# Vortex Merging in a 1½-Layer Fluid on an *f* Plane*

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ABSTRACT

Mass, angular momentum, and energy budgets are examined in an analytical model of vortex merging relevant to midlatitude mesoscale eddies. The vortices are baroclinic and cyclogeostrophic. The fluid surrounding them is assumed to remain quiescent. It is shown that due to this surrounding fluid, angular momentum is conserved when expressed in both the inertial and rotating frames of reference.

Lens-shaped solid-body vortices can conserve mass, angular momentum, and energy when they merge. If an upper-layer of thickness $H_1$ is included in the model, the merged vortex must have either less energy or mass than the sum of the original two vortices.

A more complex model of the vortex azimuthal structure is then considered, which includes a constant vorticity shell surrounding the solid-body core. If the shell is large compared to the core, the mass, angular momentum, and energy can all be conserved in the merged vortex. However, if the shell is small, the merged vortex must have less energy or mass than in the solid-body case.

## 1. Introduction

Like-sign vortices merge in rotating tanks (Nof and Simon 1987; Griffiths and Hopfinger 1987) and in the ocean (Cresswell 1982; Tokos et al. 1994). Merging has been numerically modeled in a wide range of settings (Melander et al. 1988; Verron and Valcke 1994; Carton and Bertrand 1994; Valcke and Verron 1997). Presumably, the characteristics of the merged vortex are set by those of its parents and the relevant conservation laws. But what properties are conserved in vortex merging? Gill and Griffiths (1981) demonstrated that, if potential vorticity and mass are conserved by two merging anticyclones, the final state contains more energy than the initial state. This ''energy paradox'' is at odds with the known spontaneous nature of vortex merging (Cushman-Roisin 1989). Several attempts to resolve this paradox have questioned the assumptions of potential vorticity or mass conservation (Nof 1988; Cushman-Roisin 1989; Pavia and Cushman-Roisin 1990).

Once vortices come into contact, merging takes place in two stages: fusion and axisymmetrization. During fusion, the vortices rapidly exchange fluid and homogenize into a central, elliptical vortex. In the subsequent, relatively slow axisymmetrization stage, the elongated vortex becomes S-shaped and sheds part of its mass into spiral-shaped filaments while its core becomes circular. This inviscid process is particularly dramatic in potential-vorticity-conserving numerical models (Melander et al. 1988; Pavia and Cushman-Roisin 1988) and has been observed in tank experiments (Griffiths and Hopfinger 1987). Cushman-Roisin (1989) proposed that filamentation plays a fundamental role in the merging, acting

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to relax mass conservation in the framework of the Gill and Griffiths (1981) merging model.[1] He also introduced an additional constraint, conservation of the absolute angular momentum (the angular momentum formulated in the inertial reference frame). He argued that this angular momentum must be conserved in the final vortex if the merging is complete, a condition not met by the solutions of Gill and Griffiths (1981) (Pavia and Cushman-Roisin 1990).

Cushman-Roisin's (1989) model assumes that the potential vorticity of each fluid parcel is conserved. He concludes that a significant fraction of the initial mass must be shed into filaments. However, when Nof and Simon (1987) allowed anticyclonic lenses to merge in a rotating tank, the initial vortex mass was conserved in the merged vortex (i.e., filamentation was negligible). Furthermore, the central depth of the vortices rapidly decreased as they fused. This depth must increase to conserve potential vorticity, suggesting that potential vorticity is not conserved during fusion (Nof and Simon 1987). Nof (1988) argued that highly viscous, turbulent mixing occurs along the interface of the vortices, which can cause an $O(1)$ alteration in the fluid parcel's potential vorticity (Nof 1986). This hypothesis suggests that the energy paradox can be resolved by relaxing the a priori assumption of potential vorticity conservation. However, Pavia and Cushman-Roisin (1990) examined solutions to the merging model that conserve mass and energy in the final vortex; they argued that angular momentum was not in general conserved, rendering these solutions physically unacceptable.

In this note, we examine an analytical model of vortex merging similar to those of Gill and Griffiths (1981) and Pavia and Cushman-Roisin (1990). We examine Nof's (1988) hypothesis for resolving the energy paradox by relaxing the constraint of potential vorticity conservation. We formulate angular momentum conservation in the inertial reference frame; unlike Cushman-Roisin (1989) and Pavia and Cushman-Roisin (1990), we include the fluid that surrounds the merging vortices. While this fluid is assumed to remain quiescent, it bears absolute angular momentum, which is altered when the fluid is rearranged during merging. We demonstrate that the absolute angular momentum of the total system, vortices and surrounding fluid, is equal to the background angular momentum (the absolute angular momentum in the absence of vortices), plus the sum of the vortex relative angular momentum (the angular momentum seen in the rotating reference frame). As a consequence, conservation of the total absolute angular momentum is equivalent to conservation of the relative angular momentum of the vortices. We conclude that solutions exist for the merging model that do not present an energy or angular momentum paradox. Using a generalized model of the vortex structure, we present two possible merging scenarios based on the choices of conserved quantities. In the discussion, we highlight the difference between our formulation of angular momentum conservation and that of Cushman-Roisin (1989) and discuss physical mechanisms that can account for reduced mass or energy in the final state.

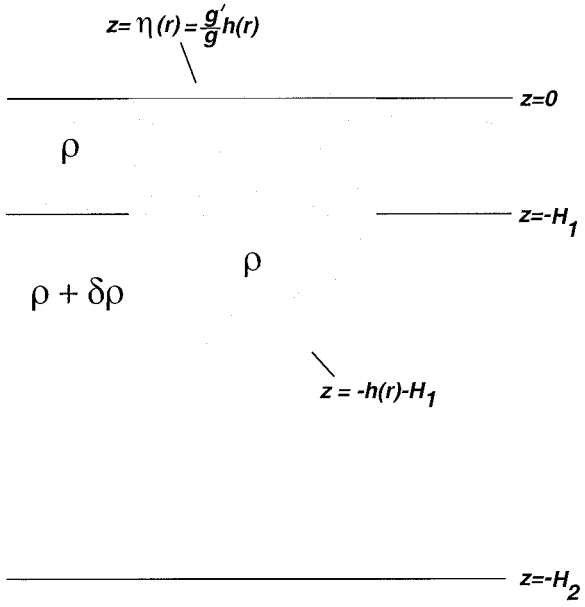


FIG. 1. Side view of a vortex and surrounding quiescent fluid in the 2-layer model. For a vortex in solid-body rotation, the surface and interface displacements are parabolic, as sketched here.

## 2. Development of the model

The model has two layers: the upper layer (density $\rho$) extends from $z = 0$ to $z = -H_1$ and the lower layer (density $\rho + \delta\rho$) extends from $z = -H_1$ to $z = -H_2$ (Fig. 1). The system is rotating at angular speed $\Omega$ (Fig. 2) and is subject to a potential that includes gravity and centrifugal force (the familiar $f$-plane approximation). Before merging, there are two identical, azimuthally symmetric vortices in the upper layer, touching at a single point. We assume that the fluid is reorganized into quiescent fluid surrounding a single vortex. We assume that the final vortex has the same velocity structure as its parents; for example, vortices in solid-body rotation merge to produce a vortex also in solid-body rotation (though not necessarily rotating at the same rate). Because we assume the vortices are in contact at time $t = 0$, we do not address the sensitivity of merging on the initial separation distance of the vortices.

### *a. Mass and angular momentum of an isolated vortex*

Consider a vortex in the upper layer with radius $r_o$, extending vertically from the surface $z = \eta(r)$ to $z =$

[1] Cushman-Roisin explicitly included an idealized model for the filaments, and concluded that when zero potential vorticity lenses merge, 23% of their mass goes into the filaments while 96% of their energy is retained in the merged vortex.

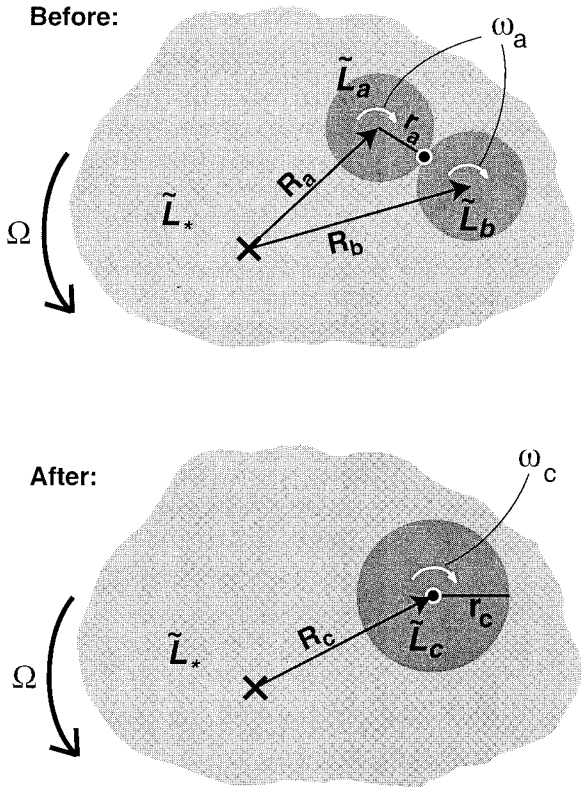


FIG. 2. Top view of the merging model. The center of rotation of the system is at the **x**.

$-H_1 - h(r)$ (Fig. 1). The surface deflection $\eta$ is related to the azimuthal speed $v$ by cyclogeostrophy:

$$\frac{v^2}{r} + 2\Omega v = g\partial_r \eta, \tag{1}$$

where $r$ is the radial distance from the vortex center and $g$ is gravity. Since there are no horizontal pressure gradients in the lower layer,

$$h = \frac{g}{g'}\eta, \tag{2}$$

where the reduced gravity $g'$ is $g'/g = \delta\rho/\rho$. The total mass of the vortex is

$$M = m_H + m_h + m_\eta, \tag{3}$$

where

$$m_H = \pi\rho r_o^2 H_1, \qquad m_h = 2\pi\rho \int_0^{r_o} dr\, rh(r),$$

$$m_\eta = 2\pi\rho \int_0^{r_o} dr\, r\eta(r). \tag{4}$$

With (2), it follows that

$$m_\eta = \frac{\delta\rho}{\rho} m_h. \tag{5}$$

In the rotating frame, the relative angular momentum[2] of the vortex is

$$L = 2\pi\rho \int_0^{r_o} dr\, r^2[H_1 + h(r) + \eta(r)]v(r). \tag{6}$$

In the inertial frame, the vortex is centered at $\mathbf{R} = \mathbf{R}_o$ (the position of $\mathbf{R} = 0$, the axis of rotation, does not affect the solutions). Its absolute angular momentum is

$$\tilde{L} = L + MR_o^2\Omega + (I_H + I_h + I_\eta)\Omega \tag{7}$$

(the tilde indicating absolute quantities expressed in the inertial frame), where the moments of inertia are

$$I_H = 2\pi\rho \int_0^{r_o} dr\, r^3 H_1 = \frac{1}{2} m_H r_o^2,$$

$$I_h = 2\pi\rho \int_0^{r_o} dr\, r^3 h(r),$$

$$I_\eta = 2\pi\rho \int_0^{r_o} dr\, r^3 \eta(r). \tag{8}$$

Using (2), it follows that

$$I_\eta = \frac{\delta\rho}{\rho} I_h. \tag{9}$$

*b. Angular momentum of vortices immersed in quiescent fluid*

In the absence of vortices, the absolute angular momentum of the quiescent background state would be

$$\tilde{L}_{\text{back}} = [H_1\rho + (H_2 - H_1)(\rho + \delta\rho)] \int_A dA\, R^2\Omega, \tag{10}$$

where the integral is taken over an arbitrary finite domain $A$.

In the presence of $N$ vortices (denoted by the subscript $\alpha = 1, \cdots, N$), the absolute angular momentum of the quiescent surrounding fluid is

$$\tilde{L}_* = \tilde{L}_{\text{back}} - \sum_{\alpha=1}^{N} \left\{ \left[ m_{H,\alpha} + \frac{\rho + \delta\rho}{\rho} m_{h,\alpha} \right] R_\alpha^2 + \left[ I_{H,\alpha} + \frac{\rho + \delta\rho}{\rho} I_{h,\alpha} \right] \right\} \Omega. \tag{11}$$

This is less than $\tilde{L}_{\text{back}}$ because the vortices occupy some volume below $z = 0$. The absolute angular momentum of the total system (with $N$ vortices) is obtained by adding (7) and (11):

[2] Specifically, the vertical component of the angular momentum vector.

$$\tilde{L}_{\text{tot}} = \tilde{L}_* + \sum_\alpha \tilde{L}_\alpha = \tilde{L}_{\text{back}} + \sum_{\alpha=1}^{N} \left\{ L_\alpha + \left[ \left( m_{\eta,\alpha} - \frac{\delta\rho}{\rho} m_{h,\alpha} \right) R_\alpha^2 + \left( I_{\eta,\alpha} - \frac{\delta\rho}{\rho} I_{h,\alpha} \right) \right] \Omega \right\}. \quad (12)$$

Vortices extend to height $\eta$ above $z = 0$; revolution and rotation of this mass at speed $\Omega$ adds angular momentum to the background state. Similarly, vortices extend a distance $h_\alpha$ into the lower layer; because this fluid is lower in density, it subtracts angular momentum from the background state. Using (5) and (9), (12) simplifies to

$$\tilde{L}_{\text{tot}} = \tilde{L}_* + \sum_\alpha \tilde{L}_\alpha = \tilde{L}_{\text{back}} + \sum_\alpha L_\alpha. \quad (13)$$

Angular momentum added by surface bulges is equal to angular momentum subtracted by layer interface bulges, and they cancel. The absolute angular momentum of the entire system differs from the angular momentum of the background state only by the sum of the relative angular momentum of the vortices.

### c. Energy of vortices

The fluid in the rotating reference frame is subject to a potential that includes the centrifugal force. In the inertial frame, the potential consists of only gravity, complicating the formulation of energy conservation in that frame. Thus, following Gill and Griffiths (1981) and Cushman-Roisin (1989), we examine the energy of the vortices in the algebraically simpler rotating reference frame, that is, using the familiar definitions of the kinetic and potential energy, the sum of which is invariant for a closed system (Gill 1982, p. 220). The kinetic energy of a vortex is

$$\text{KE} = 2\pi\rho \int_0^{r_o} dr\, r[H_1 + h(r) + \eta(r)] \frac{1}{2} v^2, \quad (14)$$

and the potential energy is

$$\text{PE} = 2\pi\rho \int_0^{r_o} dr\, r[h(r) + \eta(r)] \frac{1}{2} g\eta. \quad (15)$$

The net energy $E = \text{KE} + \text{PE}$ may be related to the relative angular momentum as follows. The relative angular momentum $L$ is given by (6). From cyclogeostrophy (1),

$$v = -\frac{1}{2\Omega}\left( \frac{v^2}{r} - g\partial_r \eta \right). \quad (16)$$

Substitution yields

$$L = -\frac{2\pi\rho}{\Omega} \int_0^{r_o} dr\, r(H_1 + h + \eta)\left( \frac{1}{2} v^2 - \frac{1}{2} r g \partial_r \eta \right). \quad (17)$$

The term containing $\partial_r \eta$ may be integrated by parts, yielding

$$-\Omega L = 2\pi\rho \int_0^{r_o} dr\, r\left[ (H_1 + h + \eta)\frac{1}{2} v^2 + (2H_1 + h + \eta)\frac{1}{2} g\eta \right]. \quad (18)$$

Using (4), (14), and (15), this becomes

$$E = -\Omega L - g H_1 m_\eta. \quad (19)$$

The total energy of $N$ vortices is then

$$E_{\text{tot}} = -\sum_{\alpha=1}^{N} (\Omega L_\alpha + g H_1 m_{\eta,\alpha}). \quad (20)$$

If $H_1 = 0$, that is, the vortices are lens-shaped, their energy is proportional to their relative angular momentum.

## 3. Vortex merging

When two identical vortices (denoted by subscripts $a$ and $b$) merge to produce a single vortex ($c$) without entraining surrounding fluid, mass conservation requires

$$M_c = M_a + M_b, \quad (21)$$

where the masses are given by (3).

By definition, the background angular momentum $\tilde{L}_{\text{back}}$ remains unchanged. Thus, from (13), conservation of angular momentum requires

$$L_c = L_a + L_b. \quad (22)$$

Conservation of the absolute angular momentum reduces to conservation of the relative angular momentum of the vortices.[3]

From (20), energy conservation $E_c = E_a + E_b$ requires

$$\Omega L_c + g H_1 m_{\eta,c} = \Omega L_a + g H_1 m_{\eta,a} + \Omega L_b + g H_1 m_{\eta,b}. \quad (23)$$

Mass, angular momentum, and energy can be simultaneously conserved when either

$$m_{\eta,c} = m_{\eta,a} + m_{\eta,b} \quad (24)$$

or $H_1 = 0$.

We now examine the merging properties of particular vortex structures.

[3] The equivalency of these invariants can also be derived directly from the momentum and continuity equations for a rotating, multilayer fluid (B. Cushman-Roisin, 1999, personal communication).

*a. Solid-body vortices*

A solid-body vortex has the velocity structure $v(r) = \omega r$, where $\omega$ is constant. By imposing $\eta = 0$ at $r = r_o$, cyclogeostrophy (1) may be integrated to yield

$$\eta(r) = -\frac{1}{2g}\omega(2\Omega + \omega)(r_o^2 - r^2), \qquad r \le r_o. \quad (25)$$

The relative angular momentum is

$$L = (I_H + I_h + I_\eta)\omega. \quad (26)$$

Consider two identical solid-body vortices ($a$ and $b$) merging to produce vortex $c$, assumed to be also in solid-body rotation. Given $\omega_a$, $\omega_b$, $r_a$, and $r_b$, two independent conservation laws determine $\omega_c$ and $r_c$. This leads to the dilemma faced by Pavia and Cushman-Roisin (1990): which of the three constraints (mass, angular momentum, and energy conservation) should apply? Following Pavia and Cushman-Roisin, we consider two separate merging scenarios; in each we conserve two of these properties and examine what happens to the third. Physical interpretations of the results will be discussed in the final section. We restrict the development to the particular case of identical merging vortices ($\omega_a = \omega_b$, $r_a = r_b$), and reject values of ($\omega_a$, $r_a$) that yield a nonpositive central thickness $H_1 + h(r) + \eta(r)$.

1) CONSERVING MASS AND ANGULAR MOMENTUM

Figure 3 shows the values of $\omega_c$ and $r_c$, which conserve mass and angular momentum. Several curves are shown, each corresponding to a fixed ratio of $r_a$ to the internal Rossby radius $R_d$:

$$R_d = \frac{1}{2\Omega}\sqrt{\frac{g'H_1}{1 + g'/g}}. \quad (27)$$

In the limit $H_1 \to 0$ or $r_a/R_d \to \infty$, the vortices become anticyclonic lenses; in particular, zero potential vorticity lenses ($\omega_1 = -\Omega$) merge such that

$$\omega_c = \frac{2}{3}\omega_a, \qquad r_c = \sqrt{\frac{3}{2}}r_a. \quad (28)$$

In the limit $H_1 \to \infty$ or $r_a/R_d \to 0$, the vortices become 2D cylinders that merge such that

$$\omega_c = \frac{1}{2}\omega_a, \qquad r_c = \sqrt{2}r_a. \quad (29)$$

The bottom panel of Fig. 3 shows the ratio of final energy $E_c$ to initial energy $2E_a$. For lens-shaped anticyclonic vortices, energy is conserved. However, for finite $r_a/R_d$, energy must be lost in the merging; the loss reaches 50% at the limit of cylindrical vortices.

2) CONSERVING ANGULAR MOMENTUM AND ENERGY

Figure 4 shows the values of $\omega_c$ and $r_c$, which conserve angular momentum and energy. Compared to the mass-conserving scenario (Fig. 3), the change in period ($\omega_a/\omega_c$) is smaller for a given $r_a/R_d$ and decreases with decreasing $r_a/R_d$; the final radius is also smaller. For finite $r_a/R_d$, the final vortex has less mass than the initial two vortices. At the limit of cylindrical vortices ($r_a/R_d \to 0$), the mass loss is $M_c/2M_a = 1/\sqrt{2}$. For lens-shaped anticyclonic vortices ($r_a/R_d = \infty$), mass is conserved.

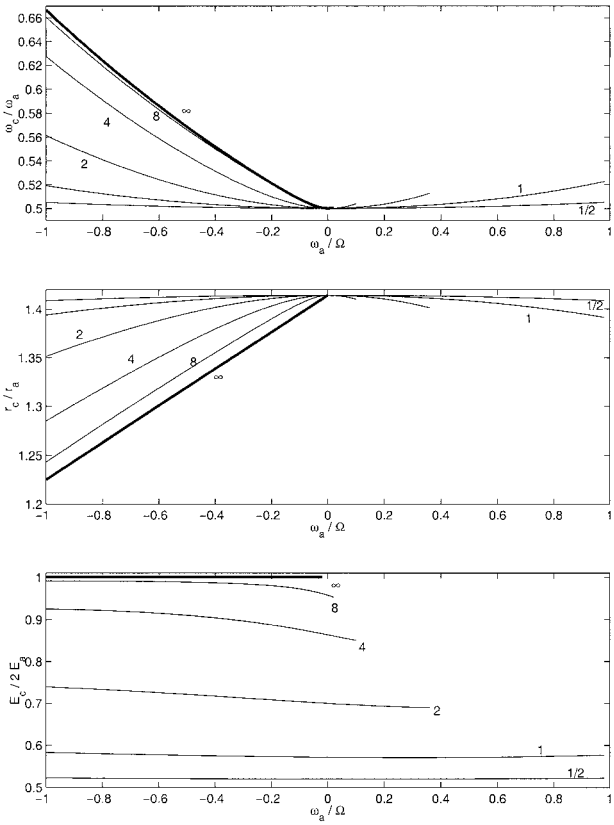


FIG. 3. Top: ratio of final rotational frequency $\omega_c$ to initial frequency $\omega_a$ as a function of $\omega_a/\Omega$, for solid-body vortices, which conserve mass and angular momentum when they merge. The label for each curve indicates the value of $r_a/R_d$. The heavy line gives the solution for a lens-shaped vortex ($R_d = H_1 = 0$). Some curves end before reaching $\omega_a = \Omega$; at greater values of $\omega_a$, the vortex has negative thickness $H_1 + h + \eta$ at its center and is thus not geometrically valid. Middle: ratio of final radius $r_c$ to initial radius $r_a$ as a function of $\omega_a/\Omega$. Bottom: ratio of final energy $E_c$ to initial energy $2E_a$. Except in the lens-shaped case, the merged vortex has less energy.

*b. Vortices with finite-shear edges*

In the solid-body model, vortices have an infinite-shear edge separating vortex fluid from surrounding upper-layer fluid of identical density. However, momentum diffusion would rapidly smear the edge vorticity into a shell surrounding the solid-body core. Within this shell, $v(r)$ decreases with increasing $r$. If the shell's width is allowed to adjust during merging, its inclusion in the model adds a degree of freedom to the solutions. Consider vortices with the velocity structure:

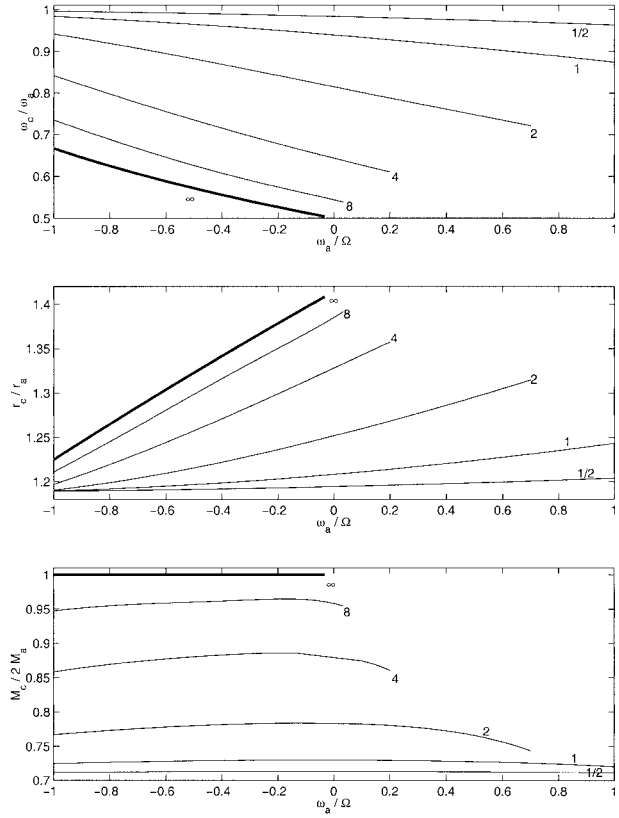


FIG. 4. Top: ratio of final rotational frequency $\omega_c$ to initial frequency $\omega_c$ as a function of $\omega_a/\Omega$, for solid-body vortices, which conserve angular momentum and energy when they merge. The label for each curve indicates the value of $r_a/R_d$. The heavy line gives the solution for a lens-shaped vortex ($R_d = H_1 = 0$). Some curves end before reaching $\omega_a = \Omega$; at greater values of $\omega_a$, the vortex has negative thickness $H_1 + h + \eta$ at its center and is thus not geometrically valid. Middle: ratio of final radius $r_c$ to initial radius $r_a$ as a function of $\omega_a/\Omega$. Bottom: ratio of final mass $M_c$ to initial mass $2M_a$. Except in the lens-shaped case, the merged vortex has less mass.

$$v(r) = \begin{cases} \omega r, & r \le r_i \\ -\left(\dfrac{r_i^2}{r_o^2 - r_i^2}\right)\omega\left(r - \dfrac{r_o^2}{r}\right), & r_i < r \le r_o \\ 0, & r > r_o. \end{cases} \quad (30)$$

The azimuthal speed $v$ is continuous at $r = r_i$, $r_o$ (Fig. 5). The solid-body core extends to $r = r_i$, and is surrounded by a constant vorticity shell extending to $r = r_o$. Outside this shell, there is no circulation induced by the vortex [i.e., this is an ''exactly shielded'' vortex as defined by Carton and Bertrand (1994)]. The vortices are completely included over an arbitrary finite domain. The ratio of shell vorticity $\zeta_o$ to core vorticity $\zeta_i$ is

$$\frac{\zeta_o}{\zeta_i} = -\frac{r_i^2}{r_o^2 - r_i^2}. \quad (31)$$

In the limit $r_o \gg r_i$, the outer shell is nearly irrotational and the vortex becomes Rankine-like. If $r_o < \sqrt{2}r_i$, the

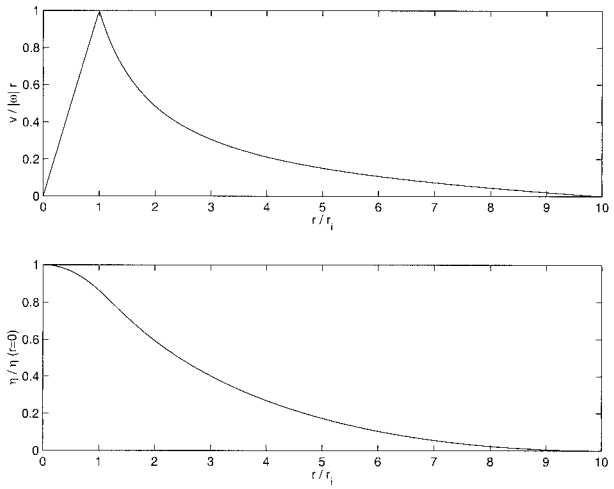


FIG. 5. Top: azimuthal velocity $v(r)$ vs radial distance for a vortex with a constant vorticity shell surrounding a solid body core. For this vortex, $\omega = -\Omega$. Bottom: sea surface displacement $\eta$ vs radial distance, calculated by integrating the cyclogeostrophic relation (1).

magnitude of the shell vorticity is larger than that of the core vorticity.

Vortices of this structure may be centrifugally unstable. If a fluid parcel orbiting a cyclogeostrophic vortex is infinitesmally perturbed, it experiences a restoring force proportional to

$$F = \partial_r(vr + \Omega r^2)^2 \quad (32)$$

(Kloosterziel and van Heijst 1991). If $F < 0$ anywhere in the vortex, perturbations will grow exponentially. Applying this stability criterion to the inner edge of the shell shows that anticyclones are unstable if $\zeta_i < -\Omega$, while cyclones are unstable if the anticyclonic shell vorticity $\zeta_o$ is less than $-\Omega$. Thus, rapidly spinning anticyclones and narrow-shelled cyclones are unstable. In this paper, we do not explore solutions in the ($\omega$, $r_i$, $r_o$) parameter space describing unstable vortices. We also discard choices of these parameters that yield nonpositive central depths for the cyclonic vortices.

There is no guarantee that the set of nonlinear conservation laws can be simultaneously satisfied by a unique combination of $\omega_c$, $r_{i,c}$, and $r_{o,c}$. Paralleling the solid-body case, we examine two merging scenarios: one in which mass is conserved a priori and the other in which energy is conserved. In both scenarios, we assume that the mass $M_i$ of the solid-body core is conserved:

$$M_i = 2\pi\rho \int_0^{r_i} dr\, r[H_1 + h_i(r) + \eta_i(r)]. \quad (33)$$

This constraint demands that fluid parcels retain the sign of their vorticity; that is, the core of the merged vortex is the fused cores of the initial vortices, and mass

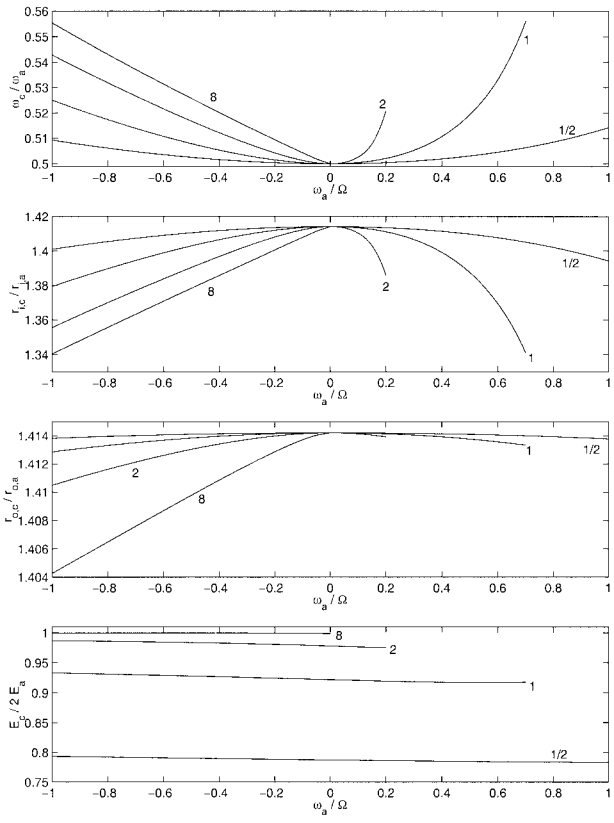


FIG. 6. Top: ratio of final rotational frequency $\omega_c$ to initial frequency $\omega_a$ as a function of $\omega_a/\Omega$, for finite-shear vortices, which conserve total mass and angular momentum when they merge ($r_{o,a} = 10r_{i,a}$). The label for each curve indicates the value of $r_{i,a}/R_d$. Some curves end before reaching $\omega_a = \Omega$; at greater values of $\omega_a$, the vortex would have negative thickness $H_1 + h + \eta$ at its center and is thus not geometrically valid. Middle, upper: ratio of final core radius $r_{i,c}$ to initial core radius $r_{i,a}$. Middle, lower: ratio of final shell radius $r_{o,c}$ to initial shell radius $r_{o,a}$. Bottom: ratio of final energy $E_c$ to initial energy $2E_a$.

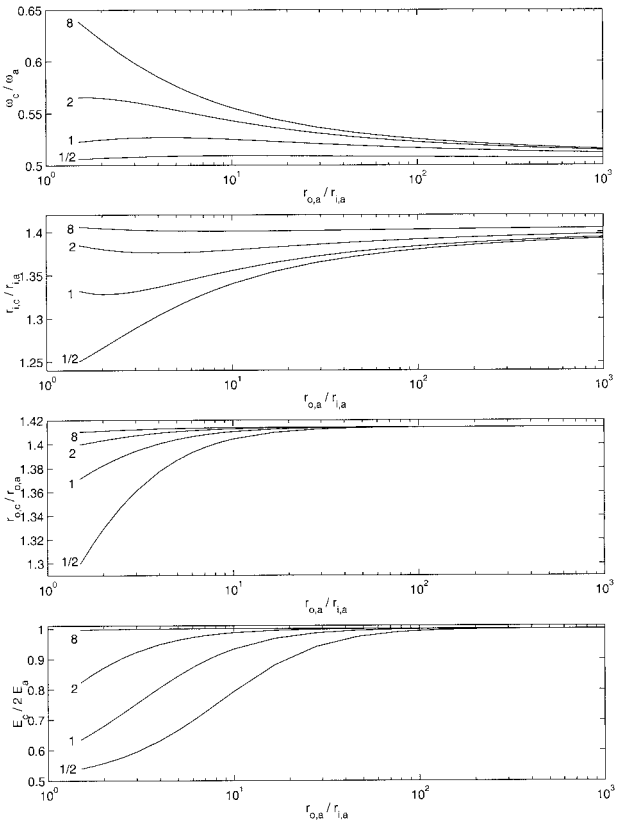


FIG. 7. Top: ratio of final rotational frequency $\omega_c$ to initial frequency $\omega_a$ as a function of $r_{o,a}/r_{i,a}$, for finite-shear anticyclonic vortices, which conserve total mass and angular momentum when they merge, at the limit $\omega_a/\Omega = -1$. The label for each curve indicates the value of $r_{i,a}/R_d$. Middle, upper: ratio of final core radius $r_{i,c}$ to initial core radius $r_{i,a}$. Middle, lower: ratio of final shell radius $r_{o,c}$ to initial shell radius $r_{o,a}$. Bottom: ratio of final energy $E_c$ to initial energy $2E_a$.

changes in the energy-conserving scenario are associated solely with shell fluid.[4]

### 1) CONSERVING TOTAL MASS AND ANGULAR MOMENTUM

Figure 6 shows the characteristics of the merged vortex ($\omega_c$, $r_{i,c}$ and $r_{o,c}$) as functions of $\omega_a/\Omega$ for $r_{o,a}/r_{i,a} = 10$. Compared to the solid-body counterpart to this scenario (Fig. 3), the period jump $\omega_a/\omega_c$ is closer to 2 and the radius jump $r_{i,c}/r_{i,a}$ is closer to $\sqrt{2}$, that is, closer to those of the limiting case of infinite cylinders.

The energy loss is nearly independent of $\omega_a/\Omega$. Energy is lost for small $r_{i,a}/R_d$, but less so than in the solid-body counterpart (Fig. 3). For $r_{i,a}/R_d \geq 8$, energy is conserved within 0.1% for all values of $\omega_a/\Omega$.

Figure 7 shows how the characteristics of the merged vortex vary as a function of shell width $r_{o,a}/r_{i,a}$, for the limiting case of a zero potential vorticity anticyclonic core ($\omega_a/\Omega = -1$). For nearly irrotational shells ($r_{o,a} \gg r_{i,a}$), the merged vortex has a core period nearly double that of the original vortices ($\omega_a/\omega_c \sim 2$) and inner and outer radii larger by $\sim\sqrt{2}$. The dependence of the solutions on $r_{i,a}/R_d$ increases as $r_{o,a} \to r_{i,a}$ (the solid-body limit), and energy is lost (particularly for small $r_{i,a}/R_d$).

### 2) CONSERVING ANGULAR MOMENTUM AND ENERGY

Figure 8 shows the characteristics of the merged vortex for $r_{o,a}/r_{i,a} = 10$. Compared to the solid-body coun-

[4] When 2D vortices of the exactly shielded structure are brought into contact, their cores merge while shell fluid is lost. However, if they have shell vorticity greater than that given by (31) (and consequently induce circulation around the edge of any arbitrarily large, finite domain), their interaction in the presence of instability may cause the shell fluid to merge while core fluid is rejected (Carton and Bertrand 1994). The possibility of such an ''inverted merging'' is not considered here.

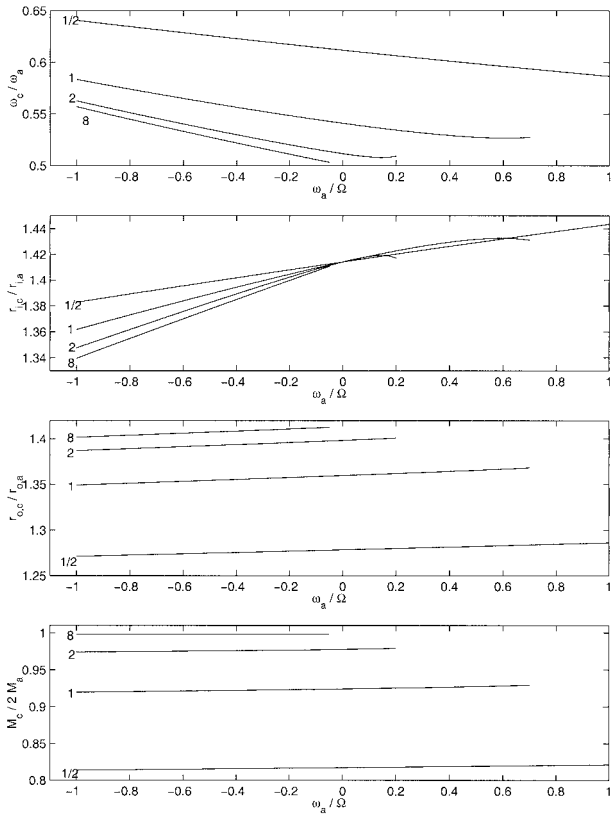

FIG. 8. Top: ratio of final rotational frequency $\omega_c$ to initial frequency $\omega_a$ as a function of $\omega_a/\Omega$, for finite-shear vortices, which conserve angular momentum and energy when they merge ($r_{o,a} = 10r_{i,a}$). The label for each curve indicates the value of $r_{i,a}/R_d$. Some curves end before reaching $\omega_a = \Omega$; at greater values of $\omega_a$, the vortex would have negative thickness $H_1 + h + \eta$ at its center and is thus not geometrically valid. Middle, upper: ratio of final core radius $r_{i,c}$ to initial core radius $r_{i,a}$. Middle, lower: ratio of final shell radius $r_{o,c}$ to initial shell radius $r_{o,a}$. Bottom: ratio of final mass $M_c$ to initial mass $2M_a$.

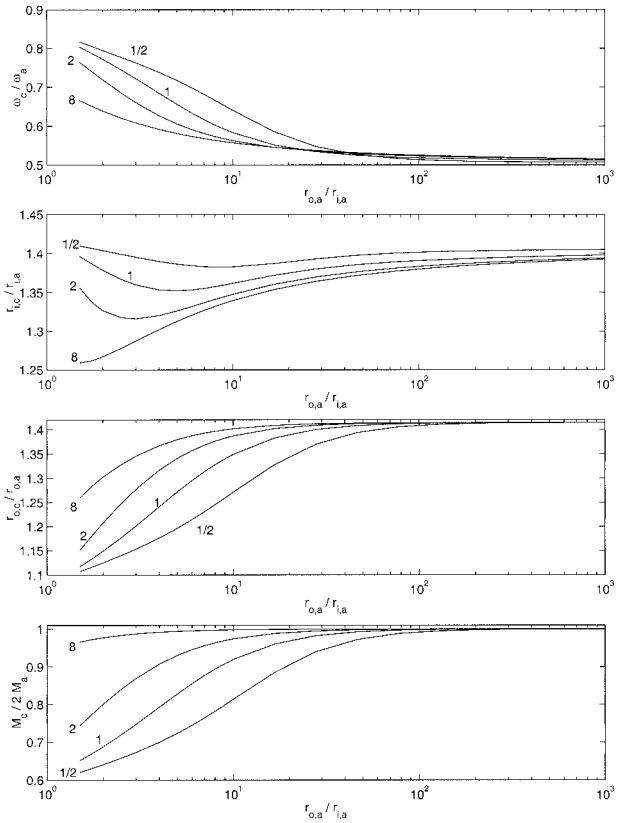

FIG. 9. Top: ratio of final rotational frequency $\omega_c$ to initial frequency $\omega_a$ as a function of $r_{o,a}/r_{i,a}$, for finite-shear vortices, which conserve angular momentum and energy when they merge ($\omega_a/\Omega = -1$). The label for each curve indicates the value of $r_{i,a}/R_d$. Middle, upper: ratio of final core radius $r_{i,c}$ to initial core radius $r_{i,a}$. Middle, lower: ratio of final shell radius $r_{o,c}$ to initial shell radius $r_{o,a}$. Bottom: ratio of final mass $M_c$ to initial mass $2M_a$.

terpart of this scenario (Fig. 4), $\omega_a/\omega_c$ is again closer to 2 and $r_{o,c}/r_{o,a}$ is closer to $\sqrt{2}$, that is, closer to those of infinite cylinders.

Mass is conserved in the limit $r_{i,a} \gg R_d$. Smaller vortices lose mass when they merge, nearly independently of $\omega_a/\Omega$. For $r_{i,a} = R_d$, 7%–8% of the mass is lost in the merging.

Figure 9 shows the characteristics of the merged vortex as a function of $r_{o,a}/r_{i,a}$, for $\omega_a/\Omega = -1$. For vortices with nearly irrotational shells, mass is conserved. The mass loss exceeds 10% for shell radius smaller than $r_{o,a}/R_d \sim 10$.

## 4. Discussion

Although we assume that the surrounding fluid remains quiescent, when the total depth $H_2$ is finite, fluid columns underlying the vortices may be compressed or stretched in the merging. A column of quiescent fluid beneath a vortex bears potential vorticity $2\Omega/(H_2 - H_1 - h)$, where $h$ is the downward displacement of the interface. If during merging $h$ increases by $\Delta h$, the underlying column must gain anticyclonic vorticity $\zeta$,

$$\frac{\zeta}{2\Omega} = -\frac{\Delta h}{H_2 - H_1 - h}, \tag{34}$$

in order to conserve its angular momentum. Thus, the assumption of a ''quiescent surrounding fluid,'' which allowed us to simplify angular momentum conservation, is valid in the limit $H_2 \gg \Delta h$ and exact in the 1½-layer limit $H_2 \to \infty$. This limit is appropriate for midlatitude open ocean baroclinic eddies, but does not apply to lens-shaped vortices whose height are an appreciable fraction of the total water depth (Nof and Simon 1987). In this case, cyclonic motion induced in the second layer contributes significantly to the total energy of the system (Dewar and Killworth 1990).

In their examination of lens-shaped vortex merging, Pavia and Cushman-Roisin (1990) conserved the mass and energy of the initial vortices, and showed that the absolute angular momentum of the vortices was not conserved. However, there is no ''angular momentum par-

adox'' inherent in their result, due to the role of the surrounding fluid in the total angular momentum budget. A simple thought experiment can serve to highlight this role: consider a pair of cylinders (density $\rho$, mass $m_a$, radius $r_a$ in solid-body rotation at relative rate $\omega_a$) touching at the center of a tank that rotates at $\Omega$. The cylinders are surrounded by fluid of density $\rho_*$. Later, they have merged to produce a single cylinder of the same density with mass $m_c = 2m_a$, radius $r_c$, and rotation rate $\omega_c$. It is centered on the contact point of the parent vortices. Conservation of mass requires $r_c = \sqrt{2}r_a$, and angular momentum conservation for the entire system requires

$$\omega_c = \frac{1}{2}\left[\left(1 - \frac{\rho_*}{\rho}\right)\Omega + \omega_a\right]. \tag{35}$$

In the absence of surrounding fluid, $\rho_* = 0$, and (35) becomes

$$\omega_c = \frac{1}{2}(\Omega + \omega_a). \tag{36}$$

This expression is analogous to the formulation of angular momentum conservation of Cushman-Roisin (1989) and Pavia and Cushman-Roisin (1990). With $\rho_* = 0$, we are essentially considering coalescing fixed-height disks on a rotating table. To an observer in the rotating frame, the rotation rate of the resulting disk is more positive (for positive $\Omega$) than it would be in the absence of rotation, due to Coriolis deflection applying a net torque on the coalescing particles (cf. Feynman et al. 1987, p. 19-8). However, if $\rho_* = \rho$, (35) simplifies to

$$\omega_c = \frac{1}{2}\omega_a, \tag{37}$$

that is, the period doubles and relative angular momentum is conserved. As the cylinders coalesce, they lose the absolute angular momentum associated with revolution about their contact point, but this is balanced by the absolute angular momentum gained by the displaced surrounding fluid. In the rotating frame, Coriolis deflection of the coalescing parcels is balanced by a pressure gradient in the surrounding fluid (Rossby 1936, 1948).

Our model demonstrates that solid-body vortices can conserve mass, angular momentum, and energy in the lens-shaped limit $H_1 = 0$. For nonzero $H_1$ either energy or mass conservation must be relaxed, leading to the two merging scenarios discussed in this paper.[5] In the first scenario, if mass is conserved, the final vortex has less total energy than its parents. These solutions are thus ''energetically allowable'' (Dewar and Killworth 1990) in the sense that an external energy source is not required, analogous to the classical Rossby adjustment problem that has a steady-state solution with one-third of the initial energy (cf. Gill 1982, pp. 191–203), the remaining energy having been removed via Poincaré wave radiation. Turbulent mixing during merging can also lower the energy state of the merged vortex (Nof 1986).

In the second scenario, if energy is conserved, the final vortex has less mass than its parents. Cushman-Roisin (1989) proposed that merging vortices eject fluid in narrow filaments as they become axisymmetric (Griffiths and Hopfinger 1987; Melander et al. 1988). Because these filaments bear negligible energy and relative angular momentum but a significant fraction of the mass, they are modeled implicitly by relaxing mass conservation, as was done in the second scenario (Pavia and Cushman-Roisin 1990).

In summary, vortices can conserve mass, angular momentum, and energy when they merge in two physically important limits of vortex structure: lens-shaped anticyclones and Rankine-like eddies. All three properties cannot be simultaneously conserved over the full range of parameters considered here; in general, mass or energy must be lost, presumably due to Poincaré wave radiation, turbulent dissipation, or filamentation. Future laboratory experiments and field observations could determine which scenario best describes the merging of oceanic vortices.

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[5] A third scenario was considered in which mass and energy were conserved. The final state contained more angular momentum than the initial state, leading us to reject this scenario as a physically meaningful description of free vortex merging.

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