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THE FIRST EIGENVALUE FOR THE p-LAPLACIAN OPERATOR

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ABSTRACT. In this paper, using the Hausdorff topology in the space of open sets under some capacity constraints on geometrical domains we prove the strong continuity with respect to the moving domain of the solutions of a p-Laplacian Dirichlet problem. We are also interested in the minimization of the first eigenvalue of the p-Laplacian with Dirichlet boundary conditions among open sets and quasi open sets of given measure.

Key words and phrases: p-Laplacian, Nonlinear eigenvalue problems, Shape optimization.

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1. INTRODUCTION

Let Ω be an open subset of a fixed ball D in $\mathbb{R}^N$, $N \geq 2$ and $1 < p < +\infty$. Consider the Sobolev space $W_0^{1,p}(\Omega)$ which is the closure of $\mathcal{C}^\infty$ functions compactly supported in Ω for the norm

$$||u||_{1,p}^p = \int_\Omega |u(x)|^p dx + \int_\Omega |\nabla u(x)|^p dx.$$

The p-Laplacian is the operator defined by

$$\begin{aligned} \Delta_p : W_0^{1,p}(\Omega) &\longrightarrow W^{-1,q}(\Omega) \\ u &\longmapsto \Delta_p u = div(|\nabla u|^{p-2}\nabla u), \end{aligned}$$

where $W^{-1,q}(\Omega)$ is the dual space of $W_0^{1,p}(\Omega)$ and we have $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

We are interested in the nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u - \lambda |u|^{p-2} u &= 0 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{cases} \tag{1.1}$$

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Let u be a function of $W_0^{1,p}(\Omega)$, not identically 0. The function u is called an eigenfunction if

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u \nabla \phi dx = \lambda \int_\Omega |u(x)|^{p-2} u\phi dx$$

for all $\phi \in \mathcal{C}_0^\infty(\Omega)$. The corresponding real number λ is called an eigenvalue.

Contrary to the Laplace operator, the p-Laplacian spectrum has not been proved to be discrete. In [15], the first eigenvalue and the second eigenvalue are described.

Let D be a bounded domain in $\mathbb{R}^N$ and $c > 0$. Let us denote $\lambda_1^p(\Omega)$ as the first eigenvalue for the p-Laplacian operator. The aim of this paper is to study the isoperimetric inequality

$$\min\{\lambda_1^p(\Omega), \Omega \subseteq D \text{ and } |\Omega| = c\}$$

and its continuous dependance with respect to the domain. We extend the Rayleigh-Faber-Khran inequality to the p-Laplacian operator and study the minimization of the first eigenvalue in two dimensions when D is a box. By considering a class of simply connected domains, we study the stability of the minimizer Ω_p of the first eigenvalue with respect to p that is if Ω_p is a minimizer of the first eigenvalue for the p-Laplacian Dirichlet, when p goes to 2, Ω_2 is also a minimizer of the first eigenvalue of the Laplacian Dirichlet. Thus we will give a formal justification of the following conjecture: "Ω is a minimizer of given volume c, contained in a fixed box D and if D is too small to contain a ball of the same volume as Ω. Are the free parts of the boundary of Ω pieces of circle?"

Henrot and Oudet solved this question and proved by using the Hölmgren uniqueness theorem, that the free part of the boundary of Ω cannot be pieces of circle, see [10].

The structure of this paper is as follows: The first section is devoted to the definition of two eigenvalues. In the second section, we study the properties of geometric variations for the first eigenvalue. The third section is devoted to the minimization of the first eigenvalue among open (or, if specified, quasi open) sets of given volume. In the fourth part we discuss the minimization of the first eigenvalue in a box in two dimensions.

Let D be a bounded open set in $\mathbb{R}^N$ which contains all the open (or, if specified, quasi open) subsets used.

2. DEFINITION OF THE FIRST AND SECOND EIGENVALUES

The first eigenvalue is defined by the nonlinear Rayleigh quotient

$$\lambda_1(\Omega) = \min_{\phi \in W_0^{1,p}(\Omega), \phi \neq 0} \frac{\int_\Omega |\nabla \phi(x)|^p dx}{\int_\Omega |\phi(x)|^p} = \frac{\int_\Omega |\nabla u_1(x)|^p dx}{\int_\Omega |u_1(x)|^p dx},$$

where the minimum is achieved by u_1 which is a weak solution of the Euler-Lagrange equation

$$\begin{cases} -\Delta_p u - \lambda |u|^{p-2} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

The first eigenvalue has many special properties, it is strictly positive, simple in any bounded connected domain see [15]. And u_1 is the only positive eigenfunction for the p-Laplacian Dirichlet see also [15].

In [15], the second eigenvalue is defined by

$$\lambda_2(\Omega) = \inf_{C \in \mathcal{C}_2} \max_C \frac{\int_\Omega |\nabla \phi(x)|^p dx}{\int_\Omega |\phi(x)|^p},$$

where

$$\mathcal{C}_2 := \{C \in W_0^{1,p}(\Omega) : C = -C \text{ such that } genus(C) \geq 2\}.$$

In [1], Anane and Tsouli proved that there does not exist any eigenvalue between the first and the second ones.

3. PROPERTIES OF THE GEOMETRIC VARIATIONS

In this section we are interested in the continuity of the map

$$\Omega \longmapsto \lambda_1(\Omega).$$

Then, we have to fix topology on the space of the open subsets of D. On the family of the open subsets of D, we define the Hausdorff complementary topology, denoted H^c given by the metric

$$d_{H^c}(\Omega_1^c, \Omega_2^c) = \sup_{x \in \mathbb{R}^N} |d(x, \Omega_1^c) - d(x, \Omega_2^c)|.$$

The H^c-topology has some good properties for example the space of the open subsets of D is compact. Moreover if $\Omega_n \xrightarrow{H^c} \Omega$, then for any compact $K \subset\subset \Omega$ we have $K \subset\subset \Omega_n$ for n large enough.

However, perturbations in this topology may be very irregular and in general situations the continuity of the mapping $\Omega \longmapsto \lambda_1(\Omega)$ fails, see [4].

In order to obtain a compactness result we impose some additional constraints on the space of the open subsets of D which are expressed in terms of the Sobolev capacity. There are many ways to define the Sobolev capacity, we use the local capacity defined in the following way.

Definition 3.1. For a compact set K contained in a ball B,

$$cap(K, B) := \inf \left\{ \int_B |\nabla \phi|^p, \phi \in \mathcal{C}_0^\infty(B), \phi \geq 1 \quad \text{on} \quad K \right\}.$$

Definition 3.2.

(1) It is said that a property holds p-quasi everywhere (abbreviated as $p - q.e$) if it holds outside a set of p-capacity zero.
(2) A set $\Omega \subset \mathbb{R}^N$ is said to be quasi open if for every $\epsilon > 0$ there exists an open set Ω_ϵ such that $\Omega \subseteq \Omega_\epsilon$, and $cap(\Omega_\epsilon \backslash \Omega) < \epsilon$.
(3) A function $u : \mathbb{R}^N \longrightarrow \mathbb{R}$ is said p-quasi continuous if for every $\epsilon > 0$ there exists an open set Ω_ϵ such that $cap(\Omega_\epsilon) < \epsilon$ and $u_{|\mathbb{R}\backslash\Omega_\epsilon}$ is continuous in $\mathbb{R}\backslash\Omega_\epsilon$.

It is well known that every Sobolev function $u \in W^{1,p}(\mathbb{R}^N)$ has a p-quasi continuous representative which we still denote u. Therefore, level sets of Sobolev functions are p-quasi open sets; in particular $\Omega_v = \{x \in D; |v(x)| > 0\}$ is quasi open subsets of D.

Definition 3.3. We say that an open set Ω has the $p - (r, c)$ capacity density condition if

$$\forall x \in \partial\Omega, \quad \forall 0 < \delta < r, \quad \frac{cap(\Omega^c \cap \bar{B}(x, \delta), B(x, 2\delta))}{cap(\bar{B}(x, \delta), B(x, 2\delta))} \geq c$$

where $B(x, \delta)$ denotes the ball of raduis δ, centred at x.

Definition 3.4. We say that the sequence of the spaces $W_0^{1,p}(\Omega_n)$ converges in the sense of Mosco to the space $W_0^{1,p}(\Omega)$ if the following conditions hold

(1) The first Mosco condition: For all $\phi \in W_0^{1,p}(\Omega)$,there exists a sequence $\phi_n \in W_0^{1,p}(\Omega_n)$ such that ϕ_n converges strongly in $W_0^{1,p}(D)$ to ϕ.
(2) The second Mosco condition: For every sequence $\phi_{n_k} \in W_0^{1,p}(\Omega_{n_k})$ weakly convergent in $W_0^{1,p}(D)$ to a function ϕ, we have $\phi \in W_0^{1,p}(\Omega)$.

Definition 3.5. We say a sequence (Ω_n) of open subsets of a fixed ball D γ_p-converges to Ω if for any $f \in W^{-1,q}(\Omega)$ the solutions of the Dirichlet problem

$$-\Delta_p u_n = f \quad \text{in} \quad \Omega_n, u_n \in W_0^{1,p}(\Omega_n)$$

converge strongly in $W_0^{1,p}(D)$, as $n \longrightarrow +\infty,$ to the solution of the corresponding problem in Ω, see [7], [8].

By $\mathcal{O}_{p-(r,c)}(D)$, we denote the family of all open subsets of D which satisfy the $p-(r,c)$ capacity density condition. This family is compact in the H^c topology see [4]. In [2], D. Bucur and P. Trebeschi, using capacity constraints analogous to those introduce in [3] and [4] for the linear case, prove the γ_p-compactness result for the p-Laplacian. In the same way, they extend the continuity result of Šveràk [19] to the p-Laplacian for $p \in (N-1, N]$, $N \geq 2$. The reason of the choice of p is that in $\mathbb{R}^N$ the curves have p positive capacity if $p > N-1$. The case $p > N$ is trivial since all functions in $W^{1,p}(\mathbb{R}^N)$ are continuous.

Let us denote by

$$\mathcal{O}_l(D) = \{\Omega \subseteq D, \sharp\Omega^c \leq l\}$$

where $\sharp$ denotes the number of the connected components. We have the following theorem.

Theorem 3.1 (Bucur-Trebeschi). *Let $N \geq p > N-1$. Consider the sequence $(\Omega_n) \subseteq \mathcal{O}_l(D)$ and assume that Ω_n converges in Hausdorff complementary topology to Ω. Then $\Omega \subseteq \mathcal{O}_l(D)$ and Ω_n γ_p-converges to Ω.*

Proof of Theorem 3.1. See [2]. □

For $N = 2$ and $p = 2$, Theorem 3.1 becomes the continuity result of Šveràk [19].

Back to the continuity result, we use the above results to prove the following theorem.

Theorem 3.2. *Consider the sequence $(\Omega_n) \subseteq \mathcal{O}_l(D)$. Assume that Ω_n converges in Hausdorff complementary topology to Ω. Then $\lambda_1(\Omega_n)$ converges to $\lambda_1(\Omega)$.*

Proof of Theorem 3.2. Let us take

$$\lambda_1(\Omega_n) = \min_{\phi_n \in W_0^{1,p}(\Omega_n), \phi_n \neq 0} \frac{\int_{\Omega_n} |\nabla \phi_n(x)|^p dx}{\int_{\Omega_n} |\phi_n(x)|^p} = \frac{\int_{\Omega_n} |\nabla u_n(x)|^p dx}{\int_{\Omega_n} |u_n(x)|^p},$$

where the minimum is attained by u_n, and

$$\lambda_1(\Omega) = \min_{\phi \in W_0^{1,p}(\Omega), \phi \neq 0} \frac{\int_{\Omega} |\nabla \phi(x)|^p dx}{\int_{\Omega} |\phi(x)|^p} = \frac{\int_{\Omega} |\nabla u_1(x)|^p dx}{\int_{\Omega} |u_1(x)|^p dx},$$

where the minimum is achieved by u_1.

By the Bucur and Trebeschi theorem, Ω_n γ_p converges to Ω. This implies $W_0^{1,p}(\Omega_n)$ converges in the sense of Mosco to $W_0^{1,p}(\Omega)$.

If the sequence (u_n) is bounded in $W_0^{1,p}(D)$, then there exists a subsequence still denoted u_n such that u_n converges weakly in $W_0^{1,p}(D)$ to a function u. The second condition of Mosco implies that $u \in W_0^{1,p}(\Omega)$.

Using the weak lower semicontinuity of the L^p-norm, we have the inequality

$$\liminf_{n \longrightarrow +\infty} \frac{\int_D |\nabla u_n(x)|^p dx}{\int_D |u_n(x)|^p} \geq \frac{\int_\Omega |\nabla u(x)|^p dx}{\int_\Omega |u(x)|^p} \geq \frac{\int_\Omega |\nabla u_1(x)|^p dx}{\int_\Omega |u_1(x)|^p},$$

then

$$\liminf_{n \to +\infty} \lambda_1(\Omega_n) \geq \lambda_1(\Omega). \tag{3.1}$$

Using the first condition of Mosco, there exists a sequence $(v_n) \in W_0^{1,p}(\Omega_n)$ such that v_n converges strongly in $W_0^{1,p}(D)$ to u_1.

We have

$$\lambda_1(\Omega_n) \leq \frac{\int_D |\nabla v_n(x)|^p dx}{\int_D |v_n(x)|^p}$$

this implies that

$$\begin{aligned}\limsup_{n \longrightarrow +\infty} \lambda_1(\Omega_n) &\leq \limsup_{n \longrightarrow +\infty} \frac{\int_D |\nabla v_n(x)|^p dx}{\int_D |v_n(x)|^p} \\ &= \lim_{n \longrightarrow +\infty} \frac{\int_D |\nabla v_n(x)|^p dx}{\int_D |v_n(x)|^p} \\ &= \frac{\int_\Omega |\nabla u_1(x)|^p dx}{\int_\Omega |u_1(x)|^p}\end{aligned}$$

then

$$\limsup_{n \to +\infty} \lambda_1(\Omega_n) \leq \lambda_1(\Omega). \tag{3.2}$$

By the relations (3.1) and (3.2) we conclude that $\lambda_1(\Omega_n)$ converges to $\lambda_1(\Omega)$. □

4. SHAPE OPTIMIZATION RESULT

We extend the classical inequality of Faber-Krahn for the first eigenvalue of the Dirichlet Laplacian to the Dirichlet p-Laplacian. We study this inequality when Ω is a quasi open subset of D.

Definition 4.1. Let Ω be an open subset and bounded in $\mathbb{R}^N$. We denote by B the ball centred at the origin with the same volume as Ω. Let u be a non negative function in Ω, which vanishes on $\partial\Omega$. For all $c > 0$, the set $\{x \in \Omega, u(x) > c\}$ is called the level set of u.

The function u^* which has the following level set

$$\forall c > 0, \quad \{x \in B, u^*(x) > c\} = \{x \in \Omega, u(x) > c\}^*$$

is called the Schwarz rearrangement of u. The level sets of u^* are the balls that we obtain by rearranging the sets of the same volume of u.

We have the following lemma.

Lemma 4.1. *Let Ω be an open subset in $\mathbb{R}^N$.*
Let ψ be any continuous function on $\mathbb{R}_+^*$, we have

(1) $\int_\Omega \psi(u(x))dx = \int_{\Omega^*} \psi(u^*(x))dx$ *u^* is equi-mesurable with u.*
(2) $\int_\Omega u(x)v(x)dx \leq \int_{\Omega^*} u^*(x)v^*(x)dx.$
(3) *If $u \in W_0^{1,p}(\Omega), p > 1$ then $u^* \in W_0^{1,p}(\Omega^*)$ and*

$$\int_\Omega |\nabla u(x)|^p dx \geq \int_{\Omega^*} |\nabla u^*(x)|^p dx \quad \textit{Pòlya inequality.}$$

Proof of Lemma 4.1. See [12]. □

The basic result for the minimization of eigenvalues is the conjecture of Lord Rayleigh: "The disk should minimize the first eigenvalue of the Laplacian Dirichlet among every open set of given measure". We extend the Rayleigh-Faber-Krahn inequality to the p-Laplacian operator.

Let Ω be any open set in $\mathbb{R}^N$ with finite measure. We denote by $\lambda_1(\Omega)$ the first eigenvalue for the p-Laplacian operator with Dirichlet boundary conditions. We have the following theorem.

Theorem 4.2. *Let B be the ball of the same volume as Ω, then*

$$\lambda_1(B) = \min\{\lambda_1(\Omega), \Omega \quad \textit{open set of} \quad \mathbb{R}^N, |\Omega| = |B|\}.$$

Proof of Theorem 4.2. Let u_1 be the first eigenfunction of $\lambda_1(\Omega)$, it is strictly positive see [15]. By Lemma 4.1, equi-mesurability of the function u_1 and its Schwarz rearrangement u_1^* gives

$$\int_\Omega |u_1(x)|^p dx = \int_B |u_1^*(x)|^p dx.$$

The Pòlya inequality implies that

$$\int_\Omega |\nabla u_1(x)|^p dx \geq \int_B |\nabla u_1^*(x)|^p dx.$$

By the two conditions, it becomes

$$\frac{\int_B |\nabla u_1^*(x)|^p dx}{\int_B |u_1^*(x)|^p dx} \leq \frac{\int_\Omega |\nabla u_1(x)|^p dx}{\int_\Omega |u_1(x)|^p dx} = \lambda_1(\Omega).$$

This implies that

$$\lambda_1(B) = \min_{v \in W_0^{1,p}(B), v \neq 0} \frac{\int_B |\nabla v(x)|^p dx}{\int_B |v(x)|^p dx} \leq \frac{\int_B |\nabla u_1^*(x)|^p dx}{\int_B |u_1^*(x)|^p dx} \leq \lambda_1(\Omega).$$

□

Remark 4.3. The solution Ω must satisfy an optimality condition. We suppose that $\Omega \mathcal{C}^2-$ regular to compute the shape derivative. We deform the domain Ω with respect to an admissible vector field V to compute the shape derivative

$$dJ(\Omega; V) = \lim_{t \longrightarrow 0} \frac{J(Id + t\Omega) - J(\Omega)}{t}.$$

We have the variation calculation

$$-div(|\nabla u|^{p-2} \nabla u) = \lambda |u|^{p-2} u$$

$$-\int_\Omega div(|\nabla u|^{p-2} \nabla u) \phi dx = \int_\Omega \lambda |u|^{p-2} u \phi dx, \quad \text{for all} \quad \phi \in \mathcal{D}(\Omega)$$

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \phi dx = \int_\Omega \lambda |u|^{p-2} u \phi dx, \quad \text{for all} \quad \phi \in \mathcal{D}(\Omega)$$

Let us take $J(\Omega) = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \phi dx$ and $J_1(\Omega) = \int_\Omega \lambda |u|^{p-2} u \phi dx$. We have $dJ(\Omega; V) = dJ_1(\Omega; V)$.

We use the classical Hadamard formula to compute the Eulerian derivative of the functional J at the point Ω in the direction V.

$$dJ(\Omega; V) = \int_\Omega (|\nabla u|^{p-2} \nabla u \nabla \phi)' dx + \int_\Omega div(|\nabla u|^{p-2} \nabla u \nabla \phi . V(0)) dx.$$

We have

$$\int_\Omega (|\nabla u|^{p-2} \nabla u \nabla \phi)' dx$$
$$= \int_\Omega (|\nabla u|^{p-2})' \nabla u \nabla \phi dx + \int_\Omega |\nabla u|^{p-2} \nabla u (\nabla \phi)' dx + \int_\Omega |\nabla u|^{p-2} (\nabla u)' \nabla \phi dx.$$

We have the expression

$$\begin{aligned}(|\nabla u|^{p-2})' &= \left((|\nabla u|^2)^{\frac{p-2}{2}}\right)' \\ &= \frac{p-2}{2}(|\nabla u|^2)'(|\nabla u|^2)^{\frac{p-4}{2}} \\ (|\nabla u|^{p-2})' &= (p-2)\nabla u \nabla u' |\nabla u|^{p-4}.\end{aligned}$$

Then

$$\begin{aligned}dJ(\Omega;V) = (p-2)\int_\Omega |\nabla u|^{p-4}|\nabla u|^2 \nabla u' \nabla \phi dx \\ - \int_\Omega div(|\nabla u|^{p-2}\nabla u)\phi' dx - \int_\Omega div(|\nabla u|^{p-2}(\nabla u)')\phi dx\end{aligned}$$

$$\begin{aligned}dJ(\Omega,V) = (p-2)\int_\Omega |\nabla u|^{p-2}\nabla u' \nabla \phi dx \\ - \int_\Omega div(|\nabla u|^{p-2}\nabla u)\phi' dx - \int_\Omega div(|\nabla u|^{p-2}(\nabla u)')\phi dx\end{aligned}$$

because

$$\int_\Omega div(|\nabla u|^{p-2}\nabla u \nabla \phi \cdot V(0))dx = \int_{\partial\Omega} |\nabla u|^{p-2}\nabla u \nabla \phi \cdot V(0)\cdot \nu ds = 0.$$

We obtain

$$dJ(\Omega;V) = -(p-1)\int_\Omega div(|\nabla u|^{p-2}\nabla u')\phi dx - \int_\Omega div(|\nabla u|^{p-2}\nabla u)\phi' dx$$

We have also

$$\begin{aligned}dJ_1(\Omega;V) = \int_\Omega \lambda'|u|^{p-2}u\phi dx + \int_\Omega \lambda|u|^{p-2}u'\phi dx \\ + \int_\Omega \lambda|u|^{p-2}u\phi' dx + (p-2)\int_\Omega \lambda|u|^{p-2}u'\phi dx,\end{aligned}$$

$$dJ_1(\Omega;V) = \int_\Omega \lambda'|u|^{p-2}u\phi dx + \int_\Omega \lambda|u|^{p-2}u\phi' dx + (p-1)\int_\Omega \lambda|u|^{p-2}u'\phi dx$$

$dJ(\Omega;V) = dJ_1(\Omega,V)$ implies

$$\begin{aligned}-(p-1)\int_\Omega div(|\nabla u|^{p-2}\nabla u')\phi dx - \int_\Omega div(|\nabla u|^{p-2}\nabla u)\phi' dx \\ = \int_\Omega \lambda'|u|^{p-2}u\phi dx + \int_\Omega \lambda|u|^{p-2}u\phi' dx + (p-1)\int_\Omega \lambda|u|^{p-2}u'\phi dx.\end{aligned}$$

By simplification we get

$$\begin{aligned}-(p-1)\int_\Omega div(|\nabla u|^{p-2}\nabla u')\phi dx - \int_\Omega div(|\nabla u|^{p-2}\nabla u)\phi' dx \\ = \int_\Omega \lambda'|u|^{p-2}u\phi dx + (p-1)\int_\Omega \lambda|u|^{p-2}u'\phi dx, \quad \text{for all} \quad \phi \in \mathcal{D}(\Omega).\end{aligned}$$

This implies that

$$\text{(4.1)} \qquad \left\{ \; -(p-1)div(|\nabla u|^{p-2}\nabla u') \;=\; \lambda'|u|^{p-2}u + (p-1)\lambda|u|^{p-2}u' \quad \text{in} \quad \mathcal{D}'(\Omega) \right.$$

We multiply the equation (4.1) by u and by Green 's formula we get

$$-(p-1)\left[\int_\Omega div(|\nabla u|^{p-2}\nabla u)u'dx + \int_{\partial\Omega}|\nabla u|^{p-2}\nabla u\cdot\nu u'ds\right] = \lambda' + (p-1)\int_\Omega \lambda|u|^{p-2}uu'dx.$$

Finally we obtain the expression of

$$\lambda'(\Omega;V) = -(p-1)\int_{\partial\Omega}|\nabla u|^p\nabla u\cdot\nu u'ds$$

where u' satisfies $u' = -\frac{\partial u}{\partial\nu}V(0)\cdot\nu$ on $\partial\Omega$. Then

$$\lambda'(\Omega,V) = -(p-1)\int_{\partial\Omega}|\nabla u|^pV\cdot\nu ds.$$

We have a similar formula for the variation of the volume $dJ_2(\Omega,V) = \int_{\partial\Omega}V\cdot\nu ds,$ where $J_2(\Omega) = \int_\Omega dx - c.$

If Ω is an optimal domain then there exists a Lagrange multiplier $a<0$ such that

$$-(p-1)\int_{\partial\Omega}|\nabla u|^pV\cdot\nu ds = a\int_{\partial\Omega}V\cdot\nu ds.$$

Then we obtain

$$|\nabla u| = \left(\frac{-a}{p-1}\right)^{\frac{1}{p}} \quad \text{on} \quad \partial\Omega.$$

Since Ω is $\mathcal{C}^2-$regular and $u=0$ on $\partial\Omega,$ then we get

$$-\frac{\partial u}{\partial\nu} = \left(\frac{-a}{p-1}\right)^{\frac{1}{p}} \quad \text{on} \quad \partial\Omega.$$

We are also interested the existence of a minimizer for the following problem

$$\min\{\lambda_1(\Omega), \Omega\in\mathcal{A}, |\Omega|\le c\},$$

where $\mathcal{A}$ is a family of admissible domain defined by

$$\mathcal{A} = \{\Omega\subseteq D, \Omega \quad \text{is quasi open}\}$$

and $\lambda_1(\Omega)$ is defined by

$$\lambda_1(\Omega) = \min_{\phi\in W_0^{1,p}(\Omega),\phi\neq 0}\frac{\int_\Omega|\nabla\phi(x)|^pdx}{\int_\Omega|\phi(x)|^p} = \frac{\int_\Omega|\nabla u_1(x)|^pdx}{\int_\Omega|u_1(x)|^pdx}.$$

The Sobolev space $W_0^{1,p}(\Omega)$ is seen as a closed subspace of $W_0^{1,p}(D)$ defined by

$$W_0^{1,p}(\Omega) = \{u\in W_0^{1,p}(D) : u = 0p-q.e \quad \text{on} \quad D\backslash\Omega\}.$$

The problem is to look for weak topology constraints which would make the class $\mathcal{A}$ sequentially compact. This convergence is called weak γ_p-convergence for quasi open sets.

Definition 4.2. We say that a sequence (Ω_n) of $\mathcal{A}$ weakly γ_p-converges to $\Omega\in\mathcal{A}$ if the sequence u_n converges weakly in $W_0^{1,p}(D)$ to a function $u\in W_0^{1,p}(D)$ (that we may take as quasi-continuous) such that $\Omega = \{u>0\}$.

We have the following theorem.

Theorem 4.4. *The problem*

$$\min\{\lambda_1(\Omega), \Omega\in\mathcal{A}, |\Omega|\le c\} \tag{4.2}$$

admits at least one solution.

Proof of Theorem 4.4. Let us take

$$\lambda_1(\Omega_n) = \min_{\phi_n \in W_0^{1,p}(\Omega_n), \phi_n \neq 0} \frac{\int_{\Omega_n} |\nabla \phi_n(x)|^p dx}{\int_{\Omega_n} |\phi_n(x)|^p dx} = \frac{\int_{\Omega_n} |\nabla u_n(x)|^p dx}{\int_{\Omega_n} |u_n(x)|^p dx}.$$

Suppose that $(\Omega_n)_{(n\in\mathbb{N})}$ is a minimizing sequence of domain for the problem (4.2). We denote by u_n a first eigenfunction on Ω_n, such that $\int_{\Omega_n} |u_n(x)|^p dx = 1$.

Since u_n is the first eigenfunction of $\lambda_1(\Omega_n)$, u_n is strictly positive, cf [15], then the sequence (Ω_n) is defined by $\Omega_n = \{u_n > 0\}$.

If the sequence (u_n) is bounded in $W_0^{1,p}(D)$, then there exists a subsequence still denoted by u_n such that u_n converges weakly in $W_0^{1,p}(D)$ to a function u. By compact injection, we have that $\int_\Omega |u(x)|^p dx = 1$.

Let Ω be quasi open and defined by $\Omega = \{u > 0\}$, this implies that $u \in W_0^{1,p}(\Omega)$. As the sequence (u_n) is bounded in $W_0^{1,p}(D)$, then

$$\liminf_{n\to+\infty} \frac{\int_{\Omega_n} |\nabla u_n(x)|^p dx}{\int_{\Omega_n} |u_n(x)|^p dx} \geq \frac{\int_\Omega |\nabla u(x)|^p dx}{\int_\Omega |u(x)|^p dx} \geq \frac{\int_\Omega |\nabla u_1(x)|^p dx}{\int_\Omega |u_1(x)|^p dx} = \lambda_1(\Omega).$$

Now we show that $|\Omega| \leq c$.

We know that if the sequence Ω_n weakly γ_p- converges to Ω and the Lebesgue measure is weakly γ_p-lower semicontinuous on the class $\mathcal{A}$ (see [5]), then we obtain $|\{u > 0\}| \leq \liminf_{n\to+\infty} |\{u_n > 0\}| \leq c$ this implies that $|\Omega| \leq c$. □

5. DOMAIN IN BOX

Now let us take $N = 2$. We consider the class of admissible domains defined by

$$\mathcal{C} = \{\Omega, \Omega \quad \text{open subsets of } D \text{ and simply connected}, |\Omega| = c\}.$$

- For $p > 2$, the p-capacity of a point is stricly positive and every $W_0^{1,p}$ function has a continuous representative. For this reason, a property which holds $p - q.e$ with $p > 2$ holds in fact everywhere. For $p > 2$, the domain Ω_p is a minimizer of the problem $\min\{\lambda_1^p(\Omega_p), \Omega_p \in \mathcal{C}\}$.

  Consider the sequence $(\Omega_{p_n}) \subseteq \mathcal{C}$ and assume that Ω_{p_n} converges in Hausdorff complementary topology to Ω_2, when p_n goes to 2 and $p_n > 2$. Then $\Omega_2 \subseteq \mathcal{C}$ and Ω_{p_n} γ_2-converges to Ω_2.

  By the Sobolev embedding theorem, we have $W_0^{1,p_n}(\Omega_{p_n}) \hookrightarrow H_0^1(\Omega_{p_n})$. The γ_2 -convergence implies that $H_0^1(\Omega_{p_n})$ converges in the sense of Mosco to $H_0^1(\Omega_2)$. For $p_n > 2$, by the Hölder inequality we have

$$\left(\int |\nabla u_{p_n}|^2 dx\right)^{\frac{1}{2}} \leq |\Omega_{p_n}|^{\frac{1}{2}-\frac{1}{p_n}} \left(\int |\nabla u_{p_n}|^{p_n} dx\right)^{\frac{1}{p_n}}$$
$$\left(\int |\nabla u_{p_n}|^2\right)^{\frac{1}{2}} dx \leq c^{\frac{1}{2}-\frac{1}{p_n}} \lambda_1^{p_n}(\Omega_{p_n}).$$

  Then the sequence (u_{p_n}) is uniformly bounded in $H_0^1(\Omega_{p_n})$. There exists a subsequence still denoted u_{p_n} such that u_{p_n} converges weakly in $H_0^1(D)$ to a function u. The second condition of Mosco implies that $u \in H_0^1(\Omega_2)$.

  For $p > 2$, we have the Sobolev embedding theorem $W_0^{1,p}(D) \hookrightarrow \mathcal{C}^{0,\alpha}(\bar{D})$.

  Ascoli's theorem implies that $u_{p_n} \longrightarrow u$ and $\nabla u_{p_n} \longrightarrow \nabla u$ locally uniformly in Ω_2, when p_n goes to 2 and $p_n > 2$.

Now show that

$$\lim_{p_n \longrightarrow 2} \int |u_{p_n}|^2 dx = 1 \quad \text{i.e.} \quad \int |u|^2 dx = 1.$$

For $\epsilon > 0$ small, we have $p_n > 2 - \epsilon$. Noting that

$$\left(\int |\nabla u_{p_n}|^{2-\epsilon} dx \right)^{\frac{1}{2-\epsilon}} \leq |\Omega_{p_n}|^{\frac{1}{2-\epsilon} - \frac{1}{p_n}} \left(\int |\nabla u_{p_n}|^{p_n} dx \right)^{\frac{1}{p_n}} = c^{\frac{1}{2-\epsilon} - \frac{1}{p_n}} \lambda_1^{p_n}(\Omega_{p_n}),$$

this implies that the sequence u_{p_n} is uniformly bounded in $W_0^{1,2-\epsilon}(\Omega_{p_n})$. Then there exists a subsequence still denoted u_{p_n} such that u_{p_n} is weakly convergent in $H_0^1(D)$ to u. By the second condition of Mosco we get $u \in W_0^{1,2-\epsilon}(\Omega_2)$. It follows that

$$\int |u|^{2-\epsilon} dx = \lim_{p_n \longrightarrow 2} \int |u_{p_n}|^{2-\epsilon} dx \leq \lim_{p_n \longrightarrow 2} |\Omega_{p_n}|^{1 - \frac{2-\epsilon}{p_n}} \left(\int |u_{p_n}|^{p_n} dx \right)^{\frac{2-\epsilon}{p_n}} = c^{\frac{\epsilon}{2}}.$$

Letting $\epsilon \longrightarrow 0$, we obtain $\int |u|^2 dx \leq 1$.

On the other hand, Lemma 4.2 of [14] implies that

$$\int |u|^{p_n} dx \geq \int |u_{p_n}|^{p_n} dx + p_n \int |u|^{p_n - 2} dx u_{p_n} (u - u_{p_n}).$$

The second integral on the right-hand side approaches 0 as $p_n \longrightarrow 2$. Thus we get $\int |u|^2 dx \geq 1$, and we conclude that $\int |u|^2 dx = 1$.

In [11, Theorem 2.1 p. 3350], λ_k^p is continuous in p for $k = 1, 2$, where λ_k^p is the $k - th$ eigenvalue for the p-Laplacian operator.

We have

$$\int |\nabla u_{p_n}|^{p_n - 2} \nabla u_{p_n} \nabla \phi dx = \int \lambda_1^{p_n} |u_{p_n}|^{p_n - 2} u_{p_n} \phi dx, \quad \text{for all} \quad \phi \in \mathcal{D}(\Omega_2). \tag{5.1}$$

Letting p_n go to 2, $p_n > 2$ in (5.1), and noting that u_{p_n} converges uniformly to u on the support of ϕ, we obtain

$$\int \nabla u \nabla \phi dx = \int \lambda_1^2 u \phi dx, \quad \text{for all} \quad \phi \in \mathcal{D}(\Omega_2),$$

whence we have

$$\begin{cases} -\Delta u &= \lambda_1^2 u & \text{in} & \mathcal{D}'(\Omega_2) \\ u &= 0 & \text{on} & \partial \Omega_2. \end{cases}$$

We conclude that when $p \longrightarrow 2$ and $p > 2$ the free parts of the boundary of Ω_p cannot be pieces of circle.

- For $p \leq 2$, we consider the sequence $(\Omega_{p_n}) \subseteq \mathcal{C}$ and assume that Ω_{p_n} converges in Hausdorff complementary topology to Ω_2, when p_n goes to 2 and $p_n \leq 2$. Then by Theorem 3.1, we get $\Omega_2 \subseteq \mathcal{C}$ and Ω_{p_n} $\gamma_{2-\epsilon}$-converges to Ω_2.

  In [16], the sequence (u_{p_n}) is bounded in $W^{1,2-\epsilon}(D)$, $0 < \epsilon < 1$ that is ∇u_{p_n} converges weakly in $L^{2-\epsilon}(D)$ to ∇u and u_{p_n} converges strongly in $L^{2-\epsilon}(D)$ to u. In [16], we get also $\int |\nabla u|^2 dx \leq \beta$ and $\int |u|^2 dx < \infty$.

  By Lemma 4.2 of [14], we have

$$\int |u|^{p_n} dx \geq \int |u_{p_n}|^{p_n} dx + p_n \int |u|^{p_n - 2} dx u_{p_n} (u - u_{p_n}).$$

The second integral on the right-hand side approaches 0 as $p_n \longrightarrow 2$. Thus we get $\int |u|^2 dx \geq 1$. This implies that

$$\lim_{p_n \longrightarrow 2} \int |u_{p_n}|^{2-\epsilon} dx = \int |u|^{2-\epsilon} dx = 1.$$

Letting $\epsilon \longrightarrow 0$, we obtain $\int |u|^2 dx = 1$.

The $\gamma_{2-\epsilon}$-convergence implies that u_{p_n} converges strongly in $W_0^{1,2-\epsilon}(\Omega_2)$ to u. According to P. Lindqvist see [16], we have $u \in H^1(D)$, and we can deduce that $u \in H_0^1(\Omega_2)$. As the first eigenvalue for the p-Laplacian operator is continuous in p cf [11], we have

$$\int |\nabla u_{p_n}|^{p_n-2} \nabla u_{p_n} \nabla \phi dx = \int \lambda_1^{p_n} |u_{p_n}|^{p_n-2} u_{p_n} \phi dx, \quad \text{for all} \quad \phi \in \mathcal{D}(\Omega_2). \tag{5.2}$$

Letting p_n go to 2, $p_n \leq 2$ in (5.2), and noting that u_{p_n} converges uniformly to u on the support of ϕ, we obtain

$$\int \nabla u \nabla \phi dx = \int \lambda_1^2 u \phi dx, \quad \text{for all} \quad \phi \in \mathcal{D}(\Omega_2),$$

whence we have

$$\begin{cases} -\Delta u &= \lambda_1^2 u \quad \text{in} \quad \mathcal{D}'(\Omega_2) \\ u &= 0 \quad \text{on} \quad \partial\Omega_2. \end{cases}$$

We conclude that when $p \longrightarrow 2$ and $p \leq 2$ the free parts of the boundary of Ω_p cannot be pieces of circle.

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