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ON INVERSES OF TRIANGULAR MATRICES WITH MONOTONE ENTRIES

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ABSTRACT. This note employs recurrence techniques to obtain entry-wise optimal inequalities for inverses of triangular matrices whose entries satisfy some monotonicity constraints. The derived bounds are easily computable.

Key words and phrases: Explicit bounds, Triangular matrix, Matrix inverse, Monotone entries, Off-diagonal decay, Recurrence relations.

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1. INTRODUCTION

Much work has been done in the recent past to understand off-diagonal decay properties of structured matrices and their inverses (cf. Benzi and Golub [1], Demko, Moss and Smith [4], Eijkhout and Polman [5], Jaffard [6], Nabben [7] and [8], Peluso and Politi [9], Robinson and Wathen [10], Strohmer [11], Vecchio [12] and the references therein).

This paper studies nonnegative triangular matrices with off-diagonal decay. In particular, let

$$\boldsymbol{L}_n = \begin{bmatrix} l_{1,1} & & & & \\ l_{2,1} & l_{2,2} & & & \\ l_{3,1} & l_{3,2} & l_{3,3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ l_{n,1} & l_{n,2} & l_{n,3} & \cdots & l_{n,n} \end{bmatrix}$$

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166-04

be an invertible lower triangular matrix, and

$$\boldsymbol{X}_n = \boldsymbol{L}_n^{-1} = \begin{bmatrix} x_{1,1} & & & & \\ x_{2,1} & x_{2,2} & & & \\ x_{3,1} & x_{3,2} & x_{3,3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ x_{n,1} & x_{n,2} & x_{n,3} & \cdots & x_{n,n} \end{bmatrix},$$

be its inverse.

We are interested in obtaining bounds on the entries in $\boldsymbol{X}_n$ under the row-wise monotonicity assumption

$$0 \le l_{i,1} \le l_{i,2} \le \cdots \le l_{i,i-1} \le l_{i,i} \tag{1.1}$$

for $2 \le i \le n$.

As an added generalization, we will consider $[l_{i,j}]$ satisfying

$$0 \le \frac{l_{i,1}}{l_{i,i}} \le \frac{l_{i,2}}{l_{i,i}} \le \cdots \le \frac{l_{i,i-1}}{l_{i,i}} \le \kappa_{i-1}, \tag{1.2}$$

for some nondecreasing sequence $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \kappa_3, \dots)$.

The paper proceeds as follows. Section 2 contains some recurrence-type lemmas, while the main result, Theorem 3.1, and its proof are contained in Section 3. The paper closes with some illustrative examples.

2. PRELIMINARY LEMMAS

In establishing our main results, we will employ recurrence techniques. In particular, suppose $\{b_i\}$ and $\{\alpha_{i,j}\}$ satisfy the linear recurrence

$$b_i = \sum_{k=0}^{i-1} (-\alpha_{i,k}) b_k, \ (1 \le i \le n), \tag{2.1}$$

with $b_0 = 1$ and

$$0 \le \alpha_{i,0} \le \alpha_{i,1} \le \alpha_{i,2} \le \cdots \le \alpha_{i,i-1} \le A_i, \tag{2.2}$$

for $i \ge 1$.

We will employ the following lemma, which reduces the scope of consideration in bounding solutions to (2.1).

Lemma 2.1. *Suppose that $\{b_i\}$ and $\{\alpha_{i,j}\}$ satisfy (2.1) and (2.2). Then, there exists a sequence $a_1, a_2, \dots, a_n$, with $0 \le a_i \le i$ for $1 \le i \le n$, such that $|b_n| \le |d_n|$, where $\{d_i\}$ satisfies $d_0 = 1$, and for $1 \le i \le n$,*

$$d_i = \begin{cases} \sum_{j=a_i}^{i-1} (-A_i) d_j, & \text{if } a_i < i \\ 0, & \text{otherwise} \end{cases}. \tag{2.3}$$

In proving Lemma 2.1, we will refer to the following result on inner products.

Lemma 2.2. *Suppose that $\boldsymbol{p} = (p_1, \dots, p_n)'$ and $\boldsymbol{q} = (q_1, \dots, q_n)'$ are n-vectors with*

$$0 \ge p_1 \ge p_2 \ge \cdots \ge p_n \ge -A. \tag{2.4}$$

Define

$$\boldsymbol{p}_n^*(\nu, A) = (\overbrace{0, 0, \dots, 0}^{\nu}, \overbrace{-A, \dots, -A, -A}^{n-\nu}) \tag{2.5}$$

for $0 \le \nu \le n$. *Then,*

$$\min_{0\le\nu\le n}\{\boldsymbol{p}_n^*(\nu, A)\cdot \boldsymbol{q}\} \le \boldsymbol{p}\cdot\boldsymbol{q} \le \max_{0\le\nu\le n}\{\boldsymbol{p}_n^*(\nu, A)\cdot\boldsymbol{q}\}, \tag{2.6}$$

where $\boldsymbol{p}\cdot\boldsymbol{q}$ *denotes the standard dot product* $\sum_{i=1}^n p_i q_i$.

Proof. Suppose $\boldsymbol{p}$ is of the form

$$(p_1, \dots, p_j, \overbrace{-k, \dots, -k}^{e_1}, \overbrace{-A, \dots, -A}^{e_2}), \tag{2.7}$$

with $0 \ge p_1 \ge p_2 \ge \dots \ge p_j > -k > -A$, $e_1 \ge 1$ and $e_2 \ge 0$. First, assume that $\boldsymbol{p}\cdot\boldsymbol{q} > 0$, and consider $S = \sum_{i=j+1}^{e_1+j} q_i$. If $S < 0$ then, since $k < A$,

$$(p_1, p_2, \dots, p_{j-1}, p_j, \overbrace{-A, \dots, -A}^{e_1}\overbrace{-A, \dots, -A}^{e_2})\cdot\boldsymbol{q} \ge \boldsymbol{p}\cdot\boldsymbol{q}. \tag{2.8}$$

Otherwise, since $-k < p_j$,

$$(p_1, p_2, \dots, p_{j-1}, p_j, \overbrace{p_j, \dots, p_j}^{e_1}, \overbrace{-A, \dots, -A}^{e_2})\cdot\boldsymbol{q} \ge \boldsymbol{p}\cdot\boldsymbol{q}. \tag{2.9}$$

In either case, there is a vector of the form in (2.7) with strictly less distinct values, whose inner product with $\boldsymbol{q}$ is at least as large as $\boldsymbol{p}\cdot\boldsymbol{q}$. Inductively, there exists a vector of the form in (2.7) with $e_2 + e_1 = n$, with as large, or larger, inner product. Hence, we have reduced to the case where $\boldsymbol{p} = (\overbrace{-k, \dots, -k}^{e_1}, \overbrace{-A, \dots, -A}^{e_2})$, where $e_1 = 0$ and $e_n = 0$ are permissible. If $k = 0$ or $e_1 = 0$, then $\boldsymbol{p} = \boldsymbol{p}_n^*(e_1, A)$. Otherwise, consider $S = \sum_{i=1}^{e_1} q_i$. If $S < 0$, then

$$\boldsymbol{p}_n^*(0, A)\cdot\boldsymbol{q} \ge \boldsymbol{p}\cdot\boldsymbol{q}. \tag{2.10}$$

If $S \ge 0$,

$$\boldsymbol{p}_n^*(e_1, A)\cdot\boldsymbol{q} \ge \boldsymbol{p}\cdot\boldsymbol{q}. \tag{2.11}$$

The result for the case $\boldsymbol{p}\cdot\boldsymbol{q} > 0$ now follows from (2.10) and (2.11).

The case when $\boldsymbol{p}\cdot\boldsymbol{q} \le 0$ is handled similarly, and the lemma follows. □

We now turn to a proof of Lemma 2.1.

Proof of Lemma 2.1. The proof, here, involves applying Lemma 2.2 to successively "scale" the rows of the coefficient matrix

$$\begin{bmatrix} -\alpha_{1,0} & 0 & \dots & 0 \\ -\alpha_{2,0} & -\alpha_{2,1} & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_{n,0} & -\alpha_{n,1} & \cdots & -\alpha_{n,n-1} \end{bmatrix},$$

while not decreasing the value of $|b_n|$ at any step.

First, define the sequences

$$\bar{\boldsymbol{\alpha}}_i = (-\alpha_{i,0}, \dots, -\alpha_{i,i-1}) \text{ and}$$
$$\boldsymbol{b}^{k,j} = (b_k, \dots, b_j),$$

for $0 \le k \le j \le n-1$ and $1 \le i \le n$.

Now, note that applying Lemma 2.2 to the vectors $\boldsymbol{p} = \bar{\boldsymbol{\alpha}}_n$ and $\boldsymbol{q} = \boldsymbol{b}^{0,n-1}$ yields a vector $\boldsymbol{p}^*(\nu_n, A_n)$ (as in (2.5)) such that either

$$\boldsymbol{p}^*(\nu_n, A_n)\cdot\boldsymbol{b}^{0,n-1} \ge \bar{\boldsymbol{\alpha}}_n\cdot\boldsymbol{b}^{0,n-1} = b_n > 0 \tag{2.12}$$

or

$$\boldsymbol{p}^*(\nu_n, A_n) \cdot \boldsymbol{b}^{0,n-1} \leq \bar{\boldsymbol{\alpha}}_n \cdot \boldsymbol{b}^{0,n-1} = b_n \leq 0 \tag{2.13}$$

Hence, suppose that the entries of the k^{th} through n^{th} rows of the coefficient matrix are of the form in (2.5), and express b_n as a linear combination of $b_1, b_2, \dots, b_k$ i.e.

$$\begin{aligned} b_n &= \sum_{i=1}^{k} C_i^k b_i \\ &= C_k^k b_k + \sum_{i=1}^{k-1} C_i^k b_i. \end{aligned} \tag{2.14}$$

Now, suppose $C_k^k > 0$. As before, applying Lemma 2.2 to the vectors $\boldsymbol{p} = \bar{\boldsymbol{\alpha}}_k$ and $\boldsymbol{q} = \boldsymbol{b}^{0,k-1}$ yields a vector $\boldsymbol{p}_k^*(\nu_k, A_k)$, such that

$$\boldsymbol{p}_k^*(\nu_k, A_k) \cdot \boldsymbol{b}^{0,k-1} \geq \bar{\boldsymbol{\alpha}}_k \cdot \boldsymbol{b}^{0,k-1} = b_k. \tag{2.15}$$

Similarly, if $C_k^k \leq 0$, we obtain a vector $\boldsymbol{p}_k^*(\nu_k, A_k)$, such that

$$\boldsymbol{p}_k^*(\nu_k, A_k) \cdot \boldsymbol{b}^{0,k-1} \leq \bar{\boldsymbol{\alpha}}_k \cdot \boldsymbol{b}^{0,k-1} = b_k. \tag{2.16}$$

Using the respective entries in $\boldsymbol{p}_k^*(\nu_k, A_k)$ in place of those in $\bar{\boldsymbol{\alpha}}_k$ in (2.1) will not decrease the value of b_n. This completes the induction for the case $b_n > 0$; the case $b_n \leq 0$ is similar, and the lemma follows. □

Remark 2.3. A version of Lemma 2.4 for $A_i \equiv 1$ was recently applied in proving that all symmetric Toeplitz matrices generated by monotone convex sequences have off-diagonal decay preserved through triangular decompositions (see [2]).

Now, For $\boldsymbol{a} = (A_1, A_2, A_3, \dots)$, with

$$0 \leq A_1 \leq A_2 \leq A_3 \leq \cdots \tag{2.17}$$

define

$$Z_i(\boldsymbol{a}) \overset{def}{=} \max\left\{ \prod_{v=j}^{i} A_v : 1 \leq j \leq i \right\}, \tag{2.18}$$

for $i \geq 1$.

We have the following result on bounds for linear recurrences.

Lemma 2.4. *Suppose that $\boldsymbol{a} = (A_j)$ satisfies the monotonicity constraint in (2.17). Then, for $i \geq 1$,*

$$\sup\{|b_i| : \{b_j\} \textit{ and } \{\alpha_{i,j}\} \textit{ satisfy (2.1) and (2.2)}\} = Z_i(\boldsymbol{a}). \tag{2.19}$$

Proof. Suppose that $\{b_i\}$ satisfies (2.1) and (2.2), and set $\zeta_i = Z_i(\boldsymbol{a})$ and $M_i = \max\{1, \zeta_i\}$, for $i \geq 1$. From (2.18), we have

$$A_{i+1} M_i = \zeta_{i+1}, \tag{2.20}$$

for $i \geq 1$. By Lemma 2.1, we may find sequences $\{d_i\}$ and $\{a_i\}$ satisfying (2.3) such that

$$|d_n| \geq |b_n|. \tag{2.21}$$

We will show that $\{d_i\}$ satisfies the inequality

$$|d_l + d_{l+1} + \cdots + d_i| \leq M_i, \tag{2.22}$$

for $0 \leq l \leq i$.

Note that (2.22) (for $i = n-1$) and (2.3) imply that $d_n = 0$ or $a_n \le n-1$ and

$$\begin{aligned} |d_n| &= \left| \sum_{j=a_n}^{n-1} (-A_n) d_j \right| \\ &= A_n \left| \sum_{j=a_n}^{n-1} d_j \right| \\ &\le A_n M_{n-1} \\ &= \zeta_n. \end{aligned} \tag{2.23}$$

Since $d_0 = 1$, $d_1 \in \{0, -A_1\}$ and

$$\begin{aligned} \max\{|d_1|, |d_0 + d_1|\} &= \max\{1, A_1, |1 - A_1|\} \\ &= \max\{1, A_1\} \\ &= M_1, \end{aligned} \tag{2.24}$$

i.e. the inequality in (2.22) holds for $i = 1$. Hence, suppose that (2.22) holds for $i < N$. Rewriting d_N, with $v = a_N$, we have for $0 \le x \le N-1$,

$$\begin{aligned} d_x + d_{x+1} + \cdots + d_N &= (d_x + d_{x+1} + \cdots + d_{N-1}) - A_n(d_v + \cdots + d_{N-1}) \\ &= \begin{cases} (1 - A_N)(d_v + \cdots + d_{N-1}) + (d_x + \cdots + d_{v-1}), & \text{if } v > x \\ (1 - A_N)(d_x + \cdots + d_{N-1}) - A_N(d_v + \cdots + d_{x-1}), & \text{if } v \le x \end{cases}. \end{aligned} \tag{2.25}$$

Let

$$S_1 = \begin{cases} d_v + \cdots + d_{N-1}, & \text{if } v > x \\ d_x + \cdots + d_{N-1}, & \text{if } v \le x \end{cases},$$

and

$$S_2 = \begin{cases} d_x + \cdots + d_{v-1}, & \text{if } v > x \\ d_v + \cdots + d_{x-1}, & \text{if } v \le x \end{cases}.$$

In showing that $|d_x + d_{x+1} + \cdots + d_N| \le M_N$, we will consider several cases depending on whether $A_N > 1$ or $A_N \le 1$, and the signs of S_1 and S_2.

Case 1 ($A_N > 1$ and $S_1 S_2 > 0$)

(1) $v > x$.

$$\begin{aligned} |d_x + d_{x+1} + \cdots + d_N| &= |(1 - A_N)S_1 + S_2| \\ &\le \max\{A_N |S_1|, A_N |S_2|\} \\ &\le A_N \max\{M_{N-1}, M_{v-1}\} \\ &\le A_N M_{N-1} \\ &= \zeta_N \\ &= M_N, \end{aligned} \tag{2.26}$$

where the first inequality follows since $(1 - A_N)S_1$ and S_2 are of opposite signs and $A_n > 1$. The second inequality follows from induction. The last equalities are direct consequences of the definition of M_N and the fact that $A_N > 1$. The monotonicity of $\{M_i\}$ is employed in obtaining the third inequality.

(2) $v \le x$.

$$\begin{aligned} |d_x + d_{x+1} + \cdots + d_N| &= |(1-A_N)S_1 - A_N S_2| \\ &\le |A_N S_1 + A_N S_2| \\ &= A_N |S_1 + S_2| \\ &= A_N |d_v + d_{v+1} + \cdots + d_{N-1}| \\ &\le A_N M_{N-1} \\ &= \zeta_N \\ &= M_N. \end{aligned} \tag{2.27}$$

In (2.27), the first inequality follows since $(1-A_N)S_1$ and $-A_N S_2$ are of the same sign.

Case 2 ($A_N > 1$ and $S_1 S_2 \le 0$)

(1) $v > x$.

$$\begin{aligned} |d_x + d_{x+1} + \cdots + d_N| &= |(1-A_N)S_1 + S_2| \\ &= |-A_N S_1 + (S_1 + S_2)|. \end{aligned} \tag{2.28}$$

If S_1 and $S_1 + S_2$ are of the same sign, then

$$\begin{aligned} |-A_N S_1 + (S_1 + S_2)| &\le \max\{A_N|S_1|, |S_1 + S_2|\} \\ &\le A_N M_{N-1} \\ &= M_N. \end{aligned} \tag{2.29}$$

Otherwise,

$$\begin{aligned} |-A_N S_1 + (S_1 + S_2)| &\le |-A_N S_1 + A_N(S_1 + S_2)| \\ &= A_N|S_2| \\ &\le A_N M_{N-1} \\ &= M_N. \end{aligned} \tag{2.30}$$

(2) $v \le x$.

$$\begin{aligned} |d_x + d_{x+1} + \cdots + d_N| &= |(1-A_N)S_1 - A_N S_2| \\ &\le \max\{A_N|S_1|, A_N|S_2|\} \\ &\le A_N M_{N-1} \\ &= M_N \end{aligned} \tag{2.31}$$

Case 3 ($A_N \le 1$ and $S_1 S_2 > 0$)

Note that for $A_N \le 1$, $M_i = 1$ for all i.

(1) $v > x$.

$$\begin{aligned} |d_x + d_{x+1} + \cdots + d_N| &= |(1-A_N)S_1 + S_2| \\ &\le |S_1 + S_2| \\ &\le M_{N-1} \\ &= M_N. \end{aligned} \tag{2.32}$$

(2) $v \le x$.

$$\begin{aligned} |d_x + d_{x+1} + \cdots + d_N| &= |(1 - A_N)S_1 - A_N S_2| \\ &\le \max\{|S_1|, |S_2|\} \\ &\le M_{N-1} \\ &= M_N. \end{aligned} \tag{2.33}$$

Case 4 ($A_N \le 1$ and $S_1 S_2 \le 0$)

(1) $v > x$.

$$\begin{aligned} |d_x + d_{x+1} + \cdots + d_N| &= |(1 - A_N)S_1 + S_2| \\ &\le \max\{|S_1|, |S_2|\} \\ &\le \max\{M_{N-1}, M_{v-1}\} \\ &= M_N. \end{aligned} \tag{2.34}$$

(2) $v \le x$.

$$\begin{aligned} |d_x + d_{x+1} + \cdots + d_N| &= |(1 - A_N)S_1 - A_N S_2| \\ &\le |S_1 + S_2| \\ &\le M_{N-1} \\ &= M_N. \end{aligned} \tag{2.35}$$

Thus, in all cases $|d_x + d_{x+1} + \cdots + d_N| \le M_N$ and hence by (2.23), $|d_N| \le \zeta_N$. Equation (2.19) now follows since, for $1 \le h \le n$, $|b_n| = A_h A_{h+1} \cdots A_n$ is attained for $[\alpha_{i,j}]$ defined by

$$\alpha_{i,j} = \begin{cases} -A_h, & \text{if } i = h \\ -A_i, & \text{if } i > h,\ j = i \\ 0, & \text{otherwise} \end{cases}. \tag{2.36}$$

□

We close this section with an elementary result (without proof) which will serve to connect entries in $\boldsymbol{L}_n^{-1}$ with solutions to (2.1).

Lemma 2.5. *Suppose $\boldsymbol{M} = [m_{i,j}]_{n\times n}$ and $\boldsymbol{y} = [y_i]_{n\times 1}$, satisfy $\boldsymbol{My} = (1, 0, \dots, 0)'$, with $\boldsymbol{M}$ an invertible lower triangular matrix. Then, $y_1 = 1/m_{1,1}$, and*

$$y_i = \sum_{j=1}^{i-1} \left(-\frac{m_{i,j}}{m_{i,i}} \right) y_j, \tag{2.37}$$

for $2 \le i \le n$.

3. THE MAIN RESULT

We are now in a position to prove our main result.

Theorem 3.1. *Suppose $\boldsymbol{\kappa} = (\kappa_i)$ satisfies*

$$0 \le \kappa_1 \le \kappa_2 \le \kappa_3 \le \cdots, \tag{3.1}$$

and set

$$S \overset{def}{=} \{i : \kappa_i > 1\}. \tag{3.2}$$

As well, define $\{W_{i,j}\}$ *by*

$$W_{i,j} \stackrel{def}{=} \prod_{v\,\in\,(S\bigcap\{j,j+1,\dots,i-2\})\ \bigcup\ \{i-1\}} \kappa_v. \tag{3.3}$$

Then, for $1 \le i \le n$, $|x_{i,i}| \le 1/l_{i,i}$ *and for* $1 \le j < i \le n$,

$$|x_{i,j}| \le \frac{W_{i,j}}{l_{j,j}}. \tag{3.4}$$

Proof. Suppose that $n \ge 1$ and $\boldsymbol{X}_n = \boldsymbol{L}_n^{-1}$. Solving for the sub-diagonal entries in the p^{th} column of $\boldsymbol{X}_n$ leads to the matrix equation

$$\begin{pmatrix} l_{p,p} & & & \\ l_{p+1,p} & l_{p+1,p+1} & & \\ \vdots & \vdots & \ddots & \\ l_{n,p} & l_{n,p+1} & \cdots & l_{n,n} \end{pmatrix} \begin{pmatrix} x_{p,p} \\ x_{p+1,p} \\ \vdots \\ x_{n,p} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Applying Lemma 2.5 gives $x_{p,p} = 1/l_{p,p}$, and

$$x_{p+i,p} = \sum_{j=0}^{i-1} \left(-\frac{l_{p+i,p+j}}{l_{p+i,p+i}} \right) x_{p+j,p}, \tag{3.5}$$

for $1 \le i \le n-p$.

Now, note that (1.2) gives

$$0 \le \frac{l_{p+i,p}}{l_{p+i,p+i}} \le \frac{l_{p+i,p+1}}{l_{p+i,p+i}} \le \cdots \le \frac{l_{p+i,p+i-1}}{l_{p+i,p+i}} \le \kappa_{p+i-1}. \tag{3.6}$$

Hence by Lemma 2.4,

$$\begin{aligned} |x_{p+i,p}| &\le |x_{p,p}| Z_i((\kappa_p, \kappa_{p+1}, \dots, \kappa_{p+i-1})) \\ &= \frac{1}{l_{p,p}} W_{p+i,p}, \end{aligned} \tag{3.7}$$

for $1 \le i \le n-p$, and the theorem follows. □

4. EXAMPLES

In this section, we provide examples to illustrate some of the structural information contained in Theorem 3.1.

Example 4.1 (Equally spaced A_i)**.** Suppose that $A_i = Ci$ for $i \ge 1$, where $C > 0$. Then, for $n \ge 1$,

$$Z_n(\boldsymbol{a}) = \begin{cases} nC, & C \in \left(0, \frac{1}{n-1}\right]; \\ (n)_k C^k, & C \in \left(\frac{1}{n-k+1}, \frac{1}{n-k}\right],\ (2 \le k \le n-1); \\ n! C^n, & C \in (1, \infty), \end{cases}$$

where $(n)_k = n(n-1)\cdots(n-k+1)$.

Consider the matrix

$$\boldsymbol{L}_7 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.25 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 1 & 0 & 0 & 0 & 0 \\ 0.75 & 0.75 & 0.75 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1.25 & 1.25 & 1.25 & 1.25 & 1 & 0 \\ 1.5 & 1.5 & 1.5 & 1.5 & 1.5 & 1.5 & 1 \end{pmatrix},$$

with (rounded to three decimal places)

$$X_7 = \boldsymbol{L}_7^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.25 & 1 & 0 & 0 & 0 & 0 & 0 \\ -0.375 & -0.5 & 1 & 0 & 0 & 0 & 0 \\ -0.281 & -0.375 & -0.75 & 1 & 0 & 0 & 0 \\ -0.094 & -0.125 & -0.25 & -1 & 1 & 0 & 0 \\ 1.25 & 0 & 0 & 0 & -1.25 & 1 & 0 \\ -1.875 & 0 & 0 & 0 & 0.375 & -1.5 & 1 \end{pmatrix}. \tag{4.1}$$

Applying Theorem 3.1, with $\boldsymbol{\kappa} = (.25, .50, .75, 1.00, 1.25, 1.50, \dots)$ gives the entry-wise bounds

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.25 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 1 & 0 & 0 & 0 & 0 \\ 0.75 & 0.75 & 0.75 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1.25 & 1.25 & 1.25 & 1.25 & 1.25 & 1 & 0 \\ 1.875 & 1.875 & 1.875 & 1.875 & 1.875 & 1.5 & 1 \end{pmatrix}. \tag{4.2}$$

Comparing (4.1) and (4.2), the absolute values of entry-wise ratios are

$$\begin{pmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 0.75 & 1 & 1 & & & & \\ 0.375 & 0.5 & 1 & 1 & & & \\ 0.094 & 0.125 & 0.25 & 1 & 1 & & \\ 1 & 0 & 0 & 0 & 1 & 1 & \\ 1 & 0 & 0 & 0 & 0.2 & 1 & 1 \end{pmatrix}. \tag{4.3}$$

Note that here $\boldsymbol{L}_7$ was constructed so that $|x_{7,1}| = W_{7,1}$. In fact, as suggested by (2.19), for each 4-tuple $(\boldsymbol{\kappa}, I, J, n)$ with $1 \le J \le I \le n$, there exists a pair $(\boldsymbol{L}_n, \boldsymbol{X}_n)$ satisfying (1.2) with $\boldsymbol{X}_n = (x_{i,j}) = \boldsymbol{L}_n^{-1}$, such that $|x_{I,J}| = W_{I,J}$.

Example 4.2 (Constant A_i)**.** Suppose that $A_i = C$ for $i \ge 1$, where $C > 0$. Then, for $n \ge 1$,

$$Z_n(\boldsymbol{a}) = \begin{cases} C, & \text{if } C \le 1 \\ C^n, & \text{if } C > 1 \end{cases}.$$

In [3], the following theorem was obtained when (2.2) is replaced with

$$0 \le \alpha_{i,j} \le A, \tag{4.4}$$

for $0 \le j \le i - 1$ and $i \ge 1$.

Theorem 4.1. *Suppose that $A > 0$ and $m = [1/A]$, where square brackets indicate the greatest integer function. If $\{\Lambda_j\}_{j=1}^{\infty}$ is defined by*

$$\Lambda_n = \max\{|b_n| : \{b_i\} \text{ and } [\alpha_{i,j}] \text{ satisfy (2.1) and (4.4)}\}, \tag{4.5}$$

for $n \geq 1$, then

$$\Lambda_n = \begin{cases} A, & \text{if } n = 1 \\ \max(A, A^2), & \text{if } n = 2 \\ \left[\frac{n-2}{2}\right]\left[\frac{n-1}{2}\right] A^3 + A, & \text{if } 3 \leq n \leq 2m+1 \\ (n-2)A^2, & \text{if } n = 2m+2 \\ A\Lambda_{n-1} + \Lambda_{n-2}, & \text{if } n \geq 2m+3 \end{cases}. \tag{4.6}$$

Proof. See [3]. □

Thus, if the monotonicity assumption in (2.2) is dropped the scenario is much different. In fact, in (4.6), $\{\Lambda_n\}$ increases at an exponential rate for all $A > 0$. This leads to the following question.

Open Question. Set

$$\Lambda_n^* = \max\{|b_n| : \{b_i\} \text{ and } [\alpha_{i,j}] \text{ satisfy (2.1) and } \alpha_{i,j} \leq A_i \text{ for } 0 \leq j \leq i-1\}. \tag{4.7}$$

What is the value of Λ_n^* in terms of the sequence $\{A_i\}$ and its assorted properties (eg. monotonicity, convexity etc.)?

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