of writing (A5) is

$$\omega = \frac{-z_t - \mathbf{u}\cdot\nabla_p z + w}{z_p}. \tag{A7}$$

Transforming the continuity equation

$$\left(\frac{\partial \rho}{\partial t}\right)_z + \nabla_z\cdot(\rho\mathbf{u}) + \frac{\partial}{\partial z}(\rho w) = 0 \tag{A8}$$

leads to

$$\left(\frac{\partial}{\partial t}\right)_p (\rho z_p) + \nabla_p\cdot(\rho z_p \mathbf{u}) + \frac{\partial}{\partial p}(-\rho z_t - \rho\mathbf{u}\cdot\nabla_p z + \rho w) = 0, \tag{A9}$$

in which one recognizes the last term as being $\partial(\rho z_p \omega)/\partial p$. This equation may be rewritten as

$$\frac{D}{Dt}(\rho z_p) + \rho z_p\left(\nabla_p\cdot\mathbf{u} + \frac{\partial \omega}{\partial p}\right) = 0. \tag{A10}$$

Nothing specific to pressure has yet been used in this derivation. If one takes $p$ to be pressure, however, then the hydrostatic relation becomes

$$\rho z_p = -g^{-1}, \quad \text{or} \quad g z_p = -\alpha \tag{A11}$$

so that the solenoidal form [(2.3)] of the continuity equation follows. The horizontal pressure gradient in the equations of motion may be written

$$\frac{1}{\rho}\nabla_z p = -\frac{1}{\rho z_p}\nabla_p z = g\nabla_p z, \tag{A12}$$

upon using (A3) and (A11). The momentum equation (2.1) results from the substitution of (A6) and (A12) in the conventional forms. If the hydrostatic balance (A11) does not hold, (A12) must contain another term,

$$\left(-\frac{1}{\rho}p_z - g\right)\nabla_p z = \frac{Dw}{Dt}\nabla_p z, \tag{A13}$$

which is the component of vertical inertia along sloping pressure surfaces.

### a. *Geographical spheroidal coordinates*

Since the centrifugo-gravitational potential is not uniform over the globe, $g$ is not strictly constant (Stacey 1977). In that case, for the implicitly Cartesian coordinates $x, y, z$ above, one might substitute the curvilinear orthogonal coordinates $\lambda$, $\theta$, $\phi$ [respectively longitude, spheroidal latitude (different from spherical latitude in that normals to spheroidal cones, $\theta$ = constant, lie in geopotential surfaces), and geopotential]. Then all of the above goes through, with appropriate curvilinear elaborations of the gradient operator (Batchelor 1967). In particular, in the equations through (A10), $z$ is replaced by geopotential $\phi$. The hydrostatic balance (A11) is replaced by

$$\rho\phi_p = -1, \quad \text{or} \quad \phi_p = -\alpha, \tag{A14}$$

and the solenoidal form of (2.3) still follows. In the horizontal momentum equations

$$\frac{1}{\rho}\nabla_\phi p = -\frac{1}{\rho\phi_p}\nabla_p\phi = \nabla_p\phi, \tag{A15}$$

instead of (A12). Indeed, throughout the main text, replacement of $gz$ by $\phi$ accomplishes the extension of the equations to nonconstant $g$.

### b. *Modified pressure*

In (5.1), we proposed a further variable shift to modified pressure $p'(x, y, p, t)$. This is of the same form as the coordinate transformation considered above, if the following substitutions are made:

$$\left.\begin{array}{ll} z := p & w := \omega \\ p := p' & \omega := \omega' \end{array}\right\}. \tag{A16}$$

By observing that

$$\frac{\partial p}{\partial p'} = 1, \qquad \left(\frac{\partial p}{\partial x_i}\right)_{p'} = \frac{\rho_0 g \partial(\eta - \hat{\eta})}{\partial x_i}, \tag{A17}$$

from (5.1), we see that the counterparts of (A3), (A5), (A6) become

$$\frac{\partial}{\partial p} = \frac{\partial}{\partial p'}, \qquad \left(\frac{\partial}{\partial x_i}\right)_p = \left(\frac{\partial}{\partial x_i}\right)_{p'} - \rho_0 g \frac{\partial}{\partial x_i}(\eta - \hat{\eta})\frac{\partial}{\partial p'}, \tag{A18}$$

$$\omega' = \frac{Dp'}{Dt} = \omega - \rho_0 g \frac{D}{Dt}(\eta - \hat{\eta}), \quad \text{and} \tag{A19}$$

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t}\right)_{p'} + \mathbf{u}\cdot\boldsymbol{\nabla}_{p'} + \omega'\frac{\partial}{\partial p'}. \tag{A20}$$

Hence we may verify that (5.6)–(5.11) result from transforming (2.1)–(2.6). In particular, we may confirm as an identity that

$$\boldsymbol{\nabla}_p \cdot \mathbf{u} + \frac{\partial \omega}{\partial p} = \boldsymbol{\nabla}_{p'} \cdot \mathbf{u} + \frac{\partial \omega'}{\partial p'}, \tag{A21}$$

so that if the hydrostatic relation (A11) holds, it follows from (A10) that both sides of (A21) vanish. The transformation of the boundary conditions (2.11), (2.12) is discussed in the main text.

## APPENDIX B

### A More Accurate Bottom Pressure

A more accurate approximation than (2.15) will be obtained linking bottom pressure $p_b$ and depth $H$. Write the equation of state (2.8) as

$$\alpha = \alpha_0(p) + \delta(S, T, p), \tag{B1}$$

where $\alpha_0(p) = \alpha(35\ \text{psu}, 0°\text{C}, p)$, is standard specific volume, and $\delta$ is the thermosteric anomaly. If one substitutes the salinity and temperature profiles $S(x, y, p)$, $T(x, y, p)$ at position $(x, y)$ into (2.14), one obtains

$$\eta + H = H(p_b) + \eta_s, \tag{B2a}$$

where

$$H(p_b) = g^{-1}\int_0^{p_b} \alpha_0(p)\, dp, \tag{B2b}$$

$$\eta_s = g^{-1}\int_0^{p_b} \delta\, dp, \tag{B2c}$$

the latter being the steric height. The function $H(p_b)$ is increasing and can be inverted. Calling this inverse function $P_b(\,\cdot\,)$, we have, from (B2a),

$$p_b = P_b(H + \eta - \eta_s) \cong P_b(H) + (\eta - \eta_s)g\rho_b(H), \tag{B3}$$

where the second equality represents the first two terms in a Taylor series with $P_b'(H) = 1/H'[P_b(H)] = g\{\alpha_0[P_b(H)]\}^{-1} \equiv g\rho_b(H)$. An accurate approximation of (B2) is then

$$\eta + P_b(H)/g\rho_b = p_b/g\rho_b + \hat{\eta}_s, \tag{B4a}$$

where

$$\hat{\eta}_s \cong g^{-1}\int_0^{P_b(H)} \delta\, dp. \tag{B4b}$$

In the latter, $P_b(H)$ has been used to approximate $p_b$ in the limit of integration for steric height, (B2c), incurring an error of millimeters or less.

## APPENDIX C

### Boundary Conditions

The first of conditions (2.12) is

$$M[x, y, p_b(x, y, t), t] = p_b/\rho_0 - gH, \tag{C1}$$

where one may take (2.15a) for $p_b$:

$$p_b = \rho_0 g(H + \eta - \hat{\eta}). \tag{C2}$$

Expanding $M$ in a Taylor series around $\rho_0 gH$, retaining only the first two terms, one obtains

$$\begin{aligned} M(p_b) &\cong M(\rho_0 gH) + \rho_0 g(\eta - \hat{\eta})M_p(\rho_0 gH) \\ &\cong M(\rho_0 gH) - g(\eta - \hat{\eta})\rho_0\hat{\delta}, \end{aligned} \tag{C3}$$

on using (2.2), and suppressing the appearance of the dependence on $x$, $y$, $t$. Combining (C1) and (C3), and neglecting the order-$\rho_0\hat{\delta}$ term in (C3),

$$M(\rho_0 gH) \cong g(\eta - \hat{\eta}). \tag{C4}$$

In the second of conditions (2.12),

$$\omega = \frac{\partial p_b}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}p_b \quad \text{at } p = p_b(x, y, t), \tag{C5}$$

one first expands $\omega(p_b)$ in a Taylor series around $p = \rho_0 gH$:

$$\begin{aligned} \omega(p_b) &\cong \omega(\rho_0 gH) + \rho_0 g(\eta - \hat{\eta})\left.\frac{\partial \omega}{\partial p}\right|_{\rho_0 gH} \\ &\cong \omega(\rho_0 gH) + \rho_0 g(\eta - \hat{\eta})(\boldsymbol{\nabla}_p \cdot \mathbf{u})|_{\rho_0 gH}. \end{aligned} \tag{C6}$$

Similarly,

$$\mathbf{u}|_{pb}\cdot\boldsymbol{\nabla}p_b \cong \mathbf{u}|_{\rho_0 gH}\cdot\boldsymbol{\nabla}p_b + \cdots, \tag{C7}$$

where the terms denoted by ellipses may be neglected as being of third order. Combining (C5)–(C7),

$$\omega \cong \rho_0 g\mathbf{u}\cdot\boldsymbol{\nabla}H + \frac{\partial}{\partial t}p_b + \boldsymbol{\nabla}\cdot[\mathbf{u}(p_b - \rho_0 gH)] \tag{C8}$$

at $p = \rho_0 gH$. The neglect of the last term in (C8) is equivalent to approximating $p_b$ in the limit of integration in the bulk continuity equation (2.13) by $\rho_0 gH$.

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