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THE DISCRETE VERSION OF OSTROWSKI'S INEQUALITY IN NORMED LINEAR SPACES

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ABSTRACT. Discrete versions of Ostrowski's inequality for vectors in normed linear spaces are given.

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1. INTRODUCTION

The following result is known in the literature as Ostrowski's inequality [10].

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with the property that $|f'(t)| \le M$ for all $t \in (a,b)$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)M \tag{1.1}$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

A simple proof of this fact can be done by using the identity:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{b-a}\int_a^b p(x,t)\,f'(t)\,dt, \quad x \in [a,b], \tag{1.2}$$

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where

$$p(x,t) := \begin{cases} t-a & \text{if} \quad a \le t \le x \\ \\ t-b & \text{if} \quad x < t \le b \end{cases}$$

which also holds for absolutely continuous functions $f:[a,b]\to\mathbb{R}$.

The following Ostrowski type result for absolutely continuous functions holds (see [6] – [8]).

Theorem 1.2. *Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$. Then, for all $x\in[a,b]$, we have:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \tag{1.3}$$

$$\le \begin{cases} \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left\|f'\right\|_\infty & \textit{if} \quad f'\in L_\infty[a,b]\,; \\ \\ \frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{x-a}{b-a}\right)^{p+1} + \left(\frac{b-x}{b-a}\right)^{p+1}\right]^{\frac{1}{p}}(b-a)^{\frac{1}{p}}\left\|f'\right\|_q & \textit{if} \quad f'\in L_q[a,b]\,, \\ & \quad \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left\|f'\right\|_1\,; \end{cases}$$

where $\|\cdot\|_r$ $(r\in[1,\infty])$ are the usual Lebesgue norms on $L_r[a,b]$, i.e.,

$$\|g\|_\infty := ess \sup_{t\in[a,b]} |g(t)|$$

and

$$\|g\|_r := \left(\int_a^b |g(t)|^r\,dt\right)^{\frac{1}{r}},\ r\in[1,\infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ and $\frac{1}{2}$ respectively are sharp in the sense presented in Theorem 1.1.

The above inequalities can also be obtained from the Fink result in [9] on choosing $n=1$ and performing some appropriate computations.

If one drops the condition of absolute continuity and assumes that f is Hölder continuous, then one may state the result (see [5]):

Theorem 1.3. *Let $f:[a,b]\to\mathbb{R}$ be of $r-H-$Hölder type, i.e.,*

$$|f(x)-f(y)| \le H|x-y|^r\,,\ \textit{for all}\ \ x,y\in[a,b]\,, \tag{1.4}$$

where $r\in(0,1]$ and $H>0$ are fixed. Then, for all $x\in[a,b]$, we have the inequality:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{H}{r+1}\left[\left(\frac{b-x}{b-a}\right)^{r+1} + \left(\frac{x-a}{b-a}\right)^{r+1}\right](b-a)^r\,. \tag{1.5}$$

The constant $\frac{1}{r+1}$ is also sharp in the above sense.

Note that if $r=1$, i.e., f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitzian functions (with L instead of H) (see [4])

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\,L. \tag{1.6}$$

Here the constant $\frac{1}{4}$ is also best.

Moreover, if one drops the condition of the continuity of the function, and assumes that it is of bounded variation, then the following result may be stated (see [2]).

Theorem 1.4. *Assume that $f:[a,b]\to\mathbb{R}$ is of bounded variation and denote by $\bigvee_a^b(f)$ its total variation. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f) \tag{1.7}$$

for all $x\in[a,b]$. The constant $\frac{1}{2}$ is the best possible.

If we assume more about f, i.e., f is monotonically increasing, then the inequality (1.7) may be improved in the following manner [3] (see also [1]).

Theorem 1.5. *Let $f:[a,b]\to\mathbb{R}$ be monotonic nondecreasing. Then for all $x\in[a,b]$, we have the inequality:*

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| &\le\frac{1}{b-a}\left\{[2x-(a+b)]f(x)+\int_a^b sgn(t-x)f(t)\,dt\right\}\\ &\le\frac{1}{b-a}\{(x-a)[f(x)-f(a)]+(b-x)[f(b)-f(x)]\}\\ &\le\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right][f(b)-f(a)].\end{aligned} \tag{1.8}$$

All the inequalities in (1.8) are sharp and the constant $\frac{1}{2}$ is the best possible.

For other recent results including Ostrowski type inequalities for n-time differentiable functions, visit the RGMIA website at `http://rgmia.vu.edu.au/database.html`.

In this paper we point out some discrete Ostrowski type inequalities for vectors in normed linear spaces.

2. SOME IDENTITIES

The following lemma holds.

Lemma 2.1. *Let x_i $(i=1,\dots,n)$ be vectors in X. Then we have the representation*

$$x_i=\frac{1}{n}\sum_{j=1}^n x_j+\frac{1}{n}\sum_{j=1}^n p(i,j)\,\Delta x_j,\quad i\in\{1,\dots,n\}, \tag{2.1}$$

where

$$p(1,j)=j-n \;\text{ if }\; 1\le j\le n-1; \tag{2.2}$$

$$p(n,j)=j \;\text{ if }\; 1\le j\le n-1; \tag{2.3}$$

and

$$p(i,j)=\begin{cases} j & \text{if }\; 1\le j\le i-1,\\ j-n & \text{if }\; i\le j\le n-1,\end{cases} \tag{2.4}$$

where $2\le i\le n-1$ and $1\le j\le n-1$.

Proof. For $i=1$, we have to prove that

$$x_1 = \frac{1}{n}\sum_{j=1}^{n} x_j + \frac{1}{n}\sum_{j=1}^{n}(j-n)\Delta x_j. \tag{2.5}$$

Using the summation by parts formula, we have

$$\begin{aligned}\sum_{j=1}^{n}(j-n)\Delta x_j &= (j-n)x_j\Big|_{j=1}^{n} - \sum_{j=1}^{n-1}\Delta(j-n)x_{j+1}\\ &= (n-1)x_1 - \sum_{j=1}^{n-1}x_{j+1}\\ &= nx_1 - \sum_{j=1}^{n}x_j\end{aligned}$$

and the formula (2.5) is proved.

For $i=n$, we can prove similarly that

$$x_n = \frac{1}{n}\sum_{j=1}^{n} x_j + \frac{1}{n}\sum_{j=1}^{n-1} j\Delta x_j. \tag{2.6}$$

Let $2 \le i \le n-1$. We have

$$\begin{aligned}\sum_{j=1}^{n-1}p(i,j)\Delta x_j &= \sum_{j=1}^{i-1}p(i,j)\Delta x_j + \sum_{j=i}^{n-1}p(i,j)\Delta x_j\\ &= \sum_{j=1}^{i-1}i\Delta x_j + \sum_{j=i}^{n-1}(j-n)\Delta x_j.\end{aligned} \tag{2.7}$$

Using the summation by parts formula, we have

$$\begin{aligned}\sum_{j=1}^{i-1}i\Delta x_j &= jx_j\Big|_{j=i}^{n} - \sum_{j=1}^{i-1}\Delta(i)x_{j+1}\\ &= ix_i - x_1 - \sum_{j=1}^{i-1}x_{j+1}\\ &= (i-1)x_i - \sum_{j=1}^{i-1}x_j\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\sum_{j=i}^{n-1}(j-n)\Delta x_j &= (j-n)x_j\Big|_{j=i}^{n} - \sum_{j=i}^{n-1}\Delta(j-n)x_{j+1}\\ &= (n-i)x_i - \sum_{j=i}^{n-1}x_{j+1}\\ &= (n-i+1)x_i - \sum_{j=i}^{n}x_j.\end{aligned} \tag{2.9}$$

Using (2.7) – (2.9), we deduce

$$\begin{aligned}\sum_{j=1}^{n-1} p(i,j)\,\Delta x_j &= (i-1)\,x_i - \sum_{j=1}^{i-1} x_j + (n-i+1)\,x_i - \sum_{j=i}^{n} x_j \\ &= nx_i - \sum_{j=1}^{n} x_j\end{aligned}$$

and the identity (2.1) is proved. □

The following corollaries hold.

Corollary 2.2. *We have the identity*

$$\frac{x_1+x_n}{2} = \frac{1}{n}\sum_{j=1}^{n} x_j + \frac{1}{n}\sum_{j=1}^{n}\left(j-\frac{n}{2}\right)\Delta x_j. \tag{2.10}$$

Corollary 2.3. *Let $n=2m+1$. Then we have*

$$x_{m+1} = \frac{1}{2m+1}\sum_{j=1}^{2m+1} x_j + \frac{1}{2m+1}\sum_{j=1}^{2m} p_m(j)\,\Delta x_j, \tag{2.11}$$

where

$$p_m(j) = \begin{cases} j & \text{if } \ 1\le j\le m, \\ \\ j-2m-1 & \text{if } \ m+1\le j\le 2m. \end{cases}$$

3. DISCRETE OSTROWSKI'S INEQUALITY

The following discrete inequality of Ostrowski type holds.

Theorem 3.1. *Let $(X,\|\cdot\|)$ be a normed linear space and x_i $(i=1,\dots,n)$ be vectors in X. Then we have the inequality*

$$\left\| x_i - \frac{1}{n}\sum_{k=1}^{n} x_k \right\| \le \frac{1}{n}\left[\left(i-\frac{n+1}{2}\right)^2 + \frac{n^2-1}{4}\right] \max_{k=1,\dots,n-1} \|\Delta x_k\|, \tag{3.1}$$

for all $i\in\{1,\dots,n\}$. The constant $c=\frac{1}{4}$ in the right hand side is best in the sense that it cannot be replaced by a smaller one.

Proof. We use the representation (2.1) and the generalised triangle inequality to obtain

$$\begin{aligned}\left\| x_i - \frac{1}{n}\sum_{k=1}^{n} x_k \right\| &= \frac{1}{n}\left\|\sum_{k=1}^{n-1} p(i,k)\,\Delta x_k\right\| \\ &\le \frac{1}{n}\sum_{k=1}^{n-1} |p(i,k)|\,\|\Delta x_k\| \\ &\le \max_{k=1,\dots,n-1}\|\Delta x_k\| \times \frac{1}{n}\sum_{k=1}^{n-1}|p(i,k)|.\end{aligned}$$

If $i=1$, then we have

$$\sum_{k=1}^{n-1}|p(1,k)| = \sum_{k=1}^{n-1}|k-n| = \sum_{k=1}^{n-1} k = \frac{n(n-1)}{2}$$

and as

$$\left(1-\frac{n+1}{2}\right)^2+\frac{n^2-1}{4}=\frac{n(n-1)}{2},\quad \text{for } n\geq 1$$

the inequality (3.1) is valid for $i=1$.

Let $2\leq i\leq n-1$. Then

$$\begin{aligned}\sum_{k=1}^{n-1}|p(i,k)| &= \sum_{k=1}^{i-1}|p(i,k)|+\sum_{k=i}^{n-1}|p(i,k)| \\ &= \sum_{k=1}^{i-1}k+\sum_{k=i}^{n-1}(n-k) \\ &= \frac{(i-1)i}{2}+n(n-1-i+1)-\left(\sum_{k=1}^{n-1}k-\sum_{k=1}^{i-1}k\right) \\ &= \frac{(i-1)i}{2}+n(n-i)-\left(\frac{n(n-1)}{2}-\frac{i(i-1)}{2}\right) \\ &= \frac{1}{2}\left(2i^2+n^2-2ni+n\right) \\ &= \left(i-\frac{n+1}{2}\right)^2+\frac{n^2-1}{4}\end{aligned}$$

and the inequality (3.1) is also proved for $i\in\{2,\dots,n-1\}$.

For $i=n$, we have $p(n,k)=k$, $k=1,\dots,n-1$ giving

$$\sum_{k=1}^{n-1}|p(n,k)|=\sum_{k=1}^{n-1}k=\frac{n(n-1)}{2}$$

and as

$$\left(n-\frac{n+1}{2}\right)^2+\frac{n^2-1}{4}=\frac{n(n-1)}{2}$$

the inequality (3.2) is also valid for $i=n$.

To prove the sharpness of the constant $c=\frac{1}{4}$, assume that (3.1) holds with a constant $c>0$, i.e.,

$$\left\|x_i-\frac{1}{n}\sum_{k=1}^{n}x_k\right\|\leq\frac{1}{n}\left[\left(i-\frac{n+1}{2}\right)^2+c\left(n^2-1\right)\right]\max_{k=1,\dots,n-1}\|\Delta x_k\| \tag{3.2}$$

for any x_k $(k=1,\dots,n)$ in X.

Let $x_k=x_1+(k-1)r$, $k=1,\dots,n$, $r\in X$, $r\neq 0$, $x_1\neq 0$ and $i=1$ in (3.2). Then we get

$$\left\|x_1-\frac{1}{n}\sum_{k=1}^{n}(x_1+(k-1)r)\right\|\leq\frac{1}{n}\left[\frac{(n-1)^2}{4}+c\left(n^2-1\right)\right]\|r\| \tag{3.3}$$

and as

$$\sum_{k=1}^{n}(x_1+(k-1)r)=nx_1+\frac{n(n-1)}{2}r,$$

then from (3.3) we deduce

$$\left\| \left(\frac{n-1}{2} \right) \cdot r \right\| \leq \frac{1}{n} \left[\frac{(n-1)^2}{4} + c\left(n^2 - 1\right) \right] \|r\|$$

from where we get

$$\frac{1}{2} \leq \frac{1}{n} \left[\frac{n-1}{4} + c\,(n+1) \right]$$

i.e.,

$$n + 1 \leq 4c\,(n+1)\,,$$

which implies that $c \geq \frac{1}{4}$, and the theorem is proved. □

Corollary 3.2. *Under the above assumptions and if* $n = 2m+1$, *then we have the inequality*

$$\left\| x_{m+1} - \frac{1}{2m+1} \sum_{k=1}^{2m+1} x_k \right\| \leq \frac{m\,(m+1)}{2m+1} \max_{k=1,\dots,2m} \|\Delta x_k\|\,. \tag{3.4}$$

The proof is obvious by the above Theorem 3.1 for $i = m+1$.

The following corollary also holds.

Corollary 3.3. *Under the above assumptions, we have:*

a) *If* $n = 2k$, *then*

$$\left\| \frac{x_1 + x_{2k}}{2} - \frac{1}{2k} \sum_{j=1}^{2k} x_j \right\| \leq \frac{1}{2}\,(k-1) \max_{j=1,\dots,2k-1} \|\Delta x_j\|\,. \tag{3.5}$$

b) *If* $n = 2k+1$, *then*

$$\left\| \frac{x_1 + x_{2k+1}}{2} - \frac{1}{2k+1} \sum_{j=1}^{2k+1} x_j \right\| \leq \frac{2k^2 + 2k + 1}{2\,(2k+1)} \max_{j=1,\dots,2k} \|\Delta x_j\|\,. \tag{3.6}$$

Proof. The proof is as follows.

a) If $n = 2k$, then by Corollary 2.2, we have

$$\begin{aligned}
\left\| \frac{x_1 + x_{2k}}{2} - \frac{1}{2k} \sum_{j=1}^{2k} x_j \right\| &\leq \frac{1}{2k} \sum_{j=1}^{2k-1} |j-k|\, \|\Delta x_j\| \\
&\leq \frac{1}{2k} \max_{j=1,\dots,2k-1} \|\Delta x_j\| \sum_{j=1}^{2k-1} |j-k| \\
&= \frac{1}{2k} \max_{j=1,\dots,2k-1} \|\Delta x_j\| \left(\sum_{j=1}^{k} (k-j) + \sum_{j=k+1}^{2k-1} (j-k) \right) \\
&= \frac{1}{k} \max_{j=1,\dots,2k-1} \|\Delta x_j\| \,\frac{(k-1)\,k}{2} \\
&= \frac{1}{2}\,(k-1) \max_{j=1,\dots,2k-1} \|\Delta x_j\|\,,
\end{aligned}$$

and the inequality (3.5) is proved.

b) If $n = 2k+1$, then by Corollary 2.2, we have

$$\begin{aligned}
&\left\| \frac{x_1 + x_{2k+1}}{2} - \frac{1}{2k+1} \sum_{j=1}^{2k+1} x_j \right\| \\
&\quad \leq \frac{1}{2k+1} \sum_{j=1}^{2k+1} \left| j - \frac{2k+1}{2} \right| \|\Delta x_j\| \\
&\quad \leq \frac{1}{2k+1} \max_{j=1,\dots,2k} \|\Delta x_j\| \sum_{j=1}^{2k+1} \left| j - k - \frac{1}{2} \right| \\
&\quad = \frac{1}{2k+1} \max_{j=1,\dots,2k} \|\Delta x_j\| \left[\sum_{j=1}^{k} \left(k + \frac{1}{2} - j \right) + \sum_{j=k+1}^{2k+1} \left(j - k - \frac{1}{2} \right) \right] \\
&\quad = \frac{1}{2k+1} \max_{j=1,\dots,2k} \|\Delta x_j\| \left[\frac{1}{2} k + \sum_{j=1}^{k} (k-j) - \frac{1}{2}(k+1) + \sum_{j=k+1}^{2k+1} (j-k) \right] \\
&\quad = \frac{1}{2k+1} \max_{j=1,\dots,2k} \|\Delta x_j\| \left[\frac{k^2 - k + k^2 + 3k + 2 - 1}{2} \right] \\
&\quad = \max_{j=1,\dots,2k} \|\Delta x_j\| \frac{2k^2 + 2k + 1}{2(2k+1)}
\end{aligned}$$

and the inequality (3.6) is proved.

□

The following result including a version of a discrete Ostrowski inequality for l_p-norms of $\{\Delta x_i\}_{i=\overline{1,n-1}}$ also holds.

Theorem 3.4. *Let* $(X, \|\cdot\|)$ *be a normed linear space and* x_i $(i = 1, \dots, n)$ *be vectors in* X*. Then we have the inequality*

$$\left\| x_i - \frac{1}{n} \sum_{k=1}^{n} x_k \right\| \leq \frac{1}{n} \left[s_\alpha (i-1) + s_\alpha (n-i) \right]^{\frac{1}{\alpha}} \left[\sum_{k=1}^{n-1} \|\Delta x_k\|^\beta \right]^{\frac{1}{\beta}} \tag{3.7}$$

for all $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, *where* $s_\alpha(\cdot)$ *denotes the sum:*

$$s_\alpha(m) := \sum_{j=1}^{m} j^\alpha.$$

When $m = 0$, *the sum is assumed to be zero.*

Proof. Using representation (2.2) and the generalised triangle inequality, we have:

$$\begin{aligned}
\left\| x_i - \frac{1}{n} \sum_{k=1}^{n} x_k \right\| &= \frac{1}{n} \left\| \sum_{k=1}^{n-1} p(i,k) \Delta x_k \right\| \\
&\leq \frac{1}{n} \sum_{k=1}^{n-1} |p(i,k)| \|\Delta x_k\| =: B.
\end{aligned} \tag{3.8}$$

Using Hölder's discrete inequality, we have

$$B \leq \frac{1}{n}\left(\sum_{k=1}^{n-1}|p(i,k)|^{\alpha}\right)^{\frac{1}{\alpha}}\left(\sum_{k=1}^{n-1}\|\Delta x_k\|^{\beta}\right)^{\frac{1}{\beta}}. \tag{3.9}$$

However,

$$\begin{aligned}\sum_{k=1}^{n-1}|p(i,k)|^{\alpha} &= \sum_{k=1}^{i-1}|p(i,k)|^{\alpha}+\sum_{k=i}^{n-1}|p(i,k)|^{\alpha}\\ &= \sum_{k=1}^{i-1}k^{\alpha}+\sum_{k=i}^{n-1}(n-k)^{\alpha}\\ &= 1^{\alpha}+\cdots+(i-1)^{\alpha}+(n-i)^{\alpha}+\cdots+1^{\alpha}\\ &= s_{\alpha}(i-1)+s_{\alpha}(n-i)\end{aligned}$$

and the inequality (3.7) then follows by (3.8) and (3.9). □

The case of $\alpha=\beta=2$ can be useful in practical applications.

Corollary 3.5. *With the assumptions of Theorem 3.4, we have*

$$\left\|x_i-\frac{1}{n}\sum_{k=1}^{n}x_k\right\| \leq \frac{1}{\sqrt{n}}\left[\left(i-\frac{n+1}{2}\right)^2+\frac{n^2-1}{12}\right]^{\frac{1}{2}}\left[\sum_{k=1}^{n-1}\|\Delta x_k\|^2\right]^{\frac{1}{2}}. \tag{3.10}$$

Proof. For $\alpha=2$, we have

$$s_2(i-1)=\sum_{k=1}^{i-1}k^2=\frac{i(i-1)(2i-1)}{6}$$

and

$$s_2(n-i)=\sum_{k=1}^{n-i}k^2=\frac{(n-i)(n-i+1)[2(n-i)+1]}{6}.$$

As simple algebra proves that

$$s_2(i-1)+s_2(n-i)=n\left[\left(i-\frac{n+1}{2}\right)^2+\frac{n^2-1}{12}\right],$$

then, by (3.7) we deduce the desired inequality (3.10). □

Corollary 3.6. *Under the above assumptions and if $n=2m+1$, then we have the inequality:*

$$\left\|x_{m+1}-\frac{1}{2m+1}\sum_{k=1}^{2m+1}x_k\right\| \leq \frac{2^{\frac{1}{\alpha}}}{2m+1}[s_{\alpha}(m)]^{\frac{1}{\alpha}}\left[\sum_{k=1}^{2m}\|\Delta x_k\|^{\beta}\right]^{\frac{1}{\beta}} \tag{3.11}$$

for $\alpha>1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$.
In particular, for $\alpha=\beta=2$, we have

$$\left\|x_{m+1}-\frac{1}{2m+1}\sum_{k=1}^{2m+1}x_k\right\| \leq \sqrt{\frac{m(m+1)}{3(2m+1)}}\left[\sum_{k=1}^{2m}\|\Delta x_k\|^2\right]^{\frac{1}{2}}. \tag{3.12}$$

The following result providing an upper bound in terms of the l_1-norm of $(\Delta x_k)_{k=\overline{1,n-1}}$ also holds.

Theorem 3.7. *Let $(X, \|\cdot\|)$ be a normed linear space and x_i $(i = 1, \dots, n)$ be vectors in X. Then we have the inequality*

$$\left\| x_i - \frac{1}{n}\sum_{k=1}^{n} x_k \right\| \leq \frac{1}{n}\left[\frac{1}{2}(n-1) + \left| i - \frac{n+1}{2} \right|\right] \sum_{k=1}^{n-1} \|\Delta x_k\| \tag{3.13}$$

for all $i \in \{1, \dots, n\}$.

Proof. As in Theorem 3.4, we have

$$\left\| x_i - \frac{1}{n}\sum_{k=1}^{n} x_k \right\| \leq B, \tag{3.14}$$

where

$$B := \frac{1}{n}\sum_{k=1}^{n-1} |p(i,k)| \, \|\Delta x_k\| .$$

It is obvious that

$$\begin{aligned} B &= \frac{1}{n}\left[\sum_{k=1}^{i-1} k \|\Delta x_k\| + \sum_{k=i}^{n-1} (n-k) \|\Delta x_k\|\right] \\ &\leq \frac{1}{n}\left[(i-1)\sum_{k=1}^{i-1} \|\Delta x_k\| + (n-i)\sum_{k=i}^{n-1} \|\Delta x_k\|\right] \\ &= \frac{1}{n}\max\{i-1, n-i\}\left[\sum_{k=1}^{i-1} \|\Delta x_k\| + \sum_{k=i}^{n-1} \|\Delta x_k\|\right] \\ &= \frac{1}{n}\left[\frac{1}{2}(n-1) + \frac{1}{2}|n-i-i+1|\right]\sum_{k=1}^{n-1} \|\Delta x_k\| \\ &= \frac{1}{n}\left[\frac{1}{2}(n-1) + \left| i - \frac{n+1}{2} \right|\right]\sum_{k=1}^{n-1} \|\Delta x_k\| \end{aligned}$$

and the inequality (3.13) is proved. □

The following corollary contains the best inequality we can get from (3.13).

Corollary 3.8. *Let $(X, \|\cdot\|)$ be as above and $n = 2m+1$. Then we have the inequality*

$$\left\| x_{m+1} - \frac{1}{2m+1}\sum_{k=1}^{2m+1} x_k \right\| \leq \frac{m}{2m+1}\sum_{k=1}^{2m} \|\Delta x_k\| . \tag{3.15}$$

4. Weighted Ostrowski Inequality

We start with the following theorem.

Theorem 4.1. *Let $(X, \|\cdot\|)$ be a normed linear space, $x_i \in X$ $(i = 1, \dots, n)$ and $p_i \geq 0$ $(i = 1, \dots, n)$ with $\sum_{i=1}^{n} p_i = 1$. Then we have the inequality:*

$$\left\|x_i - \sum_{j=1}^{n} p_j x_j\right\| \leq \sum_{j=1}^{n} p_j |j - i| \cdot \max_{k=\overline{1,n-1}} \|\Delta x_k\| \leq \max_{k=\overline{1,n-1}} \|\Delta x_k\| \times \begin{cases} \frac{n-1}{2} + \left|i - \frac{n+1}{2}\right|, \\ \left(\sum_{j=1}^{n} |j-i|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} p_j^q\right)^{\frac{1}{q}} & \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ \left[\frac{n^2-1}{4} + \left(i - \frac{n+1}{2}\right)^2\right] \max_{j=\overline{1,n}} \{p_j\} \end{cases} \tag{4.1}$$

for all $i \in \{1, \dots, n\}$.

Proof. Using the properties of the norm, we have

$$\sum_{j=1}^{n} p_j \|x_i - x_j\| \geq \left\|\sum_{j=1}^{n} p_j (x_i - x_j)\right\| = \left\|x_i \sum_{j=1}^{n} p_j - \sum_{j=1}^{n} p_j x_j\right\| = \left\|x_i - \sum_{j=1}^{n} p_j x_j\right\|, \tag{4.2}$$

for all $i \in \{1, \dots, n\}$.

On the other hand,

$$\begin{aligned} \sum_{j=1}^{n} p_j \|x_i - x_j\| &= \sum_{j=1}^{i-1} p_j \|x_i - x_j\| + \sum_{j=i+1}^{n} p_j \|x_i - x_j\| \\ &= \sum_{j=1}^{i-1} p_j \left\|\sum_{k=j}^{i-1} (x_{k+1} - x_k)\right\| + \sum_{j=i+1}^{n} p_j \left\|\sum_{l=i}^{j-1} (x_{l+1} - x_l)\right\| \\ &\leq \sum_{j=1}^{i-1} p_j \left(\sum_{k=j}^{i-1} \|\Delta x_k\|\right) + \sum_{j=i+1}^{n} p_j \left(\sum_{l=i}^{j-1} \|\Delta x_l\|\right) =: A. \end{aligned} \tag{4.3}$$

Now, as

$$\sum_{k=j}^{i-1} \|\Delta x_k\| \leq (i-j) \max_{k=\overline{j,i-1}} \|\Delta x_k\| \quad \text{(where } j \leq i-1\text{)}$$

and

$$\sum_{l=i}^{s-1} \|\Delta x_l\| \leq (s-i) \max_{l=\overline{i,n-1}} \|\Delta x_l\| \quad \text{(where } i \leq s-1\text{)},$$

then we deduce that

$$
\begin{aligned}
A &\leq \sum_{j=1}^{i-1} p_j (i-j) \cdot \max_{k=\overline{j,i-1}} \|\Delta x_k\| + \sum_{j=i+1}^{n} p_j (j-i) \cdot \max_{l=\overline{i,n-1}} \|\Delta x_l\| \\
&\leq \max_{k=\overline{1,n-1}} \|\Delta x_k\| \left[\sum_{j=1}^{i-1} p_j (i-j) + \sum_{j=i+1}^{n} p_j (j-i)\right] \\
&= \max_{k=\overline{1,n-1}} \|\Delta x_k\| \cdot \sum_{j=1}^{n} p_j |i-j|
\end{aligned}
$$

and the first inequality in (4.1) is proved.

Now, we observe that

$$
\begin{aligned}
\sum_{j=1}^{n} p_j |i-j| &\leq \max_{j=\overline{1,n}} |i-j| \sum_{j=1}^{n} p_j \\
&= \max_{j=\overline{1,n}} |i-j| \\
&= \max \{i-1, n-i\} \\
&= \frac{n-1}{2} + \left| i - \frac{n+1}{2} \right|,
\end{aligned}
$$

which proves the first part of the second inequality in (4.1).

By Hölder's discrete inequality, we also have

$$
\sum_{j=1}^{n} p_j |i-j| \leq \left(\sum_{j=1}^{n} p_j^q\right)^{\frac{1}{q}} \left(\sum_{j=1}^{n} |i-j|^p\right)^{\frac{1}{p}},
$$

where $p > q$ and $\frac{1}{p} + \frac{1}{q} = 1$, and the second part of the second inequality in (4.1) holds.

Finally, we also have

$$
\sum_{j=1}^{n} p_j |i-j| \leq \max_{j=\overline{1,n}} |p_j| \sum_{j=1}^{n} |i-j| .
$$

Now, let us observe that

$$
\begin{aligned}
\sum_{j=1}^{n} |i-j| &= \sum_{j=1}^{i} |i-j| + \sum_{j=i+1}^{n} |i-j| \\
&= \sum_{j=1}^{i} (i-j) + \sum_{j=i+1}^{n} (j-i) \\
&= i^2 - \frac{i(i+1)}{2} + \sum_{j=1}^{n} j - \sum_{j=1}^{i} j - i(n-i) \\
&= \frac{n^2-1}{4} + \left(i - \frac{n+1}{2}\right)^2
\end{aligned}
$$

and the last part of the second inequality in (4.1) is proved. □

Remark 4.2. In some practical applications the case $p = q = 2$ in the second part of the second inequality may be useful. As

$$\begin{aligned}\sum_{j=1}^{n}(j-i)^2 &= \sum_{j=1}^{n} j^2 - 2i\sum_{j=1}^{n} j + ni^2 \\ &= n\left[\frac{n^2-1}{12} + \left(i - \frac{n+1}{2}\right)^2\right],\end{aligned}$$

then we may state the inequality

$$\left\|x_i - \sum_{j=1}^{n} p_j x_j\right\| \leq \sqrt{n}\left[\frac{n^2-1}{12} + \left(i - \frac{n+1}{2}\right)^2\right]^{\frac{1}{2}} \left(\sum_{j=1}^{n} p_j^2\right)^{\frac{1}{2}} \max_{k=\overline{1,n-1}} \|\Delta x_k\| \tag{4.4}$$

for all $i \in \{1, \dots, n\}$.

The following particular case was proved in a different manner in Theorem 3.1.

Corollary 4.3. *If x_i $(i = 1, \dots, n)$ are vectors in the normed linear space $(X, \|\cdot\|)$, then we have*

$$\left\|x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right\| \leq \frac{1}{n}\left[\frac{n^2-1}{4} + \left(i - \frac{n+1}{2}\right)^2\right] \max_{k=\overline{1,n-1}} \|\Delta x_k\|. \tag{4.5}$$

The following result also holds.

Theorem 4.4. *Let $(X, \|\cdot\|)$ be a normed linear space, $x_i \in X$ $(i = 1, \dots, n)$ and $p_i \geq 0$ $(i = 1, \dots, n)$ with $\sum_{i=1}^{n} p_i = 1$. Then, for $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, we have the inequality:*

$$\begin{aligned}&\left\|x_i - \sum_{j=1}^{n} p_j x_j\right\| \\ &\leq \sum_{j=1}^{n} |i-j|^{\frac{1}{\beta}} p_j \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^{\alpha}\right)^{\frac{1}{\alpha}} \\ &\leq \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^{\alpha}\right)^{\frac{1}{\alpha}} \times \begin{cases} \left[\frac{1}{2}(n-1) + \left|i - \frac{n+1}{2}\right|\right]^{\frac{1}{\beta}}, \\ \left(\sum_{j=1}^{n} |i-j|^{\frac{\delta}{\beta}}\right)^{\frac{1}{\delta}} \left(\sum_{j=1}^{n} p_j^{\gamma}\right)^{\frac{1}{\gamma}} & \text{if } \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1, \\ \sum_{j=1}^{n} |i-j|^{\frac{1}{\beta}} \max_{j=\overline{1,n}} \{p_j\} \end{cases}\end{aligned} \tag{4.6}$$

for all $i \in \{1, \dots, n\}$.

Proof. Using Hölder's discrete inequality, we may write that

$$\sum_{k=j}^{i-1} \|\Delta x_k\| \leq (i-j)^{\frac{1}{\beta}} \left(\sum_{k=j}^{i-1} \|\Delta x_k\|^{\alpha}\right)^{\frac{1}{\alpha}}$$

and

$$\sum_{l=i}^{s-1} \|\Delta x_l\| \leq (s-i)^{\frac{1}{\beta}} \left(\sum_{l=i}^{s-1} \|\Delta x_l\|^{\alpha}\right)^{\frac{1}{\alpha}},$$

which implies for A, as defined in the proof of Theorem 4.1, that

$$\begin{aligned} A &\leq \sum_{j=1}^{i-1} (i-j)^{\frac{1}{\beta}} \left(\sum_{k=j}^{i-1} \|\Delta x_k\|^{\alpha} \right)^{\frac{1}{\alpha}} p_j + \sum_{s=i+1}^{n} (s-i)^{\frac{1}{\beta}} \left(\sum_{l=i}^{s-1} \|\Delta x_l\|^{\alpha} \right)^{\frac{1}{\alpha}} p_s \\ &\leq \left(\sum_{k=1}^{i-1} \|\Delta x_k\|^{\alpha} \right)^{\frac{1}{\alpha}} \sum_{j=1}^{i-1} (i-j)^{\frac{1}{\beta}} p_j + \left(\sum_{l=i}^{n-1} \|\Delta x_l\|^{\alpha} \right)^{\frac{1}{\alpha}} \sum_{s=i+1}^{n} (s-i)^{\frac{1}{\beta}} p_s \\ &\leq \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^{\alpha} \right)^{\frac{1}{\alpha}} \left[\sum_{j=1}^{i-1} (i-j)^{\frac{1}{\beta}} p_j + \sum_{s=i+1}^{n} (s-i)^{\frac{1}{\beta}} p_s \right] \\ &= \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^{\alpha} \right)^{\frac{1}{\alpha}} \sum_{j=1}^{n} |i-j|^{\frac{1}{\beta}} p_j, \end{aligned}$$

which proves the first inequality in (4.6).

Now it is obvious that

$$\begin{aligned} \sum_{j=1}^{n} |i-j|^{\frac{1}{\beta}} p_j &\leq \max_{j=\overline{1,n}} |i-j|^{\frac{1}{\beta}} \sum_{j=1}^{n} p_j \\ &= \max \left\{ (i-1)^{\frac{1}{\beta}}, (n-i)^{\frac{1}{\beta}} \right\} \\ &= \left[\frac{1}{2}(n-1) + \left| i - \frac{n+1}{2} \right| \right]^{\frac{1}{\beta}}, \end{aligned}$$

proving the first part of the second inequality in (4.6).

For $\gamma, \delta > 1$ with $\frac{1}{\gamma} + \frac{1}{\delta} = 1$, we have

$$\sum_{j=1}^{n} |i-j|^{\frac{1}{\beta}} p_j \leq \left(\sum_{j=1}^{n} p_j^{\gamma} \right)^{\frac{1}{\gamma}} \left(\sum_{j=1}^{n} |i-j|^{\frac{\delta}{\beta}} \right)^{\frac{1}{\delta}}$$

obtaining the second part of the second inequality in (4.6).

Finally, we observe that

$$\sum_{j=1}^{n} |i-j|^{\frac{1}{\beta}} p_j \leq \max_{j=\overline{1,n}} \{p_j\} \sum_{j=1}^{n} |i-j|^{\frac{1}{\beta}},$$

and the theorem is proved. □

Corollary 4.5. *If x_i $(i = 1, \ldots, n)$ are vectors in the normed space $(X, \|\cdot\|)$, then for all $i \in \{1, \ldots, n\}$ we have:*

$$\left\| x_i - \frac{1}{n} \sum_{j=1}^{n} x_j \right\| \leq \frac{1}{n} \sum_{j=1}^{n} |i-j|^{\frac{1}{\beta}} \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^{\alpha} \right)^{\frac{1}{\alpha}}, \quad \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1. \tag{4.7}$$

Finally, we may state the following result as well.

Theorem 4.6. *Let X, x_i and p_i $(i = 1, \dots, n)$ be as in Theorem 4.4. Then we have the inequality:*

$$\left\| x_i - \sum_{j=1}^{n} p_j x_j \right\| \leq \begin{cases} \max\left\{P_{i-1}, 1 - P_i\right\} \sum_{k=1}^{n-1} \|\Delta x_k\| \\ (1 - p_i) \max\left\{ \sum_{k=1}^{i-1} \|\Delta x_k\|, \sum_{k=i}^{n-1} \|\Delta x_k\| \right\} \end{cases} \tag{4.8}$$

$$\leq (1 - p_i) \sum_{j=1}^{n-1} \|\Delta x_k\|$$

for all $i \in \{1, \dots, n\}$, where

$$P_m := \sum_{i=1}^{m} p_i, \quad m = 1, \dots, n$$

and $P_0 := 0$.

Proof. It is obvious that

$$\sum_{k=j}^{i-1} \|\Delta x_k\| \leq \sum_{k=1}^{i-1} \|\Delta x_k\|$$

and

$$\sum_{l=i}^{s-1} \|\Delta x_l\| \leq \sum_{l=i}^{n-1} \|\Delta x_l\|,$$

Then, for A as defined in the proof of Theorem 4.1, we have that

$$\begin{aligned} A &\leq \sum_{k=1}^{i-1} \|\Delta x_k\| \sum_{j=1}^{i-1} p_j + \sum_{l=i}^{n-1} \|\Delta x_l\| \sum_{j=i+1}^{n} p_j \\ &=: B \\ &\leq \max\left\{P_{i-1}, 1 - P_i\right\} \left[\sum_{j=1}^{i-1} \|\Delta x_j\| + \sum_{j=i+1}^{n-1} \|\Delta x_j\| \right] \\ &= \max\left\{P_{i-1}, 1 - P_i\right\} \sum_{k=1}^{n-1} \|\Delta x_k\|. \end{aligned}$$

Also, we observe that

$$\begin{aligned} B &\leq \max\left\{ \sum_{j=1}^{i-1} \|\Delta x_j\|, \sum_{j=i+1}^{n-1} \|\Delta x_j\| \right\} (P_{i-1} + 1 - P_i) \\ &= (1 - p_i) \max\left\{ \sum_{k=1}^{i-1} \|\Delta x_k\|, \sum_{k=i}^{n-1} \|\Delta x_k\| \right\} \end{aligned}$$

and the theorem is thus proved. □

Corollary 4.7. *Let X and x_i $(i = 1, \ldots, n)$ be as in Corollary 4.5. Then*

$$\left\| x_i - \frac{1}{n}\sum_{j=1}^{n} x_j \right\| \leq \begin{cases} \dfrac{1}{n}\left[\frac{1}{2}(n-1) + \left|i - \frac{n+1}{2}\right|\right]\sum\limits_{k=1}^{n-1}\|\Delta x_k\|, \\[2ex] \dfrac{n-1}{n}\max\left\{\sum\limits_{k=1}^{i-1}\|\Delta x_k\|, \sum\limits_{k=i}^{n-1}\|\Delta x_k\|\right\} \end{cases} \tag{4.9}$$

for all $i \in \{1, \ldots, n\}$.

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