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SUBORDINATION RESULTS FOR THE FAMILY OF UNIFORMLY CONVEX $p-$VALENT FUNCTIONS

H.A. AL-KHARSANI AND S.S. AL-HAJIRY

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
GIRLS COLLEGE, DAMMAM
SAUDI ARABIA.

ssmh1@hotmail.com

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ABSTRACT. The object of the present paper is to introduce a class of $p-$valent uniformly functions UCV_p. We deduce a criteria for functions to lie in the class UCV_p and derive several interesting properties such as distortion inequalities and coefficients estimates. We confirm our results using the Mathematica program by drawing diagrams of extremal functions of this class.

Key words and phrases: $p-$valent, Uniformly convex functions, Subordination.

2000 *Mathematics Subject Classification.* 30C45.

1. INTRODUCTION

Denote by $A(p,n)$ the class of normalized functions

$$f(z) = z^p + \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1} \tag{1.1}$$

regular in the unit disk $D = \{z : |z| < 1\}$ and $p \in \mathbb{N}$, consider also its subclasses $C(p), S^*(p)$ consisting of $p-$valent convex and starlike functions respectively, where $C(1) \equiv C$, $S^*(1) \equiv S^*$, the classes of univalent convex and starlike functions .

It is well known that for any $f \in \mathbb{C}$, not only $f(D)$ but the images of all circles centered at 0 and lying in D are convex arcs. B. Pinchuk posed a question whether this property is still valid for circles centered at other points of D. A.W. Goodman [1] gave a negative answer to this question and introduced the class UCV of univalent uniformly convex functions, $f \in C$

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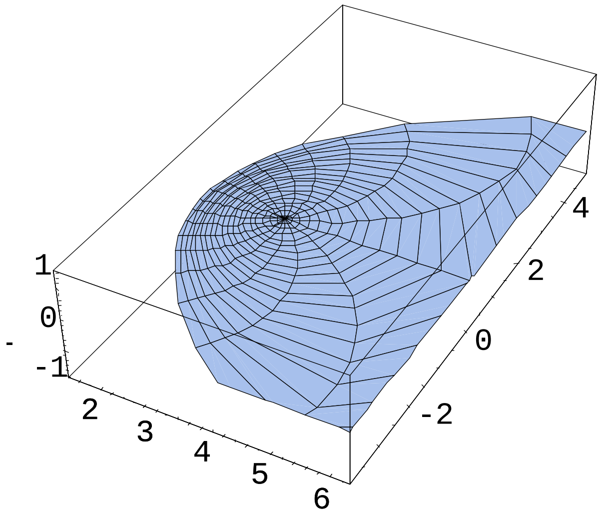


such that any circular arc γ lying in D, having the center $\zeta \in D$ is carried by f into a convex arc. A.W.Goodman [1] stated the criterion

$$\operatorname{Re}\left[1+(z-\zeta)\frac{f''(z)}{f'(z)}\right] > 0, \quad \forall z,\zeta \in D \Longleftrightarrow f \in UCV. \tag{1.2}$$

Later F. Ronning (and independently W. Ma and D. Minda) [7] obtained a more suitable form of the criterion , namely

$$\operatorname{Re}\left[1+\frac{zf''(z)}{f'(z)}\right] > \left|\frac{zf''(z)}{f'(z)}\right|, \quad \forall z \in D \Longleftrightarrow f \in UCV. \tag{1.3}$$

This criterion was used to find some sharp coefficients estimates and distortion theorems for functions in the class UCV.

2. THE CLASS PAR_p

We now introduce a subfamily PAR_p of P. Let

$$\Omega = \left\{ w = \mu + iv : \frac{v^2}{p} < 2\mu - p \right\} \tag{2.1}$$

$$= \{w : \operatorname{Re} w > |w-p|\}. \tag{2.2}$$

Note that Ω is the interior of a parabola in the right half-plane which is symmetric about the real axis and has vertex at $(p/2, 0)$. The following diagram shows Ω when $p = 3$:

Let

$$PAR_p = \{h \in p : h(D) \subseteq \Omega\}. \tag{2.3}$$

Example 2.1. It is known that $z = -\tan^2\left(\frac{\pi}{2\sqrt{2p}}\sqrt{w}\right)$ maps $\left\{w = \mu + i\nu : \frac{\nu^2}{p} < p - 2\mu\right\}$ conformally onto D. Hence, $z = -\tan^2\left(\frac{\pi}{2\sqrt{2p}}\sqrt{p-w}\right)$ maps Ω conformally onto D. Let $w = Q(z)$ be the inverse function. Then $Q(z)$ is a Riemann mapping function from D to Ω

which satisfies $Q(0) = p$; more explicitly,

$$Q(z) = p + \frac{2p}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 = \sum_{n=0}^{\infty} B_n z^n \tag{2.4}$$

$$= p + \frac{8p}{\pi^2}z + \frac{16p}{3\pi^2}z^2 + \frac{184p}{45\pi^2}z^3 + \cdots \tag{2.5}$$

Obviously, $Q(z)$ belongs to the class PAR_p. Geometrically, PAR_p consists of those holomorphic functions $h(z)$ $(h(0) = p)$ defined on D which are subordinate to $Q(z)$, written $h(z) \prec Q(z)$.

The analytic characterization of the class PAR_p is shown in the following relation:

$$h(z) \in PAR_p \Leftrightarrow \mathrm{Re}\{h(z)\} \geq |h(z) - p| \tag{2.6}$$

such that $h(z)$ is a $p-$valent analytic function on D.

Now, we can derive the following definition.

Definition 2.1. Let $f(z) \in A(p, n)$. Then $f(z) \in UCV_p$ if $f(z) \in C(p)$ and $1+z\frac{f''(z)}{f'(z)} \in PAR_p$.

3. CHARACTERIZATION OF UCV_p

We present the nessesary and sufficient condition to belong to the class UCV_p in the following theorem:

Theorem 3.1. *Let* $f(z) \in A(p, n)$. *Then*

$$f(z) \in UCV_p \Leftrightarrow 1 + \mathrm{Re}\left\{z\frac{f''(z)}{f'(z)}\right\} \geq \left|z\frac{f''(z)}{f'(z)} - (p-1)\right|, \quad z \in D. \tag{3.1}$$

Proof. Let $f(z) \in UCV_p$ and $h(z) = 1 + z\frac{f''(z)}{f'(z)}$. Then $h(z) \in PAR_p$, that is, $\mathrm{Re}\{h(z)\} \geq |h(z) - p|$. Then

$$\mathrm{Re}\left\{1 + z\frac{f''(z)}{f'(z)}\right\} \geq \left|z\frac{f''(z)}{f'(z)} - (p-1)\right|.$$

□

Example 3.1. We now specify a holomorphic function $K(z)$ in D by

$$1 + z\frac{K''(z)}{K'(z)} = Q(z), \tag{3.2}$$

where $Q(z)$ is the conformal mapping onto Ω given in Example 2.1. Then it is clear from Theorem 3.1 that $K(z)$ is in UCV_p.

Let

$$K(z) = z^p + \sum_{k=2}^{\infty} A_k z^{k+p-1}. \tag{3.3}$$

From the relationship between the functions $Q(z)$ and $K(z)$, we obtain

$$(p + n - 1)(n - 1)A_n = \sum_{k=1}^{n-1}(k + p - 1)A_k B_{n-k}. \tag{3.4}$$

Since all the coefficients B_n are positive, it follows that all of the coefficients A_n are also positive. In particular,

$$A_2 = \frac{8p^2}{\pi^2(p+1)}, \tag{3.5}$$

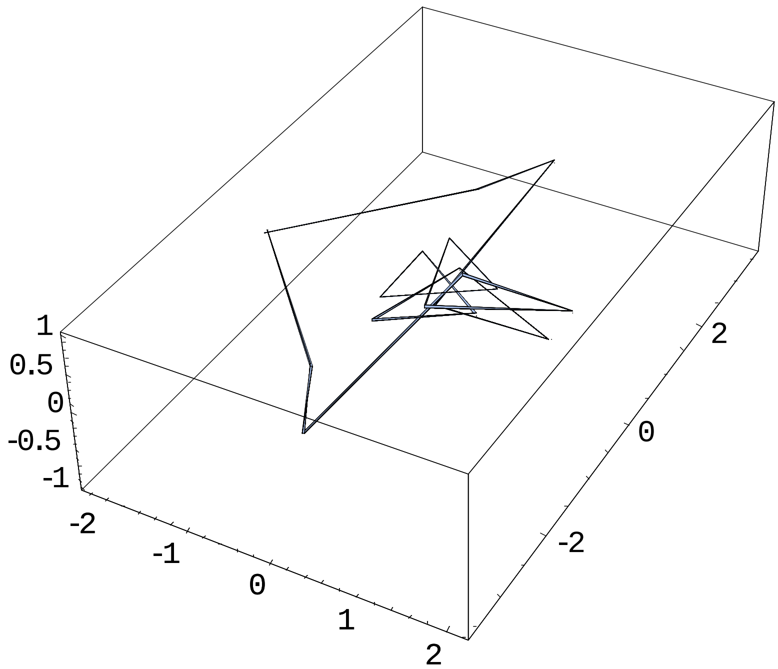
and

$$A_3 = \frac{p^2}{2(p+2)}\left(\frac{16}{3\pi^2} + \frac{64p}{\pi^4}\right). \tag{3.6}$$

Note that

$$\log \frac{k'(z)}{z^{p-1}} = \int_0^z \frac{Q(\varsigma) - p}{\varsigma} d\varsigma. \tag{3.7}$$

By computing some coefficients of $K(z)$ when $p = 3$, we can obtain the following diagram



4. SUBORDINATION THEOREM AND CONSEQUENCES

In this section, we first derive some subordination results from Theorem 4.1; as corollaries we obtain sharp distortion, growth, covering and rotation theorems from the family UCV_p.

Theorem 4.1. *Assume that* $f(z) \in UCV_p$. *Then* $1 + z\frac{f''(z)}{f'(z)} \prec 1 + z\frac{K''(z)}{K'(z)}$ *and* $\frac{f'(z)}{z^{p-1}} \prec \frac{K'(z)}{z^{p-1}}$.

Proof. Let $f(z) \in UCV_p$. Then $h(z) = 1 + z\frac{f''(z)}{f'(z)} \prec 1 + z\frac{K''(z)}{K'(z)}$ is the same as $h(z) \prec Q(z)$. Note that $Q(z) - p$ is a convex univalent function in D. By using a result of Goluzin, we may conclude that

$$\log \frac{f'(z)}{z^{p-1}} = \int_0^z \frac{h(\varsigma) - 1}{\varsigma} d\varsigma \prec \int_0^z \frac{Q(\varsigma) - p}{\varsigma} d\varsigma = \log \frac{K'(z)}{z^{p-1}}. \tag{4.1}$$

Equivalently, $\frac{f'(z)}{z^{p-1}} \prec \frac{K'(z)}{z^{p-1}}$. □

Corollary 4.2 (Distortion Theorem)**.** *Assume* $f(z) \in UCV_p$ *and* $|z| = r < 1$. *Then* $K'(-r) \leq |f'(z)| \leq K'(r)$.

Equality holds for some $z \neq 0$ if and only if $f(z)$ is a rotation of $K(z)$.

Proof. Since $Q(z) - p$ is convex univalent in D, it follows that $\log K'(z)$ is also convex univalent in D. In fact, the power series for $\log K'(z)$ has positive coefficients, so the image of D under this convex function is symmetric about the real axis. As $\log \frac{f'(z)}{z^{p-1}} \prec \log \frac{K'(z)}{z^{p-1}}$, the subordination principle shows that

$$K'(-r) = e^{\{\log K'(-r)\}} = e^{\left\{\min\limits_{|z|=r} \mathrm{Re}\{\log K'(z)\}\right\}} \tag{4.2}$$
$$\leq e^{\{\mathrm{Re} \log K'(z)\}} = |f'(z)| \leq e^{\left\{\max\limits_{|z|=r} \mathrm{Re}\{\log K'(z)\}\right\}}$$
$$= e^{\{\log K'(r)\}} = K'(r).$$

Note that for $|z_0| = r$, either

$$\mathrm{Re}\{\log f'(z_0)\} = \min_{|z|=r} \mathrm{Re}\{\log K'(z)\}$$

or

$$\mathrm{Re}\{\log f'(z_0)\} = \max_{|z|=r} \mathrm{Re}\{\log K'(z)\}$$

for some $z_0 \neq 0$ if and only if $\log f'(z) = \log K'(e^{i\theta} z)$ for some $\theta \in R$. □

Theorem 4.3. *Let* $f(z) \in UCV_p$. *Then*

$$|f'(z)| \leq \left|z^{p-1}\right| e^{\frac{14p}{\pi^2}\varsigma(3)} = \left|z^{p-1}\right| L^p \tag{4.3}$$

for $|z| < 1$. $(L \approx 5.502, \varsigma(t)$ *is the Riemann Zeta function.*$)$

Proof. Let $\phi(z) = \frac{zg'(z)}{g(z)}$, where $g(z) = zf'(z)$. Then $\phi(z) \prec Q(z)$ which means that $\phi(z) \prec p + \frac{2p}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)$. Moreover ,

$$\log \frac{g(z)}{z^p} = \int_0^z \left(\frac{\phi(s) - p}{s}\right) ds$$

and therefore, if $z = re^{i\theta}$ and $|z| = 1$,

$$\log \left|\frac{g(z)}{z^p}\right| = \int_0^r \Re e(\phi(te^{i\theta}) - p)\frac{dt}{t}$$
$$\leq \frac{2p}{\pi^2} \int_0^r \frac{1}{t} \log\left(\frac{1+\sqrt{t}}{1-\sqrt{t}}\right) dt$$
$$\leq \frac{2p}{\pi^2} \int_0^1 \frac{1}{t} \log\left(\frac{1+\sqrt{t}}{1-\sqrt{t}}\right) dt$$
$$= \frac{2p}{\pi^2}(7\varsigma(3)),$$

where

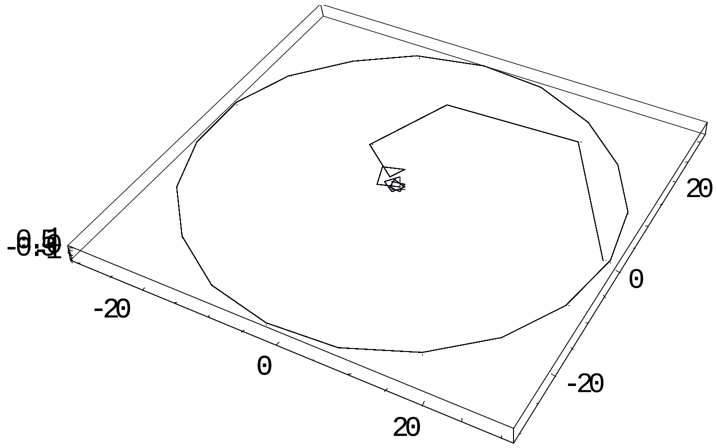
$$\int_0^1 \frac{1}{t} \log\left(\frac{1+\sqrt{t}}{1-\sqrt{t}}\right) dt = 7\varsigma(3) \qquad [8].$$

Then we find that

$$\left|\frac{zf'(z)}{z^p}\right| \leq e^{\frac{2p}{\pi^2}(7\varsigma(3))}.$$

□

The following diagram shows the boundary of $K(z)$'s dervative when $p = 2$ in a circle has the radius $(5.5)^2$:



Corollary 4.4 (Growth Theorem). *Let $f(z) \in UCV_p$ and $|z| = r < 1$. Then $-K(-r) \leq |f(z)| \leq K(r)$.*

Equality holds for some $z \neq 0$ if and only if $f(z)$ is a rotation of $K(z)$.

Corollary 4.5 (Covering Theorem). *Suppose $f(z) \in UCV_p$. Then either $f(z)$ is a rotation of $K(z)$ or $\{w : |w| \leq -K(-1)\} \subseteq f(D)$.*

Corollary 4.6 (Rotation Theorem). *Let $f(z) \in UCV_p$ and $|z_0| = r < 1$. Then*

$$|Arg\{f'(z_0)\}| \leq \max_{|z|=r} Arg\{K'(z). \tag{4.4}$$

Equality holds for some $z \neq 0$ if and only if $f(z)$ is a rotation of $K(z)$.

Theorem 4.7. *Let $f(z) = z^p + \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1}$ and $f(z) \in UCV_p$, and let $A_{n+p-1} = \max\limits_{f(z)\in UCV_p} |a_{n+p-1}|$. Then*

$$A_{p+1} = \frac{8p^2}{\pi^2(p+1)}. \tag{4.5}$$

The result is sharp. Further, we get

$$A_{n+p-1} \leq \frac{8p^2}{(n+p-1)(n-1)\pi^2} \prod_{k=3}^{n} \left(1 + \frac{8p}{(k-2)\pi^2}\right). \tag{4.6}$$

Proof. Let $f(z) = z^p + \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1}$ and $f(z) \in UCV_p$, and define

$$\phi(z) = 1 + \frac{zf''(z)}{f'(z)} = p + \sum_{k=2}^{\infty} c_k z^{k+p-1}.$$

Then $\phi(z) \prec Q(z)$. $Q(z)$ is univalent in D and $Q(D)$ is a convex region, so Rogosinski's theorem applies.

$$Q(z) = p + \frac{8p}{\pi^2} z + \frac{16p}{3\pi^2} z^2 + \frac{184p}{45\pi^2} z^3 + \cdots,$$

so we have $|c_n| \leq |B_1| = \frac{8p}{\pi^2} := B$. Now, from the relationship between functions $f(z)$ and $Q(z)$, we obtain

$$(n+p-1)(n-1)a_{n+p-1} = \sum_{k=1}^{n-1} (k+p-1) a_{k+p-1} c_{n-k}.$$

From this we get $|a_{p+1}| = \frac{pB}{(p+1)} = \frac{8p^2}{\pi^2(p+1)}$. If we choose $f(z)$ to be that function for which $Q(z) = 1 + \frac{zf''(z)}{f'(z)}$, then $f(z) \in UCV_p$ with $a_{p+1} = \frac{8p^2}{\pi^2(p+1)}$, which shows that this result is sharp. Now, when we put $|c_1| = B$, then

$$\begin{aligned} a_{p+2} &= \frac{pa_pc_2 + (p+1)a_{p+1}c_1}{2(p+2)} \\ |a_{p+2}| &\leq \frac{pB(1+Bp)}{2(p+2)}. \end{aligned}$$

When $n = 3$

$$\begin{aligned} a_{p+3} &= \frac{pa_pc_3 + (p+1)a_{p+1}c_2 + (p+2)a_{p+2}c_1}{3(p+3)} \\ |a_{p+3}| &\leq \frac{1}{2}\frac{pB(1+Bp)(2+Bp)}{3(p+3)} \\ &= \frac{1}{3(p+3)}pB(1+Bp)\left(1+\frac{Bp}{2}\right). \end{aligned}$$

We now proceed by induction. Assume we have

$$\begin{aligned} |a_{p+n-1}| &\leq \frac{1}{(n-1)(p+n-1)}pB(1+Bp)\left(1+\frac{Bp}{2}\right)\cdots\left(1+\frac{Bp}{n-2}\right) \\ &= \frac{pB}{(n-1)(p+n-1)}\prod_{k=3}^{n}\left(1+\frac{Bp}{k-2}\right). \end{aligned}$$

Corollary 4.8. *Let $f(z) = z^p + \sum_{k=2}^{\infty} a_{k+p-1}z^{k+p-1}$ and $f(z) \in UCV_p$. Then $|a_{p+n-1}| = O\left(\frac{1}{n^2}\right)$.*

□

5. GENERAL PROPERTIES OF FUNCTIONS IN UCV_p

Theorem 5.1. *Let $f(z) = z^p + \sum_{k=2}^{\infty} a_{k+p-1}z^{k+p-1}$ and $f(z) \in UCV_p$. Then $f(z)$ is a $p-$valently convex function of order β in $|z| < r_1 = r_1(p,\beta)$, where $r_1(p,\beta)$ is the largest value of r for which*

$$r^{k-1} \leq \frac{(p-\beta)(k-1)}{(k+p-\beta-1)B\prod_{j=3}^{k}\left(1+\frac{pB}{j-2}\right)}, \quad (k \in \mathbb{N} - \{1\},\ 0 \leq \beta < p). \tag{5.1}$$

Proof. It is sufficient to show that for $f(z) \in UCV_p$,

$$\left|1 + \frac{zf''(z)}{f'(z)} - p\right| \leq p - \beta, \quad |z| < r_1(p,\beta), \quad 0 \leq \beta < p,$$

where $r_1(p,\beta)$ is the largest value of r for which the inequality (5.1) holds true. Observe that

$$\left|1 + \frac{zf''(z)}{f'(z)} - p\right| = \left|\frac{\sum_{k=2}^{\infty}(k+p-1)(k-1)a_{k+p-1}z^{k-1}}{p + \sum_{k=2}^{\infty}(k+p-1)a_{k+p-1}z^{k-1}}\right|.$$

Then we have $\left|1+\frac{zf''(z)}{f'(z)}-p\right|\leq p-\beta$ if and only if

$$\frac{\sum_{k=2}^{\infty}(k+p-1)(k-1)\left|a_{k+p-1}\right|r^{k-1}}{p-\sum_{k=2}^{\infty}(k+p-1)\left|a_{k+p-1}\right|r^{k-1}}\leq p-\beta$$
$$\Rightarrow\sum_{k=2}^{\infty}(k+p-1)(k+p-1-\beta)\left|a_{k+p-1}\right|r^{k-1}\leq p^2-p\beta.$$

Then from Theorem 4.7 since $f(z)\in UCV_p$, we have

$$\left|a_{k+p-1}\right|\leq\frac{p\beta}{(k+p-1)(k-1)}\prod_{j=3}^{k}\left(1+\frac{Bp}{j-2}\right)$$

and we may set

$$\left|a_{k+p-1}\right|=\frac{p\beta}{(k+p-1)(k-1)}\prod_{j=3}^{k}\left(1+\frac{Bp}{j-2}\right)c_{k+p-1},c_{k+p-1}\geq 0,$$
$$\left\{k\in\mathbb{N}-\{1\},\sum_{k=1}^{\infty}c_{k+p-1}\leq 1\right\}.$$

Now, for each fixed r, we choose a positive integer $k_0=k_0(r)$ for which

$$\frac{(k+p-1-\beta)}{(k-1)}r^{k-1}$$

is maximal. Then

$$\sum_{k=2}^{\infty}(k+p-1)(k+p-\beta-1)\left|a_{k+p-1}\right|r^{k-1}\leq\frac{(k_0+p-\beta-1)}{(k_0-1)}r^{k_0-1}\prod_{j=3}^{k}\left(1+\frac{Bp}{j-2}\right).$$

Consequently, the function $f(z)$ is a $p-$valently convex function of order β in $|z|<r_1=r_1(p,\beta)$ provided that

$$\frac{(k_0+p-\beta-1)}{(k_0-1)}r^{k_0-1}\prod_{j=3}^{k}\left(1+\frac{Bp}{j-2}\right)\leq p(p-\beta).$$

We find the value $r_0=r_0(p,\beta)$ and the corresponding integer $k_0(r_0)$ so that

$$\frac{(k_0+p-\beta-1)}{(k_0-1)}r^{k_0-1}\prod_{j=3}^{k}\left(1+\frac{Bp}{j-2}\right)=p(p-\beta),\quad(0\leq\beta<p).$$

Then this value r_0 is the radius of $p-$valent convexity of order β for functions $f(z)\in UCV_p$. □

Theorem 5.2. *$h(z)=z^p+b_{n+p-1}z^{n+p-1}$ is in UCV_p if and only if*

$$r\leq\frac{p^2}{(p+n-1)(p+2n-2)},$$

where $|b_{n+p-1}|=r$ and $b_{n+p-1}z^{n-1}=re^{i\theta}$.

Proof. Let $w(z) = 1 + \frac{zh''(z)}{h'(z)}$. Then $h(z) \in UCV_p$ if and only if $w(z) \in PAR_p$ which means that $\operatorname{Re}\{w(z)\} \geq |w(z) - p|$. On the other side we have

$$\operatorname{Re}\left\{1 + \frac{zh''(z)}{h'(z)}\right\} \geq \left|1 + \frac{zh''(z)}{h'(z)} - p\right|,$$

then

$$\begin{aligned}\operatorname{Re}\left\{1 + \frac{zh''(z)}{h'(z)}\right\} &= \operatorname{Re}\left\{(p-1) + \frac{p + (n+p-1)nre^{i\theta}}{p + (n+p-1)re^{i\theta}}\right\} \\ &= \frac{p^3 + p(n+p-1)(n+2p-1)r\cos\theta + (n+p-1)^3r^2}{|p + (n+p-1)re^{i\theta}|^2}.\end{aligned}$$

The right-hand side is seen to have a minimum for $\theta = \pi$ and this minimal value is

$$\frac{p^3 + p(n+p-1)(n+2p-1)r + (n+p-1)^3r^2}{|p + (n+p-1)re^{i\theta}|^2}.$$

Now, by computation we see that

$$\left|1 + \frac{zh''(z)}{h'(z)} - p\right| = \frac{(n+p-1)(n-1)r}{|p + (n+p-1)re^{i\theta}|}.$$

Then

$$(n+p-1)(n-1)r \leq \frac{p^3 + p(n+p-1)(n+2p-1)r + (n+p-1)^3r^2}{p - (n+p-1)r},$$

which leads to

$$(n+p-1)(n-1)r \leq p^2 - (n+p-1)^2r.$$

Hence,

$$r \leq \frac{p^2}{(n+p-1)(2n+p-2)}.$$

$\square$

Theorem 5.3. *Let $f(z) \in UCV$, then $(f(z))^p \in UCV_p$.*

Proof. Let $w(z) = (f(z))^p$, then

$$1 + z\frac{w''(z)}{w'(z)} = 1 + z\frac{f''(z)}{f'(z)} + (p-1)z\frac{f'(z)}{f(z)}.$$

Then we find

$$\begin{aligned}&\operatorname{Re}\left\{1 + z\frac{w''(z)}{w'(z)}\right\} - \left|z\frac{w''(z)}{w'(z)} - (p-1)\right| \\ &\qquad = \operatorname{Re}\left\{1 + z\frac{f''(z)}{f'(z)} + (p-1)z\frac{f'(z)}{f(z)}\right\} \\ &\qquad\qquad - \left|z\frac{f''(z)}{f'(z)} + (p-1)z\frac{f'(z)}{f(z)} - (p-1)\right|.\end{aligned}$$

Since $f(z) \in UCV$, therefore we have

$$\begin{aligned}&\operatorname{Re}\left\{1 + z\frac{f''(z)}{f'(z)} + (p-1)z\frac{f'(z)}{f(z)}\right\} - \left|z\frac{f''(z)}{f'(z)} + (p-1)z\frac{f'(z)}{f(z)} - (p-1)\right| \\ &\qquad \geq (p-1)\operatorname{Re}\left\{z\frac{f'(z)}{f(z)}\right\} - \left|z\frac{f'(z)}{f(z)} - 1\right|\end{aligned}$$

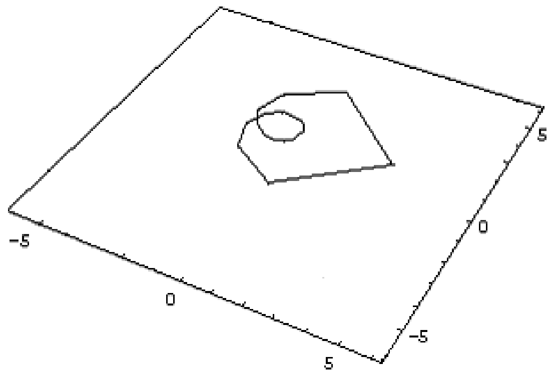
$f(z) \in UCV$, then $f(z) \in SP$ [7] which means that

$$\operatorname{Re}\left\{z\frac{f'(z)}{f(z)}\right\} - \left|z\frac{f'(z)}{f(z)} - 1\right| \geq 0.$$

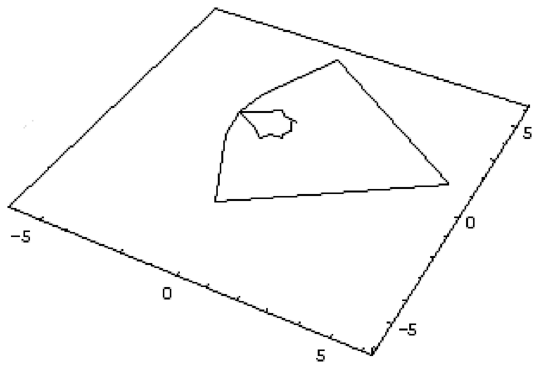
Then

$$\operatorname{Re}\left\{1 + z\frac{w''(z)}{w'(z)}\right\} - \left|z\frac{w''(z)}{w'(z)} - (p-1)\right| \geq 0.$$

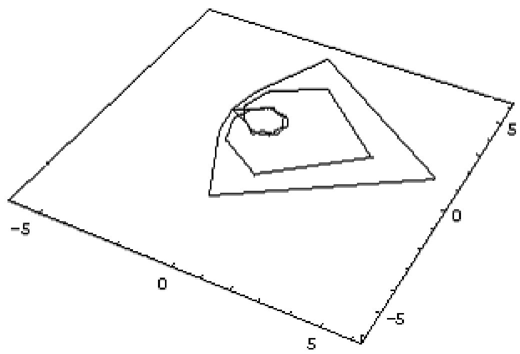
The following diagram shows the extermal function $k(z)$ of the class UCV when $(k(z))^p, p = 2$:



The following diagram shows the extermal function $K(z)$ of the class UCV_p when $p = 2$:



And the following diagram shows that $(k(z))^p \prec K(z)$:



□

5.1. **Remarks.** Taking $p = 1$ in Theorem 3.1, we obtain the corresponding Theorem 1 of [7].

Taking $p = 1$ in Theorem 4.1, we obtain the corresponding Theorem 3 of [3].

Taking $p = 1$ in inequality (4.3), we obtain Theorem 6 of [7], and in inequalities (4.5), (4.6), we obtain Theorem 5 of [7].

Taking $p = 1$ in Theorem 5.2, we obtain Theorem 2 of [4].

REFERENCES

[1] A.W. GOODMAN, On uniformly convex functions, *Ann. Polon. Math.*, **56**(1) (1991), 87–92.

[2] S. KANAS AND A. WISNIOWSKA, Conic regions and k-uniform convexity, *J. Comput. Appl. Math.*, **105**(1-2) (1999), 327–336.

[3] W.C. MA AND D. MINDA, Uniformly convex functions, *Ann. Polon. Math.*, **57**(2) (1992), 165–175.

[4] S. OWA, On uniformly convex functions, *Math. Japonica*, **48**(3) (1998), 377–384.

[5] M.S. ROBERTSON, On the theory of univalent functions, *Ann. Math.*, **2**(37) (1936), 347–408.

[6] F. RONNING, A survey on uniformly convex and uniformly starlike functions, *Ann. Univ. Mariae Curie-Sklodowska Sect. A*, **47** (1993), 123–134.

[7] F. RONNING, Uniformly convex functions and a corresponding class of starlike functions, *Proc. Amer. Math. Soc.*, **118**(1) (1993), 189–196.

[8] F. RONNING, On uniform starlikeness and related properties of univalent functions, *Complex Variables Theory Appl.*, **24**(3-4) (1994), 233–239.