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ON THE STAR PARTIAL ORDERING OF NORMAL MATRICES

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ABSTRACT. We order the space of complex $n \times n$ matrices by the star partial ordering $\leq^*$. So $\mathbf{A} \leq^* \mathbf{B}$ means that $\mathbf{A}^*\mathbf{A} = \mathbf{A}^*\mathbf{B}$ and $\mathbf{A}\mathbf{A}^* = \mathbf{B}\mathbf{A}^*$. We find several characterizations for $\mathbf{A} \leq^* \mathbf{B}$ in the case of normal matrices. As an application, we study how $\mathbf{A} \leq^* \mathbf{B}$ relates to $\mathbf{A}^2 \leq^* \mathbf{B}^2$. The results can be extended to study how $\mathbf{A} \leq^* \mathbf{B}$ relates to $\mathbf{A}^k \leq^* \mathbf{B}^k$, where $k \geq 2$ is an integer.

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1. INTRODUCTION

Throughout this paper, we consider the space of complex $n \times n$ matrices ($n \geq 2$). We order it by the star partial ordering $\leq^*$. So $\mathbf{A} \leq^* \mathbf{B}$ means that $\mathbf{A}^*\mathbf{A} = \mathbf{A}^*\mathbf{B}$ and $\mathbf{A}\mathbf{A}^* = \mathbf{B}\mathbf{A}^*$. Our motivation rises from the following

Theorem 1.1 (Baksalary and Pukelsheim [1, Theorem 3])**.** *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be Hermitian and nonnegative definite. Then* $\mathbf{A}^2 \leq^* \mathbf{B}^2$ *if and only if* $\mathbf{A} \leq^* \mathbf{B}$.

We cannot drop out the assumption on nonnegative definiteness.

Example 1.1. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $\mathbf{A}^2 \leq^* \mathbf{B}^2$, but not $\mathbf{A} \leq^* \mathbf{B}$.

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We will study how $\mathbf{A} \leq^* \mathbf{B}$ relates to $\mathbf{A}^2 \leq^* \mathbf{B}^2$ in the case of normal matrices. We will see (Theorem 3.1) that the "if" part of Theorem 1.1 remains valid. However, it is not valid for all matrices.

Example 1.2. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}.$$

Then $\mathbf{A} \leq^* \mathbf{B}$, but not $\mathbf{A}^2 \leq^* \mathbf{B}^2$.

In Section 2, we will give several characterizations of $\mathbf{A} \leq^* \mathbf{B}$. Thereafter, in Section 3, we will apply some of them in discussing our problem. Finally, in Section 4, we will complete our paper with some remarks.

2. CHARACTERIZATIONS OF $\mathbf{A} \leq^* \mathbf{B}$

Hartwig and Styan ([2, Theorem 2]) presented eleven characterizations of $\mathbf{A} \leq^* \mathbf{B}$ for general matrices. One of them uses singular value decompositions. In the case of normal matrices, spectral decompositions are more accessible.

Theorem 2.1. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be normal matrices with* $1 \leq \operatorname{rank} \mathbf{A} < \operatorname{rank} \mathbf{B}$. *Then the following conditions are equivalent:*

(a) $\mathbf{A} \leq^* \mathbf{B}$.

(b) *There is a unitary matrix* $\mathbf{U}$ *such that*

$$\mathbf{U}^*\mathbf{A}\mathbf{U} = \begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \qquad \mathbf{U}^*\mathbf{B}\mathbf{U} = \begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{E} \end{pmatrix},$$

where $\mathbf{D}$ *is a nonsingular diagonal matrix and* $\mathbf{E} \neq \mathbf{O}$ *is a diagonal matrix.*

(c) *There is a unitary matrix* $\mathbf{U}$ *such that*

$$\mathbf{U}^*\mathbf{A}\mathbf{U} = \begin{pmatrix} \mathbf{F} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \qquad \mathbf{U}^*\mathbf{B}\mathbf{U} = \begin{pmatrix} \mathbf{F} & \mathbf{O} \\ \mathbf{O} & \mathbf{G} \end{pmatrix},$$

where $\mathbf{F}$ *is a nonsingular square matrix and* $\mathbf{G} \neq \mathbf{O}$.

(d) *If a unitary matrix* $\mathbf{U}$ *satisfies*

$$\mathbf{U}^*\mathbf{A}\mathbf{U} = \begin{pmatrix} \mathbf{F} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \qquad \mathbf{U}^*\mathbf{B}\mathbf{U} = \begin{pmatrix} \mathbf{F}' & \mathbf{O} \\ \mathbf{O} & \mathbf{G} \end{pmatrix},$$

where $\mathbf{F}$ *is a nonsingular square matrix,* $\mathbf{F}'$ *is a square matrix of the same dimension, and* $\mathbf{G} \neq \mathbf{O}$, *then* $\mathbf{F} = \mathbf{F}'$.

(e) *If a unitary matrix* $\mathbf{U}$ *satisfies*

$$\mathbf{U}^*\mathbf{A}\mathbf{U} = \begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \qquad \mathbf{U}^*\mathbf{B}\mathbf{U} = \begin{pmatrix} \mathbf{D}' & \mathbf{O} \\ \mathbf{O} & \mathbf{E} \end{pmatrix},$$

where $\mathbf{D}$ *is a nonsingular diagonal matrix,* $\mathbf{D}'$ *is a diagonal matrix of the same dimension, and* $\mathbf{E} \neq \mathbf{O}$ *is a diagonal matrix, then* $\mathbf{D} = \mathbf{D}'$.

(f) *If a unitary matrix* $\mathbf{U}$ *satisfies*

$$\mathbf{U}^*\mathbf{A}\mathbf{U} = \begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix},$$

where $\mathbf{D}$ *is a nonsingular diagonal matrix, then*

$$\mathbf{U}^*\mathbf{B}\mathbf{U} = \begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{G} \end{pmatrix},$$

where $\mathbf{G} \neq \mathbf{O}$.

(g) *All eigenvectors corresponding to nonzero eigenvalues of* $\mathbf{A}$ *are eigenvectors of* $\mathbf{B}$ *corresponding to the same eigenvalues.*

The reason to assume $1 \leq \operatorname{rank} \mathbf{A} < \operatorname{rank} \mathbf{B}$ is to omit the trivial cases $\mathbf{A} = \mathbf{O}$ and $\mathbf{A} = \mathbf{B}$.

Proof. We prove this theorem in four parts.

Part 1. (a) ⇒ (b) ⇒ (c) ⇒ (a).

(a) ⇒ (b). Assume (a). Then, by normality, $\mathbf{A}^*$ and $\mathbf{B}$ commute and are therefore simultaneously diagonalizable (see, e.g., [3, Theorem 1.3.19]). Since $\mathbf{A}$ and $\mathbf{A}^*$ have the same eigenvectors (see, e.g., [3, Problem 2.5.20]), also $\mathbf{A}$ and $\mathbf{B}$ are simultaneously diagonalizable. Hence (recall the assumption on the ranks) there exists a unitary matrix $\mathbf{U}$ such that

$$\mathbf{U}^*\mathbf{A}\mathbf{U} = \begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \qquad \mathbf{U}^*\mathbf{B}\mathbf{U} = \begin{pmatrix} \mathbf{D}' & \mathbf{O} \\ \mathbf{O} & \mathbf{E} \end{pmatrix},$$

where $\mathbf{D}$ is a nonsingular diagonal matrix, $\mathbf{D}'$ is a diagonal matrix of the same dimension, and $\mathbf{E} \neq \mathbf{O}$ is a diagonal matrix. Now $\mathbf{A}^*\mathbf{A} = \mathbf{A}^*\mathbf{B}$ implies $\mathbf{D}^*\mathbf{D} = \mathbf{D}^*\mathbf{D}'$ and further $\mathbf{D} = \mathbf{D}'$. Hence (b) is valid.

(b) ⇒ (c). Trivial.

(c) ⇒ (a). Direct calculation.

Part 2. (a) ⇒ (d) ⇒ (e) ⇒ (a).

This is a trivial modification of Part 1.

Part 3. (b) ⇔ (f).

(b) ⇒ (f). Assume (b). Let $\mathbf{U}$ be a unitary matrix satisfying

$$\mathbf{U}^*\mathbf{A}\mathbf{U} = \begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}.$$

By (b), there exists a unitary matrix $\mathbf{V}$ such that

$$\mathbf{V}^*\mathbf{A}\mathbf{V} = \begin{pmatrix} \mathbf{D}' & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \qquad \mathbf{V}^*\mathbf{B}\mathbf{V} = \begin{pmatrix} \mathbf{D}' & \mathbf{O} \\ \mathbf{O} & \mathbf{E} \end{pmatrix},$$

where $\mathbf{D}'$ is a nonsingular diagonal matrix and $\mathbf{E} \neq \mathbf{O}$ is a diagonal matrix. Interchanging the columns of $\mathbf{V}$ if necessary, we can assume $\mathbf{D}' = \mathbf{D}$.

Let $\mathbf{U} = \begin{pmatrix} \mathbf{U}_1 & \mathbf{U}_2 \end{pmatrix}$ be such a partition that

$$\mathbf{U}^*\mathbf{A}\mathbf{U} = \begin{pmatrix} \mathbf{U}_1^* \\ \mathbf{U}_2^* \end{pmatrix} \mathbf{A} \begin{pmatrix} \mathbf{U}_1 & \mathbf{U}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{U}_1^*\mathbf{A}\mathbf{U}_1 & \mathbf{U}_1^*\mathbf{A}\mathbf{U}_2 \\ \mathbf{U}_2^*\mathbf{A}\mathbf{U}_1 & \mathbf{U}_2^*\mathbf{A}\mathbf{U}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}.$$

Then, for the corresponding partition $\mathbf{V} = \begin{pmatrix} \mathbf{V}_1 & \mathbf{V}_2 \end{pmatrix}$, we have

$$\mathbf{V}^*\mathbf{A}\mathbf{V} = \begin{pmatrix} \mathbf{V}_1^* \\ \mathbf{V}_2^* \end{pmatrix} \mathbf{A} \begin{pmatrix} \mathbf{V}_1 & \mathbf{V}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{V}_1^*\mathbf{A}\mathbf{V}_1 & \mathbf{V}_1^*\mathbf{A}\mathbf{V}_2 \\ \mathbf{V}_2^*\mathbf{A}\mathbf{V}_1 & \mathbf{V}_2^*\mathbf{A}\mathbf{V}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}$$

and

$$\mathbf{V}^*\mathbf{B}\mathbf{V} = \begin{pmatrix} \mathbf{V}_1^* \\ \mathbf{V}_2^* \end{pmatrix} \mathbf{B} \begin{pmatrix} \mathbf{V}_1 & \mathbf{V}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{V}_1^*\mathbf{B}\mathbf{V}_1 & \mathbf{V}_1^*\mathbf{B}\mathbf{V}_2 \\ \mathbf{V}_2^*\mathbf{B}\mathbf{V}_1 & \mathbf{V}_2^*\mathbf{B}\mathbf{V}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{E} \end{pmatrix}.$$

Noting that

$$\mathbf{A} = \begin{pmatrix} \mathbf{V}_1 & \mathbf{V}_2 \end{pmatrix} \begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{V}_1^* \\ \mathbf{V}_2^* \end{pmatrix} = \begin{pmatrix} \mathbf{V}_1 & \mathbf{V}_2 \end{pmatrix} \begin{pmatrix} \mathbf{D}\mathbf{V}_1^* \\ \mathbf{O} \end{pmatrix} = \mathbf{V}_1\mathbf{D}\mathbf{V}_1^*,$$

we therefore have

$$\begin{aligned}\mathbf{U}^*\mathbf{B}\mathbf{U} &= \begin{pmatrix}\mathbf{U}_1^*\\ \mathbf{U}_2^*\end{pmatrix}\begin{pmatrix}\mathbf{V}_1 & \mathbf{V}_2\end{pmatrix}\begin{pmatrix}\mathbf{D} & \mathbf{O}\\ \mathbf{O} & \mathbf{E}\end{pmatrix}\begin{pmatrix}\mathbf{V}_1^*\\ \mathbf{V}_2^*\end{pmatrix}\begin{pmatrix}\mathbf{U}_1 & \mathbf{U}_2\end{pmatrix}\\ &= \begin{pmatrix}\mathbf{U}_1^*\\ \mathbf{U}_2^*\end{pmatrix}\begin{pmatrix}\mathbf{V}_1 & \mathbf{V}_2\end{pmatrix}\begin{pmatrix}\mathbf{D}\mathbf{V}_1^*\\ \mathbf{E}\mathbf{V}_2^*\end{pmatrix}\begin{pmatrix}\mathbf{U}_1 & \mathbf{U}_2\end{pmatrix}\\ &= \begin{pmatrix}\mathbf{U}_1^*\mathbf{V}_1 & \mathbf{U}_1^*\mathbf{V}_2\\ \mathbf{U}_2^*\mathbf{V}_1 & \mathbf{U}_2^*\mathbf{V}_2\end{pmatrix}\begin{pmatrix}\mathbf{D}\mathbf{V}_1^*\mathbf{U}_1 & \mathbf{D}\mathbf{V}_1^*\mathbf{U}_2\\ \mathbf{E}\mathbf{V}_2^*\mathbf{U}_1 & \mathbf{E}\mathbf{V}_2^*\mathbf{U}_2\end{pmatrix}\\ &= \begin{pmatrix}\mathbf{U}_1^*\mathbf{V}_1 & \mathbf{O}\\ \mathbf{O} & \mathbf{U}_2^*\mathbf{V}_2\end{pmatrix}\begin{pmatrix}\mathbf{D}\mathbf{V}_1^*\mathbf{U}_1 & \mathbf{O}\\ \mathbf{O} & \mathbf{E}\mathbf{V}_2^*\mathbf{U}_2\end{pmatrix}\\ &= \begin{pmatrix}\mathbf{U}_1^*\mathbf{V}_1\mathbf{D}\mathbf{V}_1^*\mathbf{U}_1 & \mathbf{O}\\ \mathbf{O} & \mathbf{U}_2^*\mathbf{V}_2\mathbf{E}\mathbf{V}_2^*\mathbf{U}_2\end{pmatrix}\\ &= \begin{pmatrix}\mathbf{U}_1^*\mathbf{A}\mathbf{U}_1 & \mathbf{O}\\ \mathbf{O} & \mathbf{U}_2^*\mathbf{V}_2\mathbf{E}\mathbf{V}_2^*\mathbf{U}_2\end{pmatrix} = \begin{pmatrix}\mathbf{D} & \mathbf{O}\\ \mathbf{O} & \mathbf{U}_2^*\mathbf{V}_2\mathbf{E}\mathbf{V}_2^*\mathbf{U}_2\end{pmatrix},\end{aligned}$$

and so (f) follows.

(f) $\Rightarrow$ (b). Assume (f). Let $\mathbf{U}$ be a unitary matrix such that

$$\mathbf{U}^*\mathbf{A}\mathbf{U} = \begin{pmatrix}\mathbf{D} & \mathbf{O}\\ \mathbf{O} & \mathbf{O}\end{pmatrix},$$

where $\mathbf{D}$ is a nonsingular diagonal matrix. Then, by (f),

$$\mathbf{U}^*\mathbf{B}\mathbf{U} = \begin{pmatrix}\mathbf{D} & \mathbf{O}\\ \mathbf{O} & \mathbf{G}\end{pmatrix},$$

where $\mathbf{G} \neq \mathbf{O}$. Since $\mathbf{G}$ is normal, there exists a unitary matrix $\mathbf{W}$ such that $\mathbf{E} = \mathbf{W}^*\mathbf{G}\mathbf{W}$ is a diagonal matrix. Let

$$\mathbf{V} = \mathbf{U}\begin{pmatrix}\mathbf{I} & \mathbf{O}\\ \mathbf{O} & \mathbf{W}\end{pmatrix}.$$

Then

$$\begin{aligned}\mathbf{V}^*\mathbf{A}\mathbf{V} &= \begin{pmatrix}\mathbf{I} & \mathbf{O}\\ \mathbf{O} & \mathbf{W}^*\end{pmatrix}\mathbf{U}^*\mathbf{A}\mathbf{U}\begin{pmatrix}\mathbf{I} & \mathbf{O}\\ \mathbf{O} & \mathbf{W}\end{pmatrix}\\ &= \begin{pmatrix}\mathbf{I} & \mathbf{O}\\ \mathbf{O} & \mathbf{W}^*\end{pmatrix}\begin{pmatrix}\mathbf{D} & \mathbf{O}\\ \mathbf{O} & \mathbf{O}\end{pmatrix}\begin{pmatrix}\mathbf{I} & \mathbf{O}\\ \mathbf{O} & \mathbf{W}\end{pmatrix} = \begin{pmatrix}\mathbf{D} & \mathbf{O}\\ \mathbf{O} & \mathbf{O}\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\mathbf{V}^*\mathbf{B}\mathbf{V} &= \begin{pmatrix}\mathbf{I} & \mathbf{O}\\ \mathbf{O} & \mathbf{W}^*\end{pmatrix}\mathbf{U}^*\mathbf{B}\mathbf{U}\begin{pmatrix}\mathbf{I} & \mathbf{O}\\ \mathbf{O} & \mathbf{W}\end{pmatrix}\\ &= \begin{pmatrix}\mathbf{I} & \mathbf{O}\\ \mathbf{O} & \mathbf{W}^*\end{pmatrix}\begin{pmatrix}\mathbf{D} & \mathbf{O}\\ \mathbf{O} & \mathbf{G}\end{pmatrix}\begin{pmatrix}\mathbf{I} & \mathbf{O}\\ \mathbf{O} & \mathbf{W}\end{pmatrix} = \begin{pmatrix}\mathbf{D} & \mathbf{O}\\ \mathbf{O} & \mathbf{E}\end{pmatrix}.\end{aligned}$$

Thus (b) follows.

Part 4. (b) $\Leftrightarrow$ (g).

This is an elementary fact. □

Corollary 2.2. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be normal matrices. If* $\mathbf{A} \leq^* \mathbf{B}$, *then* $\mathbf{A}\mathbf{B} = \mathbf{B}\mathbf{A}$.

Proof. Apply (b). □

The converse does not hold (even assuming $\operatorname{rank}\mathbf{A} < \operatorname{rank}\mathbf{B}$), see Example 2.1. The normality assumption cannot be dropped out, see Example 2.2.

Example 2.1. Let

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\mathbf{AB} = \mathbf{BA}$ and $\operatorname{rank} \mathbf{A} < \operatorname{rank} \mathbf{B}$, but $\mathbf{A} \leq^* \mathbf{B}$ does not hold. However, $\frac{1}{2}\mathbf{A} \leq^* \mathbf{B}$, which makes us look for a counterexample such that $c\mathbf{A} \leq^* \mathbf{B}$ does not hold for any $c \neq 0$. It is easy to see that we must have $n \geq 3$. The matrices

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

obviously have the required properties.

Example 2.2. Let

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $\mathbf{A} \leq^* \mathbf{B}$, but $\mathbf{AB} \neq \mathbf{BA}$.

3. RELATIONSHIP BETWEEN $\mathbf{A} \leq^* \mathbf{B}$ AND $\mathbf{A}^2 \leq^* \mathbf{B}^2$

We will see that $\mathbf{A} \leq^* \mathbf{B} \Rightarrow \mathbf{A}^2 \leq^* \mathbf{B}^2$ for normal matrices, but the converse needs an extra condition, which we first present using eigenvalues.

Theorem 3.1. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be normal matrices with* $1 \leq \operatorname{rank} \mathbf{A} < \operatorname{rank} \mathbf{B}$. *Then*

(a) $$\mathbf{A} \leq^* \mathbf{B}$$

is equivalent to the following:

(b) $$\mathbf{A}^2 \leq^* \mathbf{B}^2$$

and if $\mathbf{A}$ *and* $\mathbf{B}$ *have nonzero eigenvalues* α *and respectively* β *such that* α^2 *and* β^2 *are eigenvalues of* $\mathbf{A}^2$ *and respectively* $\mathbf{B}^2$ *with a common eigenvector* $\mathbf{x}$, *then* $\alpha = \beta$ *and* $\mathbf{x}$ *is a common eigenvector of* $\mathbf{A}$ *and* $\mathbf{B}$.

Proof. Assuming (a), we have

$$\mathbf{U}^*\mathbf{A}\mathbf{U} = \begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \qquad \mathbf{U}^*\mathbf{B}\mathbf{U} = \begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{E} \end{pmatrix}$$

as in (b) of Theorem 2.1, and so

$$\mathbf{U}^*\mathbf{A}^2\mathbf{U} = \begin{pmatrix} \mathbf{D}^2 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \qquad \mathbf{U}^*\mathbf{B}^2\mathbf{U} = \begin{pmatrix} \mathbf{D}^2 & \mathbf{O} \\ \mathbf{O} & \mathbf{E}^2 \end{pmatrix}.$$

Hence, by Theorem 2.1, the first part of (b) follows. The second part is trivial.

Conversely, assume (b). Then

$$\mathbf{U}^*\mathbf{A}^2\mathbf{U} = \begin{pmatrix} \mathbf{\Delta} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \qquad \mathbf{U}^*\mathbf{B}^2\mathbf{U} = \begin{pmatrix} \mathbf{\Delta} & \mathbf{O} \\ \mathbf{O} & \mathbf{\Gamma} \end{pmatrix},$$

where $\mathbf{U}$, $\mathbf{\Delta}$, and $\mathbf{\Gamma}$ are matrices obtained by applying (b) of Theorem 2.1 to $\mathbf{A}^2$ and $\mathbf{B}^2$. Let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ be the column vectors of $\mathbf{U}$ and denote $r = \operatorname{rank} \mathbf{A}$.

For $i = 1, \ldots, r$, we have $\mathbf{A}^2\mathbf{u}_i = \mathbf{B}^2\mathbf{u}_i = \delta_i\mathbf{u}_i$, where $(\delta_i) = \operatorname{diag} \mathbf{\Delta}$. So, by the second part of (b), there exist complex numbers $d_1, \ldots, d_r$ such that, for all $i = 1, \ldots, r$, we have $d_i^2 = \delta_i$ and $\mathbf{A}\mathbf{u}_i = \mathbf{B}\mathbf{u}_i = \delta_i\mathbf{u}_i$. Let $\mathbf{D}$ be the diagonal matrix with $(d_i) = \operatorname{diag} \mathbf{D}$.

For $i = r+1, \dots, n$, we have $\mathbf{B}^2\mathbf{u}_i = \gamma_{i-r}\mathbf{u}_i$, where $(\gamma_j) = \operatorname{diag}\mathbf{\Gamma}$. Take complex numbers $e_1, \dots, e_{n-r}$ satisfying $e_i^2 = \gamma_i$ for $i = 1, \dots, n-r$. Let $\mathbf{E}$ be the diagonal matrix with $(e_i) = \operatorname{diag}\mathbf{E}$. Then

$$\mathbf{U}^*\mathbf{A}\mathbf{U} = \begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \qquad \mathbf{U}^*\mathbf{B}\mathbf{U} = \begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{E} \end{pmatrix},$$

and (a) follows from Theorem 2.1. □

As an immediate corollary, we obtain a generalization of Theorem 1.1.

Corollary 3.2. *Let $\mathbf{A}$ and $\mathbf{B}$ be normal matrices whose all eigenvalues have nonnegative real parts. Then $\mathbf{A}^2 \le^* \mathbf{B}^2$ if and only if $\mathbf{A} \le^* \mathbf{B}$.*

Next, we present the extra condition using diagonalization.

Theorem 3.3. *Let $\mathbf{A}$ and $\mathbf{B}$ be normal matrices with $1 \le \operatorname{rank}\mathbf{A} < \operatorname{rank}\mathbf{B}$. Then*

(a) $$\mathbf{A} \le^* \mathbf{B}$$

is equivalent to the following:

(b) $$\mathbf{A}^2 \le^* \mathbf{B}^2$$

and if

$$\mathbf{U}^*\mathbf{A}\mathbf{U} = \begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \qquad \mathbf{U}^*\mathbf{B}\mathbf{U} = \begin{pmatrix} \mathbf{D}\mathbf{H} & \mathbf{O} \\ \mathbf{O} & \mathbf{E} \end{pmatrix},$$

where $\mathbf{U}$ is a unitary matrix, $\mathbf{D}$ is a nonsingular diagonal matrix, $\mathbf{H}$ is a unitary diagonal matrix, and $\mathbf{E} \ne \mathbf{O}$ is a diagonal matrix, then $\mathbf{H} = \mathbf{I}$.

(Note that the second part of (b) is weaker than (e) of Theorem 2.1. Otherwise Theorem 3.3 would be nonsense.)

Proof. For (a) $\Rightarrow$ the first part of (b), see the proof of Theorem 3.1. For (a) $\Rightarrow$ the second part of (b), see (e) of Theorem 2.1.

Conversely, assume (b). As in the proof of Theorem 3.1, we have

$$\mathbf{U}^*\mathbf{A}^2\mathbf{U} = \begin{pmatrix} \mathbf{\Delta} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \qquad \mathbf{U}^*\mathbf{B}^2\mathbf{U} = \begin{pmatrix} \mathbf{\Delta} & \mathbf{O} \\ \mathbf{O} & \mathbf{\Gamma} \end{pmatrix}.$$

Hence

$$\mathbf{U}^*\mathbf{A}\mathbf{U} = \begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \qquad \mathbf{U}^*\mathbf{B}\mathbf{U} = \begin{pmatrix} \mathbf{D}' & \mathbf{O} \\ \mathbf{O} & \mathbf{E} \end{pmatrix},$$

where $\mathbf{D}$ and $\mathbf{D}'$ are diagonal matrices satisfying $\mathbf{D}^2 = (\mathbf{D}')^2 = \mathbf{\Delta}$ and $\mathbf{E}$ is a diagonal matrix satisfying $\mathbf{E}^2 = \mathbf{\Gamma}$.

Denoting $(d_i) = \operatorname{diag}\mathbf{D}$, $(d_i') = \operatorname{diag}\mathbf{D}'$, $r = \operatorname{rank}\mathbf{A}$, we therefore have $d_i^2 = (d_i')^2$ for all $i = 1, \dots, r$. Hence there are complex numbers $h_1, \dots, h_r$ such that $|h_1| = \cdots = |h_r| = 1$ and $d_i' = d_i h_i$ for all $i = 1, \dots, r$. Let $\mathbf{H}$ be the diagonal matrix with $(h_i) = \operatorname{diag}\mathbf{H}$. Then $\mathbf{D}' = \mathbf{D}\mathbf{H}$, and so $\mathbf{D}' = \mathbf{D}$ by the second part of (b). Thus (b) of Theorem 2.1 is satisfied, and so (a) follows. □

4. REMARKS

We complete our paper with four remarks.

Remark 4.1. Let $k \geq 2$ be an integer. A natural further question is whether our discussion can be extended to describe how $\mathbf{A} \leq^* \mathbf{B}$ relates to $\mathbf{A}^k \leq^* \mathbf{B}^k$. As noted by Baksalary and Pukelsheim [1], Theorem 1.1 can be generalized in a similar way. In other words, for Hermitian nonnegative definite matrices, $\mathbf{A}^k \leq^* \mathbf{B}^k$ if and only if $\mathbf{A} \leq^* \mathbf{B}$. It can be seen also that Theorems 3.1 and 3.3 can be, with minor modifications, extended correspondingly.

Remark 4.2. Let $\mathbf{A}$ and $\mathbf{B}$ be arbitrary $n \times n$ matrices with $\operatorname{rank} \mathbf{A} < \operatorname{rank} \mathbf{B}$. Hartwig and Styan ([2, Theorem 2]) proved that $\mathbf{A} \leq^* \mathbf{B}$ if and only if there are unitary matrices $\mathbf{U}$ and $\mathbf{V}$ such that

$$\mathbf{U}^*\mathbf{A}\mathbf{V} = \begin{pmatrix} \mathbf{\Sigma} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \qquad \mathbf{U}^*\mathbf{B}\mathbf{V} = \begin{pmatrix} \mathbf{\Sigma} & \mathbf{O} \\ \mathbf{O} & \mathbf{\Theta} \end{pmatrix},$$

where $\mathbf{\Sigma}$ is a positive definite diagonal matrix and $\mathbf{\Theta} \neq \mathbf{O}$ is a nonnegative definite diagonal matrix. This is analogous to (a) $\Leftrightarrow$ (b) of Theorem 2.1. Actually it can be seen that all the characterizations of $\mathbf{A} \leq^* \mathbf{B}$ listed in Theorem 2.1 have singular value analogies in the general case.

Remark 4.3. The singular values of a normal matrix are absolute values of its eigenvalues (see e.g., [3, p. 417]). Hence it is relatively easy to see that if (and only if) $\mathbf{A}$ and $\mathbf{B}$ are normal, then $\mathbf{U}$ and $\mathbf{V}$ above can be chosen so that $\mathbf{U}^*\mathbf{V}$ is a diagonal matrix.

Remark 4.4. For normal matrices, it can be shown that Theorems 3.1 and 3.3 have singular value analogies. In the proof, it is crucial that $\mathbf{U}^*\mathbf{V}$ is a diagonal matrix. So these results do not remain valid without the normality assumption.

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