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SUBORDINATION RESULTS FOR A CLASS OF ANALYTIC FUNCTIONS DEFINED BY A LINEAR OPERATOR

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ABSTRACT. In this paper, we derive several interesting subordination results for certain class of analytic functions defined by the linear operator $\mathcal{L}(a,c)f(z)$ which introduced and studied by Carlson and Shaffer [2].

Key words and phrases: Analytic functions, Hadamard product, Subordinating factor sequence.

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1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\Delta = \{z : |z| < 1\}$. For two functions $f(z)$ and $g(z)$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} c_n z^n \tag{1.2}$$

their Hadamard product (or convolution) is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n c_n z^n. \tag{1.3}$$

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Define the function $\phi(a,c;z)$ by

$$\phi(a,c;z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \qquad (c \notin \mathbb{Z}_0^- := \{0,-1,-2,\ldots\},\ z \in \Delta), \tag{1.4}$$

where $(\lambda)_n$ is the Pochhammer symbol given, in terms of Gamma functions,

$$\begin{aligned}(\lambda)_n &:= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \\ &= \begin{cases} 1, & n=0, \\ \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+n-1), & n \in \mathbb{N} : \{1,2,\ldots\}. \end{cases}\end{aligned} \tag{1.5}$$

Corresponding to the function $\phi(a,c;z)$, Carlson and Shaffer [2] introduced a linear operator $\mathcal{L}(a,c) : \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{L}(a,c)f(z) := \phi(a,c;z) * f(z), \tag{1.6}$$

or, equivalently, by

$$\mathcal{L}(a,c)f(z) := z + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} a_{n+1} z^{n+1} \qquad (z \in \Delta).$$

Note that $\mathcal{L}(1,1)f(z) = f(z)$, $\mathcal{L}(2,1)f(z) = zf'(z)$ and $\mathcal{L}(3,1)f(z) = zf'(z) + \frac{1}{2}z^2 f''(z)$.

For $-1 \leq \alpha < 1$, $\beta \geq 0$, we let $\mathcal{L}(a,c;\alpha,\beta)$ consist of functions f in $\mathcal{A}$ satisfying the condition

$$\operatorname{Re}\left\{ \frac{a\mathcal{L}(a+1,c)f(z)}{\mathcal{L}(a,c)f(z)} - (a-1) \right\} > \beta \left| \frac{a\mathcal{L}(a+1,c)f(z)}{\mathcal{L}(a,c)f(z)} - a \right| + \alpha,\ (z \in \Delta) \tag{1.7}$$

The family $\mathcal{L}(a,c;\alpha,\beta)$ is of special interest for it contains many well-known as well as many new classes of analytic univalent functions. For $\mathcal{L}(1,1;\alpha,0)$, we obtain the family of starlike functions of order α $(0 \leq \alpha < 1)$ and $\mathcal{L}(2,1;\alpha,0)$ is the family of convex functions of order α $(0 \leq \alpha < 1)$. For $\mathcal{L}(1,1;0,\beta)$ and $\mathcal{L}(2,1;0,\beta)$, we obtain the class of uniformly β- starlike functions and uniformly β- convex functions, respectively, introduced by Kanas and Winsiowska ([3],[4]) (see also the work of Kanas and Srivastava [5], Goodman ([7],[8]), Rønning ([10],[11]), Ma and Minda [9] and Gangadharan et al. [6]).

Before we state and prove our main result we need the following definitions and lemmas.

Definition 1.1 (Subordination Principle)**.** Let $g(z)$ be analytic and univalent in Δ. If $f(z)$ is analytic in Δ, $f(0) = g(0)$, and $f(\Delta) \subset g(\Delta)$, then we see that the function $f(z)$ is subordinate to $g(z)$ in Δ, and we write $f(z) \prec g(z)$.

Definition 1.2 (Subordinating Factor Sequence)**.** A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is called a subordinating factor sequence if, whenever $f(z)$ is analytic , univalent and convex in Δ, we have the subordination given by

$$\sum_{n=2}^{\infty} b_n a_n z^n \prec f(z) \qquad (z \in \Delta,\ a_1 = 1). \tag{1.8}$$

Lemma 1.1 ([14])**.** *The sequence* $\{b_n\}_{n=1}^{\infty}$ *is a subordinating factor sequence if and only if*

$$\operatorname{Re}\left\{ 1 + 2\sum_{n=1}^{\infty} b_n z^n \right\} > 0 \qquad (z \in \Delta). \tag{1.9}$$

Lemma 1.2. *If*

$$\sum_{n=2}^{\infty} \sigma_n(a,c;\alpha,\beta)\,|a_n| \le 1-\alpha \tag{1.10}$$

where, for convenience,

$$\sigma_n(a,c;\alpha,\beta) := \frac{(1+\beta)(a)_n + [1-\alpha-a(1+\beta)](a)_{n-1}}{(c)_{n-1}} \tag{1.11}$$
$$(-1 \le \alpha < 1;\ \beta \ge 0,\ n \ge 2),$$

then $f(z) \in \mathcal{L}(a,c;\alpha,\beta)$.

Proof. It suffices to show that

$$\beta\left|\frac{a\mathcal{L}(a+1,c)f(z)}{\mathcal{L}(a,c)f(z)} - a\right| - \operatorname{Re}\left\{\frac{a\mathcal{L}(a+1,c)f(z)}{\mathcal{L}(a,c)f(z)} - a\right\} \le 1-\alpha.$$

We have

$$\begin{aligned}\beta\left|\frac{a\mathcal{L}(a+1,c)f(z)}{\mathcal{L}(a,c)f(z)} - a\right| &- \operatorname{Re}\left\{\frac{a\mathcal{L}(a+1,c)f(z)}{\mathcal{L}(a,c)f(z)} - a\right\}\\ &\le (1+\beta)\left|\frac{a\mathcal{L}(a+1,c)f(z)}{\mathcal{L}(a,c)f(z)} - a\right|\\ &\le \frac{(1+\beta)\sum_{n=2}^{\infty}\left(\frac{a(a+1)_{n-1}-a(a)_{n-1}}{(c)_{n-1}}\right)|a_n|\,|z|^{n-1}}{1-\sum_{n=2}^{\infty}\frac{(a)_{n-1}}{(c)_{n-1}}|a_n|\,|z|^{n-1}}\\ &\le \frac{(1+\beta)\sum_{n=2}^{\infty}\left(\frac{(a)_n-a(a)_{n-1}}{(c)_{n-1}}\right)|a_n|}{1-\sum_{n=2}^{\infty}\frac{(a)_{n-1}}{(c)_{n-1}}|a_n|}.\end{aligned}$$

The last expression is bounded above by $1-\alpha$ if

$$\sum_{n=2}^{\infty}\frac{(1+\beta)(a)_n + [1-\alpha-a(1+\beta)](a)_{n-1}}{(c)_{n-1}}|a_n| \le 1-\alpha$$

and the proof is complete. □

Let $\mathcal{L}^\star(a,c;\alpha,\beta)$ denote the class of functions $f(z) \in \mathcal{A}$ whose coefficients satisfy the condition (1.10). We note that $\mathcal{L}^\star(a,c;\alpha,\beta) \subseteq \mathcal{L}(a,c;\alpha,\beta)$.

2. MAIN THEOREM

Employing the techniques used earlier by Srivastava and Attiya [13], Attiya [1] and Singh [12], we state and prove the following theorem.

Theorem 2.1. *Let the function* $f(z)$ *defined by (1.1) be in the class* $\mathcal{L}^\star(a,c;\alpha,\beta)$ *where* $-1 \le \alpha < 1$; $\beta \ge 0$; $a > 0$; $c > 0$. *Also let* $\mathcal{K}$ *denote the familiar class of functions* $f(z) \in \mathcal{A}$ *which are also univalent and convex in* Δ. *Then*

$$\frac{\sigma_2(a,c;\alpha,\beta)}{2[1-\alpha+\sigma_2(a,c;\alpha,\beta)]}(f*g)(z) \prec g(z) \qquad (z\in\Delta;\ g\in\mathcal{K}), \tag{2.1}$$

and

$$\operatorname{Re}(f(z)) > -\frac{1-\alpha+\sigma_2(a,c;\alpha,\beta)}{\sigma_2(a,c;\alpha,\beta)}, \qquad (z\in\Delta). \tag{2.2}$$

The constant $\frac{\sigma_2(a,c;\alpha,\beta)}{2[1-\alpha+\sigma_2(a,c;\alpha,\beta)]}$ *is the best estimate.*

Proof. Let $f(z) \in \mathcal{L}^\star(a,c;\alpha,\beta)$ and let $g(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{K}$. Then

$$\frac{\sigma_2(a,c;\alpha,\beta)}{2[1-\alpha+\sigma_2(a,c;\alpha,\beta)]}(f*g)(z) = \frac{\sigma_2(a,c;\alpha,\beta)}{2[1-\alpha+\sigma_2(a,c;\alpha,\beta)]}\left(z + \sum_{n=2}^{\infty} a_n c_n z^n\right). \tag{2.3}$$

Thus, by Definition 1.2, the assertion of our theorem will hold if the sequence

$$\left\{\frac{\sigma_2(a,c;\alpha,\beta)}{2[1-\alpha+\sigma_2(a,c;\alpha,\beta)]}a_n\right\}_{n=1}^{\infty} \tag{2.4}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1.1, this will be the case if and only if

$$\operatorname{Re}\left\{1 + 2\sum_{n=1}^{\infty}\frac{\sigma_2(a,c;\alpha,\beta)}{2[1-\alpha+\sigma_2(a,c;\alpha,\beta)]}a_n z^n\right\} > 0 \qquad (z \in \Delta). \tag{2.5}$$

Now

$$\begin{aligned}
\operatorname{Re}&\left\{1 + \frac{\sigma_2(a,c;\alpha,\beta)}{1-\alpha+\sigma_2(a,c;\alpha,\beta)}\sum_{n=1}^{\infty} a_n z^n\right\} \\
&= \operatorname{Re}\left\{1 + \frac{\sigma_2(a,c;\alpha,\beta)}{1-\alpha+\sigma_2(a,c;\alpha,\beta)}z \right.\\
&\qquad \left. + \frac{1}{1-\alpha+\sigma_2(a,c;\alpha,\beta)}\sum_{n=1}^{\infty}\sigma_2(a,c;\alpha,\beta)a_n z^n\right\} \\
&\geq 1 - \left\{\frac{\sigma_2(a,c;\alpha,\beta)}{1-\alpha+\sigma_2(a,c;\alpha,\beta)}r \right.\\
&\qquad \left. - \frac{1}{1-\alpha+\sigma_2(a,c;\alpha,\beta)}\sum_{n=1}^{\infty}\sigma_n(a,c;\alpha,\beta)a_n r^n\right\}.
\end{aligned}$$

Since $\sigma_n(a,c;\alpha,\beta)$ is an increasing function of n $(n \geq 2)$

$$\begin{aligned}
1-&\left\{\frac{\sigma_2(a,c;\alpha,\beta)}{1-\alpha+\sigma_2(a,c;\alpha,\beta)}r \right.\\
&\qquad \left. - \frac{1}{1-\alpha+\sigma_2(a,c;\alpha,\beta)}\sum_{n=1}^{\infty}\sigma_n(a,c;\alpha,\beta)a_n r^n\right\} \\
&> 1 - \frac{\sigma_2(a,c;\alpha,\beta)}{1-\alpha+\sigma_2(a,c;\alpha,\beta)}r - \frac{1-\alpha}{1-\alpha+\sigma_2(a,c;\alpha,\beta)}r \quad (|z| = r) \\
&> 0.
\end{aligned}$$

Thus (2.5) holds true in Δ. This proves the inequality (2.1). The inequality (2.2) follows by taking the convex function $g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n$ in (2.1). To prove the sharpness of the constant $\frac{\sigma_2(a,c;\alpha,\beta)}{2[1-\alpha+\sigma_2(a,c;\alpha,\beta)]}$, we consider the function $f_0(z) \in \mathcal{L}^\star(a,c;\alpha,\beta)$ given by

$$f_0(z) = z - \frac{1-\alpha}{\sigma_2(a,c;\alpha,\beta)}z^2 \qquad (-1 \leq \alpha < 1;\ \beta \geq 0). \tag{2.6}$$

Thus from (2.1), we have

$$\frac{\sigma_2(a,c;\alpha,\beta)}{2[1-\alpha+\sigma_2(a,c;\alpha,\beta)]}f_0(z) \prec \frac{z}{1-z}. \tag{2.7}$$

It can easily verified that

$$\min\left\{\operatorname{Re}\left(\frac{\sigma_2(a,c;\alpha,\beta)}{2[1-\alpha+\sigma_2(a,c;\alpha,\beta)]}f_0(z)\right)\right\}=-\frac{1}{2}\qquad(z\in\Delta),\tag{2.8}$$

This shows that the constant $\frac{\sigma_2(a,c;\alpha,\beta)}{2[1-\alpha+\sigma_2(a,c;\alpha,\beta)]}$ is best possible. □

Corollary 2.2. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{L}^\star(1,1;\alpha,\beta)$ and satisfy the condition*

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)]\,|a_n|\leq 1-\alpha\tag{2.9}$$

then

$$\frac{\beta+2-\alpha}{2(\beta+3-2\alpha)}(f*g)(z)\prec g(z)\tag{2.10}$$
$$(-1\leq\alpha<1;\ \beta\geq 0;\ z\in\Delta;\ g\in\mathcal{K})$$

and

$$\operatorname{Re}(f(z))>-\frac{\beta+3-2\alpha}{\beta+2-\alpha},\qquad(z\in\Delta).\tag{2.11}$$

The constant $\frac{\beta+2-\alpha}{2(\beta+3-2\alpha)}$ is the best estimate.

Corollary 2.3. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{L}^\star(1,1;\alpha,0)$ and satisfy the condition*

$$\sum_{n=2}^{\infty}(n-\alpha)\,|a_n|\leq 1-\alpha,\tag{2.12}$$

then

$$\frac{2-\alpha}{6-4\alpha}(f*g)(z)\prec g(z)\qquad(z\in\Delta;\ g\in\mathcal{K})\tag{2.13}$$

and

$$\operatorname{Re}(f(z))>-\frac{3-2\alpha}{2-\alpha},\qquad(z\in\Delta).\tag{2.14}$$

The constant $\frac{2-\alpha}{6-4\alpha}$ is the best estimate.

Putting $\alpha=0$ in Corollary 2.3, we obtain

Corollary 2.4 ([12])**.** *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{L}^\star(1,1;0,0)$ and satisfy the condition*

$$\sum_{n=2}^{\infty}n\,|a_n|\leq 1\tag{2.15}$$

then

$$\frac{1}{3}(f*g)(z)\prec g(z)\qquad(z\in\Delta;\ g\in\mathcal{K})\tag{2.16}$$

and

$$\operatorname{Re}(f(z))>-\frac{3}{2},\qquad(z\in\Delta).\tag{2.17}$$

The constant $1/3$ is the best estimate.

Corollary 2.5. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{L}^{\star}(2,1;\alpha,\beta)$ and satisfy the condition*

$$\sum_{n=2}^{\infty} n[n(1+\beta)-(\alpha+\beta)]\,|a_n| \le 1-\alpha, \tag{2.18}$$

then

$$\frac{\beta+2-\alpha}{2\beta+5-3\alpha}(f*g)(z) \prec g(z) \tag{2.19}$$
$$(-1 \le \alpha < 1;\ \beta \ge 0;\ z \in \Delta;\ g \in \mathcal{K})$$

and

$$\mathrm{Re}(f(z)) > -\frac{2\beta+5-3\alpha}{2(\beta+2-\alpha)}, \qquad (z \in \Delta). \tag{2.20}$$

The constant $\frac{\beta+2-\alpha}{2\beta+5-3\alpha}$ is the best estimate.

Corollary 2.6. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{L}^{\star}(2,1;\alpha,0)$ and satisfy the condition*

$$\sum_{n=2}^{\infty} n(n-\alpha)\,|a_n| \le 1-\alpha, \tag{2.21}$$

then

$$\frac{2-\alpha}{5-3\alpha}(f*g)(z) \prec g(z) \qquad (z \in \Delta;\ g \in \mathcal{K}) \tag{2.22}$$

and

$$\mathrm{Re}(f(z)) > -\frac{5-3\alpha}{2(2-\alpha)}, \qquad (z \in \Delta). \tag{2.23}$$

The constant $\frac{2-\alpha}{5-3\alpha}$ is the best estimate.

Putting $\alpha = 0$ in Corollary 2.6, we obtain

Corollary 2.7. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{L}^{\star}(2,1;0,0)$ and satisfy the condition*

$$\sum_{n=2}^{\infty} n^2\,|a_n| \le 1 \tag{2.24}$$

then

$$\frac{2}{5}(f*g)(z) \prec g(z) \qquad (z \in \Delta;\ g \in \mathcal{K}) \tag{2.25}$$

and

$$\mathrm{Re}(f(z)) > \frac{-5}{4}, \qquad (z \in \Delta). \tag{2.26}$$

The constant $2/5$ is the best estimate.

REFERENCES

[1] A.A. ATTIYA, On some application of a subordination theorems, *J. Math. Anal. Appl.*, **311** (2005), 489–494.

[2] B.C. CARLSON AND D.B. SHAFFER, Starlike and prestarlike hypergeometric functions, *SIAM J. Math. Anal.*, **15**(4) (1984), 737–745.

[3] S. KANAS AND A. WISNIOWSKA, Conic regions and k- uniform convexity, *J. Comput. Appl. Math.*, **105** (1999), 327–336.

[4] S. KANAS AND A. WISNIOWSKA, Conic regions and k- starlike functions, *Rev. Roumaine Math. Pures Appl.*, **45**(4) (2000), 647–657.

[5] S. KANAS AND H.M. SRIVASTAVA, Linear operators associated with k-uniformly convex functions, *Integral Transform. Spec. Funct.*, **9** (2000), 121–132.

[6] A. GANGADHARAN, T.N. SHANMUGAN AND H.M. SRIVASTAVA, Generalized Hypergeometric functions associated with $k-$uniformly convex functions, *Comput. Math. App.*, **44** (2002), 1515–1526.

[7] A.W. GOODMAN, On uniformly convex functions, *Ann. Polon. Math.*, **56** (1991), 87–92.

[8] A.W. GOODMAN, On uniformly starlike functions, *J. Math. Anal. Appl.*, **155** (1991), 364–370.

[9] W.C. MA AND D. MINDA, Uniformly convex functions, *Ann. Polon. Math.*, **57**(2) (1992), 165–175.

[10] F. RØNNING, Uniformly convex functions and a corresponding class of starlike functions, *Proc. Amer. Math. Soc.*, **118**(1) (1993), 189–196.

[11] F. RØNNING, On starlike functions associated with parabolic regions, *Ann Univ. Mariae Curie-Sklodowska Sect. A*, **45** (1991), 117–122.

[12] S. SINGH, A subordination theorems for spirallike functions, *IJMMS*, **24**(7) (2000), 433–435.

[13] H.M. SRIVASTAVA AND A.A. ATTIYA, Some subordination results associated with certain subclasses of analytic functions, *J. Inequal. Pure Appl. Math.*, **5**(4) (2004), Art. 82. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=434`].

[14] H.S. WILF, Subordinating factor sequence for convex maps of the unit circle, *Proc. Amer. Math. Soc.*, **12** (1961), 689–693, MR0125214 (23 #A2519).