Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 5, Issue 2, Article 45, 2004

SOME PROBLEMS AND SOLUTIONS INVOLVING MATHIEU'S SERIES AND ITS GENERALIZATIONS

H.M. SRIVASTAVA AND ŽIVORAD TOMOVSKI

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF VICTORIA
VICTORIA, BRITISH COLUMBIA V8W 3P4
CANADA
harimsri@math.uvic.ca

INSTITUTE OF MATHEMATICS
ST. CYRIL AND METHODIUS UNIVERSITY
MK-1000 SKOPJE
MACEDONIA
tomovski@iunona.pmf.ukim.edu.mk

Received 14 October, 2003; accepted 1 May, 2004
Communicated by P. Cerone

Dedicated to Professor Blagoj Sazdo Popov on the Occasion of his Eightieth Birthday

ABSTRACT. The authors investigate several recently posed problems involving the familiar Mathieu series and its various generalizations. For certain families of generalized Mathieu series, they derive a number of integral representations and investigate several one-sided inequalities which are obtainable from some of these general integral representations or from sundry other considerations. Relevant connections of the results and open problems (which are presented or considered in this paper) with those in earlier works are also indicated. Finally, a conjectured generalization of one of the Mathieu series inequalities proven here is posed as an open problem.

Key words and phrases: Mathieu's series, Integral representations, Bessel functions, Hypergeometric functions, One-sided inequalities, Fourier transforms, Riemann and Hurwitz Zeta functions, Eulerian integral, Polygamma functions, Laplace integral representation, Euler-Maclaurin summation formula, Riemann-Liouville fractional integral, Lommel function of the first kind.

2000 *Mathematics Subject Classification.* Primary 26D15, 33C10, 33C20, 33C60; Secondary 33E20, 40A30.

ISSN (electronic): 1443-5756

The present investigation was supported, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353.

146-03

1. INTRODUCTION, DEFINITIONS, AND PRELIMINARIES

The following familiar infinite series:

$$S(r) := \sum_{n=1}^{\infty} \frac{2n}{(n^2 + r^2)^2} \qquad (r \in \mathbb{R}^+) \tag{1.1}$$

is named after Émile Leonard Mathieu (1835-1890), who investigated it in his 1890 work [13] on elasticity of solid bodies.

For the Mathieu series $S(r)$ defined by (1.1), Alzer *et al.* [2] showed that the best constants κ_1 and κ_2 in the following two-sided inequality:

$$\frac{1}{\kappa_1 + r^2} < S(r) < \frac{1}{\kappa_2 + r^2} \qquad (r \neq 0) \tag{1.2}$$

are given by

$$\kappa_1 = \frac{1}{2\zeta(3)} \qquad \text{and} \qquad \kappa_2 = \frac{1}{6},$$

where $\zeta(s)$ denotes the Riemann Zeta function defined by (see, for details, [20, Chapter 2])

$$\zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (\Re(s) > 1) \\[2ex] \displaystyle(1-2^{1-s})^{-1} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & (\Re(s) > 0;\ s \neq 1). \end{cases} \tag{1.3}$$

A remarkably useful integral representation for $S(r)$ in the elegant form:

$$S(r) = \frac{1}{r} \int_0^{\infty} \frac{x \sin(rx)}{e^x - 1}\, dx \tag{1.4}$$

was given by Emersleben [6]. In fact, by applying (1.4) in conjunction with the generating function:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \qquad (|z| < 2\pi) \tag{1.5}$$

for the Bernoulli numbers

$$B_n \qquad (n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}),$$

Elbert [5] derived the following asymptotic expansion for $S(r)$:

$$S(r) \sim \sum_{k=0}^{\infty} (-1)^k \frac{B_{2k}}{r^{2k+2}} = \frac{1}{r^2} - \frac{1}{6r^4} - \frac{1}{30r^6} - \cdots \qquad (r \to \infty). \tag{1.6}$$

More recently, Guo [10] made use of the integral representation (1.4) in order to obtain a number of interesting results including (for example) bounds for $S(r)$. For various subsequent developments using (1.4), the interested reader may be referred to the works by (among others) Qi *et al.* ([16] to [19]). (See also an *independent* derivation of the asymptotic expansion (1.6) by Wang and Wang [24]).

Several interesting problems and solutions dealing with integral representations and bounds for the following *mild* generalization of the Mathieu series (1.1):

$$\mathbb{S}_\mu(r) := \sum_{n=1}^{\infty} \frac{2n}{(n^2+r^2)^\mu} \qquad \left(r \in \mathbb{R}^+;\ \mu > 1\right) \tag{1.7}$$

can be found in the recent works by Diananda [4], Guo [10], Tomovski and Trenčevski [23], and Cerone and Lenard [3]. Motivated essentially by the works of Cerone and Lenard [3] (and Qi [17]), we propose to investigate the corresponding problems involving a family of generalized Mathieu series, which is defined here by

$$\mathcal{S}_\mu^{(\alpha,\beta)}(r;\mathbf{a}) = \mathcal{S}_\mu^{(\alpha,\beta)}(r;\{a_k\}_{k=1}^{\infty}) := \sum_{n=1}^{\infty} \frac{2a_n^\beta}{(a_n^\alpha+r^2)^\mu} \tag{1.8}$$

$$\left(r,\alpha,\beta,\mu \in \mathbb{R}^+\right),$$

where (*and throughout this paper*) it is *tacitly* assumed that the *positive* sequence

$$\mathbf{a} := \{a_k\}_{k=1}^{\infty} = \{a_1,a_2,a_3,\ldots,a_k,\ldots\} \qquad \left(\lim_{k\to\infty} a_k = \infty\right)$$

is so chosen (and then the *positive* parameters α,β, and μ are so constrained) that the infinite series in the definition (1.8) converges, that is, that the following auxiliary series:

$$\sum_{n=1}^{\infty} \frac{1}{a_n^{\mu\alpha-\beta}}$$

is convergent. We remark in passing that, in a very recent research report (which appeared after the submission of this paper to *JIPAM*), Pogány [14] considered a substantially more general form of the definition (1.8). As a matter of fact, Pogány's investigation [14] was based largely upon such *main* mathematical tools as the Laplace integral representation of general Dirichlet series and the familiar Euler-Maclaurin summation formula (*cf.*, *e.g.*, [20, p. 36 *et seq.*]).

Clearly, by comparing the definitions (1.1), (1.7), and (1.8), we obtain

$$\mathbb{S}_2(r) = S(r) \qquad \text{and} \qquad \mathbb{S}_\mu(r) = \mathcal{S}_\mu^{(2,1)}(r;\{k\}_{k=1}^{\infty}). \tag{1.9}$$

Furthermore, the special cases

$$\mathcal{S}_2^{(2,1)}(r;\{a_k\}_{k=1}^{\infty}), \qquad \mathcal{S}_\mu^{(2,1)}(r;\{k^\gamma\}_{k=1}^{\infty}), \qquad \text{and} \qquad \mathcal{S}_\mu^{(\alpha,\alpha/2)}(r;\{k\}_{k=1}^{\infty})$$

were investigated by Qi [17], Tomovski [22], and Cerone and Lenard [3].

2. A Class of Integral Representations

First of all, we find from the definition (1.8) that

$$\mathcal{S}_\mu^{(\alpha,\beta)}(r;\{a_k\}_{k=1}^{\infty}) = 2\sum_{m=0}^{\infty} \binom{\mu+m-1}{m} (-r^2)^m \sum_{n=1}^{\infty} \frac{1}{a_n^{(\mu+m)\alpha-\beta}},$$

so that

$$\mathcal{S}_{\mu}^{(\alpha,\beta)}\left(r;\{k^{\gamma}\}_{k=1}^{\infty}\right)=2\sum_{m=0}^{\infty}\binom{\mu+m-1}{m}\left(-r^{2}\right)^{m}\,\zeta\left(\gamma\left[\left(\mu+m\right)\alpha-\beta\right]\right) \tag{2.1}$$
$$\left(r,\alpha,\beta,\gamma\in\mathbb{R}^{+};\ \gamma\left(\mu\alpha-\beta\right)>1\right)$$

in terms of the Riemann Zeta function defined by (1.3).

Now, by making use of the familiar integral representation (*cf*., *e.g*., [20, p. 96, Equation 2.3 (4)]):

$$\zeta\left(s\right)=\frac{1}{\Gamma\left(s\right)}\int_{0}^{\infty}\frac{x^{s-1}}{e^{x}-1}\,dx\qquad\left(\Re\left(s\right)>1\right) \tag{2.2}$$

in (2.1), we obtain

$$\mathcal{S}_{\mu}^{(\alpha,\beta)}\left(r;\{k^{\gamma}\}_{k=1}^{\infty}\right)=\frac{2}{\Gamma\left(\mu\right)}\int_{0}^{\infty}\frac{x^{\gamma(\mu\alpha-\beta)-1}}{e^{x}-1}\cdot\,{}_{1}\Psi_{1}\left[\left(\mu,1\right);\left(\gamma\left(\mu\alpha-\beta\right),\gamma\alpha\right);-r^{2}\,x^{\gamma\alpha}\right]dx, \tag{2.3}$$
$$\left(r,\alpha,\beta,\gamma\in\mathbb{R}^{+};\ \gamma\left(\mu\alpha-\beta\right)>1\right),$$

where ${}_{p}\Psi_{q}$ denotes the Fox-Wright generalization of the hypergeometric ${}_{p}F_{q}$ function with p numerator and q denominator parameters, defined by [21, p. 50, Equation 1.5 (21)]

$${}_{p}\Psi_{q}\left[\left(\alpha_{1},A_{1}\right),\ldots,\left(\alpha_{p},A_{p}\right);\left(\beta_{1},B_{1}\right),\ldots,\left(\beta_{q},B_{q}\right);z\right]:=\sum_{m=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma\left(\alpha_{j}+A_{j}m\right)}{\prod_{j=1}^{q}\Gamma\left(\beta_{j}+B_{j}m\right)}\cdot\frac{z^{m}}{m!}, \tag{2.4}$$
$$\left(A_{j}\in\mathbb{R}^{+}\ \left(j=1,\ldots,p\right);\ B_{j}\in\mathbb{R}^{+}\ \left(j=1,\ldots,q\right);\ 1+\sum_{j=1}^{q}B_{j}-\sum_{j=1}^{p}A_{j}>0\right),$$

so that, obviously,

$${}_{p}\Psi_{q}\left[\left(\alpha_{1},1\right),\ldots,\left(\alpha_{p},1\right);\left(\beta_{1},1\right),\ldots,\left(\beta_{q},1\right);z\right]=\frac{\Gamma\left(\alpha_{1}\right)\cdots\Gamma\left(\alpha_{p}\right)}{\Gamma\left(\beta_{1}\right)\cdots\Gamma\left(\beta_{q}\right)}\,{}_{p}F_{q}\left(\alpha_{1},\ldots,\alpha_{p};\beta_{1},\ldots,\beta_{q};z\right). \tag{2.5}$$

In its special case when

$$\gamma\alpha=q\qquad\left(q\in\mathbb{N}:=\{1,2,3,\ldots\}\right),$$

we can apply the Gauss-Legendre multiplication formula [21, p. 23, Equation 1.1 (27)]:

$$\Gamma\left(mz\right)=\left(2\pi\right)^{\frac{1}{2}(1-m)}\ m^{mz-\frac{1}{2}}\ \prod_{j=1}^{m}\Gamma\left(z+\frac{j-1}{m}\right) \tag{2.6}$$
$$\left(z\in\mathbb{C}\setminus\left\{0,-\frac{1}{m},-\frac{2}{m},\ldots\right\};\ m\in\mathbb{N}\right)$$

on the right-hand side of our integral representation (2.3). We thus find that

$$\begin{aligned}(2.7)\quad \mathcal{S}_{\mu}^{(\alpha,\beta)}\left(r;\left\{k^{q/\alpha}\right\}_{k=1}^{\infty}\right) = {} & \frac{2}{\Gamma\left(q\left[\mu-\frac{\beta}{\alpha}\right]\right)}\int_{0}^{\infty}\frac{x^{q\left[\mu-\frac{\beta}{\alpha}\right]-1}}{e^{x}-1} \\ & \cdot\ {}_{1}F_{q}\left(\mu;\Delta\left(q;q\left[\mu-\frac{\beta}{\alpha}\right]\right);-r^{2}\left(\frac{x}{q}\right)^{q}\right)dx \\ & \left(r,\alpha,\beta\in\mathbb{R}^{+};\ \mu-\frac{\beta}{\alpha}>q^{-1};\ q\in\mathbb{N}\right),\end{aligned}$$

where, for convenience, $\Delta(q;\lambda)$ abbreviates the array of q parameters

$$\frac{\lambda}{q},\frac{\lambda+1}{q},\ldots,\frac{\lambda+q-1}{q}\qquad(q\in\mathbb{N}).$$

For $q=2,$ (2.7) can easily be simplified to the form:

$$\begin{aligned}(2.8)\quad \mathcal{S}_{\mu}^{(\alpha,\beta)}\left(r;\left\{k^{2/\alpha}\right\}_{k=1}^{\infty}\right) = {} & \frac{2}{\Gamma\left(2\left[\mu-(\beta/\alpha)\right]\right)}\int_{0}^{\infty}\frac{x^{2[\mu-(\beta/\alpha)]-1}}{e^{x}-1} \\ & \cdot\ {}_{1}F_{2}\left(\mu;\mu-\frac{\beta}{\alpha},\mu-\frac{\beta}{\alpha}+\frac{1}{2};-\frac{r^{2}x^{2}}{4}\right)dx \\ & \left(r,\alpha,\beta\in\mathbb{R}^{+};\ \mu-\frac{\beta}{\alpha}>\frac{1}{2}\right).\end{aligned}$$

A *further* special case of (2.8) can be deduced in terms of the Bessel function $J_{\nu}(z)$ of order ν:

$$\begin{aligned}(2.9)\quad J_{\nu}(z) & :=\sum_{m=0}^{\infty}\frac{(-1)^{m}\left(\frac{1}{2}z\right)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)} \\ & =\frac{\left(\frac{1}{2}z\right)^{\nu}}{\Gamma(\nu+1)}\,{}_{0}F_{1}\left(-;\nu+1;-\frac{z^{2}}{4}\right).\end{aligned}$$

Thus, by setting $\beta=\frac{1}{2}\alpha$ and $\mu\mapsto\mu+1$ in (2.8), and applying (2.9) as well as (2.6) *with* $m=2$, we obtain the following known result [3, p. 3, Theorem 2.1]:

$$\begin{aligned}(2.10)\quad \mathcal{S}_{\mu+1}^{(\alpha,\alpha/2)}\left(r;\left\{k^{2/\alpha}\right\}_{k=1}^{\infty}\right) & =\mathcal{S}_{\mu+1}^{(2,1)}\left(r;\{k\}_{k=1}^{\infty}\right)=\mathbb{S}_{\mu+1}(r) \\ & =\frac{\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}}\,\Gamma(\mu+1)}\int_{0}^{\infty}\frac{x^{\mu+\frac{1}{2}}}{e^{x}-1}\,J_{\mu-\frac{1}{2}}(rx)\,dx \\ & \left(r,\mu\in\mathbb{R}^{+}\right).\end{aligned}$$

In a similar manner, a limit case of (2.8) when $\beta\to0$ would formally yield the formula:

$$\begin{aligned}(2.11)\quad \mathcal{S}_{\mu}^{(\alpha,0)}\left(r;\left\{k^{2/\alpha}\right\}_{k=1}^{\infty}\right) & =\sum_{n=1}^{\infty}\frac{2}{(n^{2}+r^{2})^{\mu}} \\ & =\frac{2\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}}\,\Gamma(\mu)}\int_{0}^{\infty}\frac{x^{\mu-\frac{1}{2}}}{e^{x}-1}\,J_{\mu-\frac{1}{2}}(rx)\,dx \\ & \left(r\in\mathbb{R}^{+};\ \mu>\frac{1}{2}\right),\end{aligned}$$

which is, in fact, equivalent to the following 1906 result of Willem Kapteyn (1849-1927) [25, p. 386, Equation 13.2 (9)]:

$$\int_0^\infty \frac{x^\nu}{e^{\pi x}-1} J_\nu(\lambda x)\, dt = \frac{(2\lambda)^\nu}{\sqrt{\pi}} \Gamma\left(\nu+\frac{1}{2}\right) \sum_{n=1}^\infty \frac{1}{(n^2\,\pi^2+\lambda^2)^{\nu+\frac{1}{2}}} \tag{2.12}$$

$$(\Re(\nu)>0;\ |\mathcal{J}(\lambda)|<\pi).$$

Furthermore, a rather simple consequence of (2.11) *or* (2.12) in the form:

$$\sum_{n=-\infty}^\infty \frac{1}{(n^2+c^2)^s} = c^{-2s} + \frac{2\sqrt{\pi}}{(2c)^{s-\frac{1}{2}}\ \Gamma(s)} \int_0^\infty \frac{x^{s-\frac{1}{2}}}{e^x-1} J_{s-\frac{1}{2}}(cx)\, dx \tag{2.13}$$
$$\left(\Re(s)>\frac{1}{2};\ |c|<1\right).$$

appears erroneously in the works by (for example) Hansen [11, p. 122, Entry (6.3.59)] and Prudnikov *et al*. [15, p. 685, Entry 5.1.25.1]. And, by making use of the Trigamma function $\psi'(z)$ defined, in general, by [20, p. 22, Equation 1.2 (52)]

$$\psi^{(m)}(z) := \frac{d^{m+1}}{dz^{m+1}}\{\log\Gamma(z)\} = \frac{d^m}{dz^m}\{\psi(z)\} \tag{2.14}$$
$$\left(m\in\mathbb{N}_0:=\mathbb{N}\cup\{0\};\ z\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ \mathbb{Z}_0^-:=\{0,-1,-2,\ldots\}\right)$$

or, equivalently, by

$$\psi^{(m)}(z) := (-1)^{m+1}\ m!\sum_{k=0}^\infty \frac{1}{(k+z)^{m+1}} =: (-1)^{m+1}\ m!\,\zeta(m+1,z) \tag{2.15}$$
$$\left(m\in\mathbb{N};\ z\in\mathbb{C}\setminus\mathbb{Z}_0^-\right)$$

in terms of the Hurwitz (or generalized) Zeta function $\zeta(s,a)$ [20, p. 88, Equation 2.2 (1) *et seq.*], both Hansen [11, p. 111, Entry (6.1.137)] and Prudnikov *et al*. [15, p. 687, Entry 5.1.25.28] have recorded the following *explicit* evaluation of the classical Mathieu series:

$$S(r) := \sum_{k=1}^\infty \frac{2k}{(k^2+r^2)^2} = \frac{\psi'(-ir)-\psi'(ir)}{2ir} \qquad \left(i:=\sqrt{-1}\right). \tag{2.16}$$

We remark in passing that, in light of one of the familiar relationships:

$$J_{\mp\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\cdot\left\{\begin{array}{l}\cos z\\ \\ \sin z\end{array}\right., \tag{2.17}$$

a special case of (2.10) when $\mu=1$ would immediately yield the well-exploited integral representation (1.4).

Next, in the theory of Bessel functions, it is fairly well known that (*cf.*, *e.g.*, [7, p. 49, Equation 7.7.3 (16)])

$$\int_0^\infty e^{-st}\, t^{\lambda-1}\, J_\nu\left(\rho t\right) dt = \left(\frac{\rho}{2s}\right)^\nu s^{-\lambda}\, \frac{\Gamma\left(\nu+\lambda\right)}{\Gamma\left(\nu+1\right)}\; {}_2F_1\left[\begin{array}{c} \frac{1}{2}\left(\nu+\lambda\right), \frac{1}{2}\left(\nu+\lambda+1\right); \\ \nu+1; \end{array} -\frac{\rho^2}{s^2}\right] \qquad \left(\Re\left(s\right) > \left|\mathcal{J}\left(\rho\right)\right|;\; \Re\left(\nu+\lambda\right) > 0\right). \tag{2.18}$$

Since

$$ {}_1F_0\left(\lambda; —; z\right) = \left(1-z\right)^{-\lambda} \qquad \left(\left|z\right| < 1;\; \lambda \in \mathbb{C}\right), \tag{2.19}$$

the integral formula (2.18) would simplify considerably when $\lambda = \nu+1$ *and* when $\lambda = \nu+2$, giving us [see also Equations (2.10) and (2.11) above]

$$\int_0^\infty e^{-st}\, t^{\nu}\, J_\nu\left(\rho t\right) dt = \frac{\left(2\rho\right)^\nu}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\left(s^2+\rho^2\right)^{\nu+\frac{1}{2}}} \qquad \left(\Re\left(s\right) > \left|\mathcal{J}\left(\rho\right)\right|;\; \Re\left(\nu\right) > -\frac{1}{2}\right) \tag{2.20}$$

and

$$\int_0^\infty e^{-st}\, t^{\nu+1}\, J_\nu\left(\rho t\right) dt = \frac{2s\left(2\rho\right)^\nu}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\nu+\frac{3}{2}\right)}{\left(s^2+\rho^2\right)^{\nu+\frac{3}{2}}} \qquad \left(\Re\left(s\right) > \left|\mathcal{J}\left(\rho\right)\right|;\; \Re\left(\nu\right) > -1\right), \tag{2.21}$$

respectively. While each of the special cases (2.20) and (2.21), too, together with the parent formula (2.18), are readily accessible in many different places in various mathematical books and tables (*cf.*, *e.g.*, [26, p. 72]), (2.20) appears slightly erroneously in [7, p. 49, Equation 7.7.3 (17)]. The integral formula (2.21) would follow also when we differentiate both sides of (2.20) partially with respect to the parameter s.

Now we turn once again to our definition (1.8) which, for $\alpha = 2$, yields

$$\mathcal{S}_\mu^{(2,\beta)}\left(r; \{a_k\}_{k=1}^\infty\right) = \sum_{n=1}^\infty \frac{2a_n^\beta}{\left(a_n^2+r^2\right)^\mu} \qquad \left(r, \beta, \mu \in \mathbb{R}^+\right). \tag{2.22}$$

Making use of the integral formulas (2.20) and (2.21), we find from (2.16) that

$$\mathcal{S}_\mu^{(2,\beta)}\left(r; \{a_k\}_{k=1}^\infty\right) = \frac{2\sqrt{\pi}}{\left(2r\right)^{\mu-\frac{1}{2}}\; \Gamma\left(\mu\right)} \int_0^\infty \left(\sum_{n=1}^\infty a_n^\beta\, e^{-a_n x}\right) x^{\mu-\frac{1}{2}}\; J_{\mu-\frac{1}{2}}\left(rx\right) dx \qquad \left(r, \beta, \mu \in \mathbb{R}^+\right) \tag{2.23}$$

and

$$\mathcal{S}_\mu^{(2,\beta)}\left(r; \{a_k\}_{k=1}^\infty\right) = \frac{\sqrt{\pi}}{\left(2r\right)^{\mu-\frac{3}{2}}\; \Gamma\left(\mu\right)} \int_0^\infty \left(\sum_{n=1}^\infty a_n^{\beta-1}\, e^{-a_n x}\right) x^{\mu-\frac{1}{2}}\; J_{\mu-\frac{3}{2}}\left(rx\right) dx \qquad \left(r, \beta, \mu \in \mathbb{R}^+\right), \tag{2.24}$$

respectively.

A special case of the integral representation (2.24) when

$$\beta = 1 \qquad \text{and} \qquad \mu \longmapsto \mu + 1$$

was given by Cerone and Lenard [3, p. 9, Equation (4.5)].

Finally, in view of the Eulerian integral formula:

$$\int_0^\infty e^{-st}\, t^{\lambda-1}\, dt = \frac{\Gamma(\lambda)}{s^\lambda} \qquad (\Re(s) > 0;\ \Re(\lambda) > 0)\,, \tag{2.25}$$

we find from the definition (1.8) that

$$\mathcal{S}_\mu^{(\alpha,\beta)}\left(r; \{a_k\}_{k=1}^\infty\right) = \frac{2}{\Gamma(\mu)} \int_0^\infty x^{\mu-1}\, e^{-r^2 x}\, \varphi(x)\, dx \tag{2.26}$$
$$\left(r, \alpha, \beta, \mu \in \mathbb{R}^+\right),$$

where, for convenience,

$$\varphi(x) := \sum_{n=1}^\infty a_n^\beta \exp\left(-a_n^\alpha\, x\right). \tag{2.27}$$

In terms of the generalized Mathieu series $\mathbb{S}_\mu(r)$ defined by (1.7), a special case of the integral representation (2.26) when

$$\alpha = 2, \qquad \beta = 1, \qquad \text{and} \qquad a_k = k \qquad (k \in \mathbb{N})\,,$$

was given by Tomovski and Trenčevski [23, p. 6, Equation (2.3)].

3. BOUNDS DERIVABLE FROM THE INTEGRAL REPRESENTATION (2.8)

For the generalized hypergeometric ${}_pF_q$ function of p numerator and q denominator parameters, which is defiend by (2.4) and (2.5), we first recall here the following equivalent form of a familiar Riemann-Liouville fractional integral formula (*cf.*, *e.g.*, [8, p. 200, Entry 13.1 (95)]:

$$\begin{aligned} {}_{p+1}F_{q+1}&(\rho, \alpha_1, \ldots, \alpha_p; \rho+\sigma, \beta_1, \ldots, \beta_q; z) \\ &= \frac{\Gamma(\rho+\sigma)}{\Gamma(\rho)\,\Gamma(\sigma)} \int_0^1 t^{\rho-1} (1-t)^{\sigma-1}\, {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; zt)\, dt \end{aligned} \tag{3.1}$$

$$\left(p \leqq q+1;\ \min\{\Re(\rho), \Re(\sigma)\} > 0;\ |z| < 1 \quad \text{when} \quad p = q+1\right),$$

which, for

$$p = q - 1 = 0 \quad \left(\beta_1 = \mu - \frac{\beta}{\alpha}\right), \quad \rho = \mu, \quad \sigma = \frac{1}{2} - \frac{\beta}{\alpha}, \quad \text{and} \quad z = -\frac{r^2x^2}{4},$$

immediately yields

$$\begin{aligned}(3.2)\quad {}_1F_2\left(\mu;\mu-\frac{\beta}{\alpha},\mu-\frac{\beta}{\alpha}+\frac{1}{2};-\frac{r^2x^2}{4}\right) &= \frac{\Gamma\left(\mu-\frac{\beta}{\alpha}\right)\Gamma\left(\mu-\frac{\beta}{\alpha}+\frac{1}{2}\right)}{\Gamma(\mu)\,\Gamma\left(\frac{1}{2}-\frac{\beta}{\alpha}\right)}\left(\frac{2}{rx}\right)^{\mu-(\beta/\alpha)-1} \\ &\quad\cdot\int_0^1\left(\sqrt{t}\right)^{\mu+(\beta/\alpha)-1}(1-t)^{-(\beta/\alpha)-\frac{1}{2}}\; J_{\mu-(\beta/\alpha)-1}\left(rx\sqrt{t}\right)dt\end{aligned}$$

$$\left(r,x,\mu\in\mathbb{R}^+;\ \frac{\beta}{\alpha}<\frac{1}{2}\right).$$

In terms of the Lommel function $s_{\mu,\nu}(z)$ of the first kind, defined by [7, p. 40, Equation 7.5.5 (69)]

$$(3.3)\qquad s_{\mu,\nu}(z)=\frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\,{}_1F_2\left(1;\frac{1}{2}\mu-\frac{1}{2}\nu+\frac{3}{2};\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{3}{2};-\frac{z^2}{4}\right),$$

the special case $\mu=1$ of (3.2) can be found recorded as a Riemann-Liouville fractional integral formula by Erdélyi *et al*. [8, p. 194, Entry 13.1 (64)] (see also [8, p. 195, Entry 13.1 (65)]).

Now we turn to a recent investigation by Landau [12] in which several best possible uniform bounds for the Bessel functions were obtained by using monotonicity arguments. Following also the work of Cerone and Lenard [3, Section 3], we choose to recall here two of Landau's inequalities given below. The first inequality:

$$(3.4)\qquad |J_\nu(x)|\leqq\frac{b_L}{\nu^{1/3}}$$

holds true uniformly in the argument x and is the best possible in the exponent $\frac{1}{3}$, with the constant b_L given by

$$(3.5)\qquad b_L=2^{1/3}\sup_x\{\mathrm{Ai}(x)\}\cong 0.674885\ldots,$$

where $\mathrm{Ai}(z)$ denotes the Airy function satisfying the differential equation:

$$(3.6)\qquad \frac{d^2w}{dz^2}-zw=0\qquad \big(w=\mathrm{Ai}(z)\big).$$

The second inequality:

$$(3.7)\qquad |J_\nu(x)|\leqq\frac{c_L}{x^{1/3}}$$

holds true uniformly in the order $\nu\in\mathbb{R}^+$ and is the best possible in the exponent $\frac{1}{3}$, with the constant c_L given by

$$(3.8)\qquad c_L=\sup_x\left\{x^{1/3}\,J_0(x)\right\}\cong 0.78574687\ldots.$$

By appealing appropriately to the bounds in (3.4) and (3.7), we find from (3.2) that

$$\tag{3.9} \begin{aligned} &\left|{}_1F_2\left(\mu;\mu-\frac{\beta}{\alpha},\mu-\frac{\beta}{\alpha}+\frac{1}{2};-\frac{r^2x^2}{4}\right)\right| \\ &\leqq b_L\left(\frac{2}{rx}\right)^{\mu-(\beta/\alpha)-1}\left(\mu-\frac{\beta}{\alpha}-1\right)^{-\frac{1}{3}}\cdot\frac{\Gamma\left(\mu-\frac{\beta}{\alpha}\right)\Gamma\left(\mu-\frac{\beta}{\alpha}+\frac{1}{2}\right)\Gamma\left(\frac{\mu}{2}+\frac{\beta}{2\alpha}+\frac{1}{2}\right)}{\Gamma(\mu)\Gamma\left(\frac{\mu}{2}-\frac{\beta}{2\alpha}+1\right)} \\ &\left(r,x\in\mathbb{R}^+;\ \mu-\frac{\beta}{\alpha}>1;\ \mu+\frac{\beta}{\alpha}>-1;\ \frac{\beta}{\alpha}<\frac{1}{2}\right) \end{aligned}$$

and

$$\tag{3.10} \begin{aligned} &\left|{}_1F_2\left(\mu;\mu-\frac{\beta}{\alpha},\mu-\frac{\beta}{\alpha}+\frac{1}{2};-\frac{r^2x^2}{4}\right)\right| \\ &\leqq \frac{c_L}{(rx)^{1/3}}\left(\frac{2}{rx}\right)^{\mu-(\beta/\alpha)-1}\cdot\frac{\Gamma\left(\mu-\frac{\beta}{\alpha}\right)\Gamma\left(\mu-\frac{\beta}{\alpha}+\frac{1}{2}\right)\Gamma\left(\frac{\mu}{2}+\frac{\beta}{2\alpha}+\frac{1}{3}\right)}{\Gamma(\mu)\Gamma\left(\frac{\mu}{2}-\frac{\beta}{2\alpha}+\frac{5}{6}\right)} \\ &\left(r,x\in\mathbb{R}^+;\mu-\frac{\beta}{\alpha}>1;\ \mu+\frac{\beta}{\alpha}>-\frac{2}{3};\ \frac{\beta}{\alpha}<\frac{1}{2}\right), \end{aligned}$$

where b_L and c_L are given by (3.5) and (3.8), respectively.

Finally, we apply the inequalities (3.9) and (3.10) in our integral representation (2.8). We thus obtain the following bounds for the generalized Mathieu series occurring in (2.8):

$$\tag{3.11} \begin{aligned} \mathcal{S}_\mu^{(\alpha,\beta)}\left(r;\left\{k^{2/\alpha}\right\}_{k=1}^\infty\right) &\leqq \frac{b_L\sqrt{\pi}}{(2r)^{\mu-(\beta/\alpha)-1}}\left(\mu-\frac{\beta}{\alpha}-1\right)^{-\frac{1}{3}} \\ &\cdot\frac{\Gamma\left(\mu-\frac{\beta}{\alpha}+1\right)\Gamma\left(\frac{\mu}{2}+\frac{\beta}{2\alpha}+\frac{1}{2}\right)}{\Gamma(\mu)\Gamma\left(\frac{\mu}{2}-\frac{\beta}{2\alpha}+1\right)}\zeta\left(\mu-\frac{\beta}{\alpha}+1\right) \\ &\left(r,x,\alpha,\beta\in\mathbb{R}^+;\ \frac{\beta}{\alpha}<\frac{1}{2};\ \mu-\frac{\beta}{\alpha}>1\right) \end{aligned}$$

and

$$\tag{3.12} \begin{aligned} \mathcal{S}_\mu^{(\alpha,\beta)}\left(r;\left\{k^{2/\alpha}\right\}_{k=1}^\infty\right) &\leqq \frac{c_L\sqrt{\pi}}{2^{\mu-(\beta/\alpha)-1}\,r^{\mu-(\beta/\alpha)-\frac{2}{3}}} \\ &\cdot\frac{\Gamma\left(\mu-\frac{\beta}{\alpha}+\frac{2}{3}\right)\Gamma\left(\frac{\mu}{2}+\frac{\beta}{2\alpha}+\frac{1}{2}\right)}{\Gamma(\mu)\Gamma\left(\frac{\mu}{2}-\frac{\beta}{2\alpha}+1\right)}\zeta\left(\mu-\frac{\beta}{\alpha}+\frac{2}{3}\right) \\ &\left(r,x,\alpha,\beta\in\mathbb{R}^+;\ \frac{\beta}{\alpha}<\frac{1}{2};\ \mu-\frac{\beta}{\alpha}>1\right), \end{aligned}$$

where we have employed the integral representation (2.2) for the Riemann Zeta function $\zeta(s)$, b_L and c_L being given (as before) by (3.5) and (3.8), respectively.

In their special case when

$$\beta\longrightarrow\frac{1}{2}\alpha \qquad \text{and} \qquad \mu\longmapsto\mu+1,$$

the bounds in (3.11) and (3.12) would correspond naturally to those given earlier by Cerone and Lenard [3, p. 7, Theorem 3.1]. The *second* bound asserted by Cerone and Lenard [3, p.

7, Equation (3.12)] should, in fact, be corrected to include $\Gamma(\mu+1)$ in the *denominator* on the right-hand side.

4. INEQUALITIES ASSOCIATED WITH GENERALIZED MATHIEU SERIES

We first prove the following inequality which was recently posed as an *open* problem by Qi [17, p. 7, Open Problem 2]:

$$\left(\int_0^\infty \frac{x\sin(rx)}{e^x-1}\,dx\right)^2 > 2r^2\int_0^\infty x^2 e^{-r^2x} f(x)\,dx \qquad (4.1)$$
$$\left(r\in\mathbb{R}^+;\ f(x) := \sum_{n=1}^\infty n e^{-n^2x}\right),$$

which, in view of the integral representation (1.4), is equivalent to the inequality:

$$[S(r)]^2 > 2\int_0^\infty x^2\, e^{-r^2x}\, f(x)\,dx, \qquad (4.2)$$

where $f(x)$ is defined as in (4.1).

Proof. Since the infinite series:

$$\sum_{n=1}^\infty n e^{-\left(n^2+r^2\right)x}$$

is uniformly convergent when $x\in\mathbb{R}^+$, for the right-hand side of the inequality (4.2), we have

$$\begin{aligned} 2\int_0^\infty x^2\, e^{-r^2x}\, f(x)\,dx &= 2\int_0^\infty x^2\left(\sum_{n=1}^\infty n e^{-\left(n^2+r^2\right)x}\right)dx \\ &= 2\sum_{n=1}^\infty n\int_0^\infty x^2\, e^{-\left(n^2+r^2\right)x}\,dx \\ &= 4\sum_{n=1}^\infty \frac{n}{\left(n^2+r^2\right)^3} =: 2\mathbb{S}_3(r), \end{aligned}$$

where we have used the Eulerian integral formula (2.25). Hence it is sufficient to prove the following inequality:

$$[S(r)]^2 > 2\mathbb{S}_3(r), \qquad (4.3)$$

which was, in fact, conjectured by Alzer and Brenner [2] and proven by Wilkins [27] by remarkably applying series and integral representations for the Trigamma function $\psi'(z)$ defined by (2.14) for $m=1$. □

We conclude our present investigation by remarking that it seems to be very likely that the inequality (4.1) can be generalized to the following form:

Open Problem. Prove or disprove that

$$\left(\int_0^\infty \frac{x\sin(rx)}{e^x-1}\,dx\right)^\mu > r^\mu\,\Gamma(\mu+1)\int_0^\infty x^\mu\,e^{-r^2x}\,f(x)\,dx \tag{4.4}$$
$$\left(r,\mu\in\mathbb{R}^+;\ f(x):=\sum_{n=1}^\infty ne^{-n^2x}\right)$$

or, equivalently, that

$$[S(r)]^\mu > \frac{\{\Gamma(\mu+1)\}^2}{2}\,\mathbb{S}_{\mu+1}(r) \tag{4.5}$$
$$\left(r,\mu\in\mathbb{R}^+\right),$$

since

$$\int_0^\infty x^\mu\,e^{-r^2x}\,f(x)\,dx = \Gamma(\mu+1)\sum_{n=1}^\infty \frac{n}{(n^2+r^2)^{\mu+1}} =: \frac{\Gamma(\mu+1)}{2}\,\mathbb{S}_{\mu+1}(r), \tag{4.6}$$

by virtue of the Eulerian integral formula (2.25) once again.

The open problem (4.1), which we have completely solved here, corresponds to the *special* case $\mu=2$ of the Open Problem (4.4) posed in this paper.

References

[1] H. ALZER AND J. BRENNER, An inequality (Problem 97-1), *SIAM Rev.*, **39** (1997), 123.

[2] H. ALZER, J.L. BRENNER, AND O.G. RUEHR, On Mathieu's inequality, *J. Math. Anal. Appl.*, **218** (1998), 607–610.

[3] P. CERONE AND C.T. LENARD, On integral forms of generalised Mathieu series, *RGMIA Res. Rep. Coll.*, **6** (2) (2003), Art. 19, 1–11; see also *J. Inequal. Pure Appl. Math.*, **4** (5) (2003), Art. 100, 1–11 (electronic). ONLINE [`http://jipam.vu.edu.au/article.php?sid=341`].

[4] P.H. DIANANDA, Some inequalities related to an inequality of Mathieu, *Math. Ann.*, **250** (1980), 95-98.

[5] Á. ELBERT, Asymptotic expansion and continued fraction for Mathieu's series, *Period. Math. Hungar.*, **13** (1982), 1–8.

[6] O. EMERSLEBEN, Über die Reihe $\sum_{k=1}^\infty k/\left(k^2+c^2\right)^2$, *Math. Ann.*, **125** (1952), 165–171.

[7] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER, AND F.G. TRICOMI, *Higher Transcendental Functions*, Vol. **II**, McGraw-Hill Book Company, New York, Toronto, and London, 1953.

[8] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER, AND F.G. TRICOMI, *Tables of Integral Transforms*, Vol. **II**, McGraw-Hill Book Company, New York, Toronto, and London, 1954.

[9] I. GAVREA, Some remarks on Mathieu's series, in *Mathematical Analysis and Approximation Theory* (Fifth Romanian-German Seminar on Approximation Theory and Its Applications; Sibiu, Romania, June 12-15, 2002) (A. Lupaş, H. Gonska, and L. Lupaş, Editors), pp. 113–117, Burg Verlag, Sibiu, Romania, 2002.

[10] B.-N. GUO, Note on Mathieu's inequality, *RGMIA Res. Rep. Coll.*, **3** (3) (2000), Article 5, 1–3. ONLINE [`http://rgmia.vu.edu.au/v3n3.html`].

[11] E.R. HANSEN, *A Table of Series and Products*, Prentice-Hall, Englewood Cliffs, New Jersey, 1975.

[12] L. LANDAU, Monotonicity and bounds on Bessel functions, in *Proceedings of the Symposium on Mathematical Physics and Quantum Field Theory* (Berkeley, California; June 11-13, 1999) (H. Warchall, Editor), pp. 147-154; *Electron. J. Differential Equations Conf.*, **4**, Southwest Texas State University, San Marcos, Texas, 2000.

[13] É.L. MATHIEU, *Traité de Physique Mathématique*. VI-VII: *Théory de l'Élasticité des Corps Solides* (Part 2), Gauthier-Villars, Paris, 1890.

[14] T.K. POGÁNY, Integral representation of Mathieu ($\mathbf{a}$,λ)-series, *RGMIA Res. Rep. Coll.*, **7** (1) (2004), Article 9, 1–5. ONLINE [`http://rgmia.vu.edu.au/v7n1.html`].

[15] A.P. PRUDNIKOV, YU. A. BRYČKOV, AND O.I. MARIČEV, *Integrals and Series* (*Elementary Functions*), "Nauka", Moscow, 1981 (Russian); English translation: *Integrals and Series*, Vol. **1**: *Elementary Functions*, Gordon and Breach Science Publishers, New York, 1986.

[16] F. QI, Inequalities for Mathieu's series, *RGMIA Res. Rep. Coll.*, **4** (2) (2001), Article 3, 1–7. ONLINE [`http://rgmia.vu.edu.au/v4n2.html`].

[17] F. QI, Integral expression and inequalities of Mathieu type series, *RGMIA Res. Rep. Coll.*, **6** (2) (2003), Article 10, 1–8. ONLINE [`http://rgmia.vu.edu.au/v6n2.html`].

[18] F. QI, An integral expression and some inequalities of Mathieu type series, *Rostock. Math. Kolloq.*, **58** (2004), 37–46.

[19] F. QI AND C.-P. CHEN, Notes on double inequalities of Mathieu's series. Preprint 2003.

[20] H.M. SRIVASTAVA AND J. CHOI, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, Boston, and London, 2001.

[21] H.M. SRIVASTAVA AND H.L. MANOCHA, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane, and Toronto, 1984.

[22] Ž. TOMOVSKI, New double inequalities for Mathieu type series, *RGMIA Res. Rep. Coll.*, **6** (2) (2003), Article 17, 1–4. ONLINE [`http://rgmia.vu.edu.au/v6n2.html`].

[23] Ž. TOMOVSKI AND K. TRENČEVSKI, On an open problem of Bai-Ni Guo and Feng Qi, *J. Inequal. Pure Appl. Math.*, **4** (2) (2003), Article 29, 1–7 (electronic). ONLINE [`http://jipam.vu.edu.au/article.php?sid=267`].

[24] C.-L. WANG AND X.-H. WANG, A refinement of the Mathieu inequality, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., No.* **716-734** (1981), 22–24.

[25] G.N. WATSON, *A Treatise on the Theory of Bessel Functions*, Second Edition, Cambridge University Press, Cambridge, London, and New York, 1944.

[26] A.D. WHEELON, *Tables of Summable Series and Integrals Involving Bessel Functions*, Holden-Day, San Francisco, London, and Amsterdam, 1968.

[27] J.E. WILKINS, Jr., An inequality (Solution of Problem 97-1 posed by H. Alzer and J. Brenner), *SIAM Rev.*, **40** (1998), 126–128.