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REGULARITY PROPERTIES OF SOME STOKES OPERATORS ON AN INFINITE STRIP

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ABSTRACT. In this paper, we try to solve the problem which arises in connection with the stability theory of a periodic equilibrium solution of Navier-Stokes equations on an infinite strip $\mathbb{R} \times \left]-\frac{1}{2}, \frac{1}{2}\right[$.

Key words and phrases: Navier-Stokes equations, Regularity, Fourier series.

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1. INTRODUCTION

This problem arises in connection with the stability theory of a periodic equilibrium solution of Navier-Stokes on infinite strip $\Omega = \mathbb{R} \times \left]-\frac{1}{2}, \frac{1}{2}\right[$.

Consider the Navier-Stokes equation on an infinite strip $\Omega = \mathbb{R} \times \left]-\frac{1}{2}, \frac{1}{2}\right[$:

$$\partial_t U = \nu \Delta U - (U \cdot \nabla)U + \nabla p + f \tag{1.1}$$

with $f = f(x, y)$ a smooth time independent outer force on Ω ,which is L -periodic in x for some L.

Let a smooth equilibrium solution $U_0 = (u_0, v_0), p_0$ of (1.1) be given, which is L -periodic in x and $U_0 = 0$ on $\partial\Omega$. The stability of $U_0 = (u_0, v_0), p_0$ can be studied against small perturbations under two aspects:

(I) The perturbations are themselves L -periodic in x.
(II) The perturbations are in $(\mathcal{L}^2(\Omega))^2$.

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The relation between (I) and (II) is the mathematical tools used by physicists in connection with Schroedinger equations with periodic potentials [3]. The main tool thereby is the notion of direct integrals (see [1] , [3] , [5], [8]). This notion is based on Θ-Periodic functions (ie. generalisation of periodic functions).

In this paper we study the Stokes operators which arise in the so-called Bloch space theory of equation (1.1). This theory, well established in the case of Schroedinger equations with periodic potentials [3] extends to the Stokes operators which occur in Navier-Stokes and related equations, but the corresponding theory is now more involved, see [8] where the three dimensional case (3d) is treated. The Stokes operators which appear in connection with (1.1), either 2d or 3d, are of the form:

$$P\Delta U - P(V\cdot\nabla) - P(U\cdot\nabla)V. \tag{1.2}$$

Here V is a fixed velocity field, periodic in the unbounded space directions (x or x, y), U is the argument on which the operator acts, while P is the orthogonal projection onto the space of divergence free fields. Three cases are of interest:

(a) $U \in (H^2(\Omega)\cap H_0^1(\Omega))^3$, $divU = 0$.
(b) U is periodic in the unbounded space directions.
(c) U is Floquet - periodic in the unbounded space directions.

Case (b) subsumes under case (c) [2]; case (a) is handled in [4]. Case (a) and (c) are related by certain spectral formulas, well known in case of the Schroedinger equations with periodic potentials. In the 3d-case however, these spectral formulas associated with (1.2) are more complicated than in the Schroedinger case due to the appearance of singularities ([8, Sect 9.4, 9.5]). The purpose of the present paper is to show that in the 2d-case these singularities are absent and that the spectral formulas associated with (1.2) have precisely the same formula as in the Schroedinger case. To this effect we study first the most important special, ie. $V = 0$. We have to perform estimates similar to those in Sections 6.4–6.7 of [8]. In our estimates, which are considerably simpler, singularities do not appear.

How this fact can be exploited so as to obtain the mentioned spectral formulas is outlined in subsequent sections.

2. NOTATION

For $\mathcal{X}, \mathcal{Y}$ Banach spaces, $\|\cdot\|_{\mathcal{X}}$, $\|\cdot\|_{\mathcal{Y}}$ are their respective norms. $L(\mathcal{X}, \mathcal{Y})$ is the space of bounded operators from $\mathcal{X}$ to $\mathcal{Y}$ with $\|T\|$ the operator norm.

For A a linear operator on $\mathcal{X}$ and $E \subseteq \mathcal{X}$ a subspace, $A\mid_E$ is the restriction of A to E.

For any Ω, $H^p(\Omega)$ is the Sobolev space of functions having square integrable derivatives up to order p with $(\cdot,\cdot)_p$ and $\|\cdot\|_{H^p(\Omega)}$ the usual scalar product and norm on $H^p(\Omega)$.We set $\mathcal{L}^2(\Omega) = H^0(\Omega)$ and $\|\cdot\|_{H^p} = \|\cdot\|_{H^p(\Omega)}$ and extend this notation to vectors and set:

$$\|u\|_{\mathcal{L}^2}^2 = \|u_1\|_{\mathcal{L}^2}^2 + \|u_2\|_{\mathcal{L}^2}^2 ,$$

where $u = (u_1, u_2) \in (\mathcal{L}^2)^2$, Likewise with the Sobolev norms. The scalar product on $(H^p(\Omega))^2$ is $\langle\cdot,\cdot\rangle_p$, with:

$$\langle u, v\rangle_p = \sum_{i=1}^{2}(u_i, v_i)_p, \quad u_i, v_i \in H^p(\Omega),$$

where $u = (u_1, u_2),\ v = (v_1, v_2)$ we set $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_0$.

$C^p(\bar\Omega)$ is the space of functions p times continuously differentiable on $\bar\Omega$ and $C_0^p(\bar\Omega)$ is the space of functions $f \in C^p(\bar\Omega)$ with $\operatorname{supp} f$ compact.

3. Θ -PERIODIC FUNCTION

We fix a period $L > 0$, set $Q_L =]0, L[$ and $Q = Q_L \times \left]-\frac{1}{2}, \frac{1}{2}\right[$, for some small $\epsilon > 0$ and put $M_\epsilon =]-\epsilon, 2\pi + \epsilon[$ with $M = [0, 2\pi]$. Also, let $\dot{M}_\epsilon$ be M_ϵ minus the numbers 0 and 2π.

We define a Θ-Periodic function: For Θ in M_ϵ; $f \in C^p_\Theta(Q)$ if $f \in C^p(Q)$ and

$$f(x + jL, y) = e^{ij\Theta} f(x, y), j \in \mathbb{Z}, (x, y) \in \overline{\Omega}.$$

We define the functional spaces: $H^p_\Theta(Q)$ is the set of $f \in \mathcal{L}^2(Q)$ such that $\lim_n \|f_n - f\|_{H^p} = 0$ for some sequence $f_n \in C^p_\Theta(Q)$.

We also let $\mathcal{L}^2_g$ be the subspace of $\mathcal{L}^2(Q)$ containing the elements f such that $f(x, -y) = f(x, y)$ a.e. Likewise with $\mathcal{L}^2_u$ and $f(x, -y) = -f(x, y)$ a.e. Finally, we put $L^2 = (\mathcal{L}^2)^2$, $L^2_g = \mathcal{L}^2_g \times \mathcal{L}^2_u$ and $L^2_u = \mathcal{L}^2_u \times \mathcal{L}^2_g$.

It is easy to prove that:

$$L^2 = L^2_g \oplus L^2_u.$$

4. FOURIER SERIES

We consider the eigenvalue problem: $y" + \lambda y = 0$ on $\left]-\frac{1}{2}, \frac{1}{2}\right[$ with Neumann resp. Dirichlet boundary conditions.

In the first case we have a complete orthonormal (C.O.N) system in $\mathcal{L}^2(Q)$:

$$\varphi_{2k} = (-1)^k \sqrt{2} \cos 2\pi k y \quad \text{for} \quad k \geq 1, \varphi_0 = 1,$$
$$\varphi_{2k+1} = (-1)^k \sqrt{2} \sin(2k + 1)\pi y \quad \text{for} \quad k \geq 0,$$

$\Lambda_p = p^2\pi^2$ is an eigenvalue associated to φ_p, φ_{2k} is even, φ_{2k+1} odd and moreover $\varphi_p(1/2) = \sqrt{2}$ for $p \geq 1$. For the other case we have a (C.O.N) system given by $\sqrt{\Lambda_p}\psi_p = \varphi'_p$, where $\psi'_p = -\sqrt{\Lambda_p}\varphi_p$ for $p \geq 1$.

Since parity in y will be important we introduce notations: $\sigma_k = \varphi_{2k+1}$, $\tau_k = \psi_{2k+1}$, $\lambda_k = \Lambda_{2k+1}$, $k \geq 0$, and $\rho_k = \varphi_{2k}$, $\pi_k = \psi_{2k}$ for $k \geq 1, \varphi_0 = 1$ and $\mu_k = \lambda_{2k}$. For $\theta \in M_\epsilon$ we set: $\hat{\alpha} = (2\pi\alpha + \theta)L^{-1}$, $\alpha \in \mathbb{Z}$ and $e_\alpha = e^{i\hat{\alpha}x}$.

We have a characterization of spaces $H^1_{\theta,0}, H^1_\theta, H^2_\theta$ with the Fourier series:

Let $f \in \mathcal{L}^2(Q)$ have Fourier series:

$$f = \sum f_{\alpha,i} e_\alpha \varphi_i = \sum \tilde{f}_{\alpha,i} e_\alpha \psi_i.$$

With respect to $\{e_\alpha \varphi_i\}$ resp $\{e_\alpha \psi_i\}$.

Proposition 4.1. (a) $f \in H^1_\theta$ *iff*

$$\sum (\hat{\alpha}^2 + \Lambda_i) |f_{\alpha,i}|^2 < \infty.$$

(b) $f \in H^1_{\theta,0}$ *iff*

$$\sum (\hat{\alpha}^2 + \Lambda_i) \left|\tilde{f}_{\alpha,i}\right|^2 < \infty.$$

For a proof see [6]. We have the characterization of space H^2_θ too:

Proposition 4.2. *Let $f \in \mathcal{L}^2(Q)$ satisfy $\sum (\hat{\alpha}^2 + \Lambda_i)^2 |f_{\alpha,i}|^2 < \infty$. then $f \in H^2_\theta$ and*

$$\|F\|^2_{H^2} \leq C \left(\sum (\hat{\alpha}^2 + \Lambda_i)^2 |f_{\alpha,i}|^2\right)$$

for a C independent of $\theta \in M_\epsilon$. Likewise with $\sum (\hat{\alpha}^2 + \Lambda_i)^2 \left|\tilde{f}_{\alpha,i}\right|^2$.

For a proof see [6].

Our aim is to prove:

Theorem 4.3. (a) *There is $C > 0$ as follows. If $U \in \operatorname{dom}(A_s(\theta)) \cap E_\theta^g$ and $A_s(\theta)U = f$ for some $\theta \in M_\epsilon$, $f \in E_\theta^g$ then $U \in (H_\theta^2)^2$ and*

$$\|U\|_{H^2} \le C \|f\|_{\mathcal{L}^2}.$$

(b) *Under the conditions $U \in \operatorname{dom}(A_s(\theta)) \cap E_\theta^u$ or $U \in \operatorname{dom}(A_s(\theta)) \cap E_\theta$ the assertion (a) holds.*

Proposition 4.4. *If $f \in H_\theta^1$ has Fourier series $\sum_{\alpha,j} a_{\alpha,j} e_\alpha \sigma_j$ then $\sum_j |a_{\alpha,j}| \le \infty$ and $f \in H_{\theta,0}^1$ iff $\sum_j a_{\alpha,j} = 0, \alpha$ in $\mathbb{Z}$.*

Remark 4.5. Proposition 4.4 is a consequence of Propositions 6.1 and 6.3 in [8].

For the proof of this theorem we need the Proposition 6 used in [7]; we recall $\lambda_k = \frac{(2k+1)^2}{\pi^2}$:

Proposition 4.6. *There are Γ_0, Γ_1 such that for $s \ge 0$:*

(i): $\Gamma_0(1+s)^{-3} \le \sum(\lambda_k + s^2)^{-2} \le \Gamma_1(1+s)^{-3}$;
(ii): $\sum(\lambda_k + s^2)^{-1} \le \Gamma_1(1+s)^{-1}$;
(iii): $\sum \lambda_k^{-1}(\lambda_k + s^2)^{-2} \le \Gamma_1(1+s)^{-4}$;
(iv): $\sum \lambda_k(\lambda_k + s^2)^{-2} \le \Gamma_1(1+s)^{-1}$.

Proof of Theorem 4.3. Since, in the first part of the proof, the factor $\hat{\alpha}^{-1}$ appears which is later cancelled, it is advantageous to assume first that $\theta \in \dot{M}_\epsilon$.

We take $U = (A, B) \in (H_{\theta,0}^1) \cap L_g^2$ such that $divU = 0$.

We know that if $L_g^2 = \mathcal{L}_g^2 \times \mathcal{L}_u^2$ then $A \in \mathcal{L}_g^2$ and $B \in \mathcal{L}_u^2$ and with the characterization of space $H_{\theta,0}^1$ by Fourier series we have

$$A = \sum A_{j\alpha} e_\alpha \tau_j \text{ and } B = \sum B_{j\alpha} e_\alpha \sigma_j$$

such that $\sum(\lambda_j + \hat{\alpha}^2)|A_{j\alpha}|^2 < \infty$, likewise for B, the components of $f = (a, b)$ admit expansions too,

$$a = \sum a_{j\alpha} e_\alpha \tau_j \text{ and } b = \sum b_{j\alpha} e_\alpha \sigma_j.$$

U is a weak solution of $A_s(\theta)U = f$ for $f \in E_\theta$ if and only if:

$$\sum_{j=1}^{2} \langle \nabla U_j, \nabla V_j \rangle + \langle f, V \rangle = 0 \tag{4.1}$$

for all $V \in (H_{\theta,0}^1)^2$.

As a test vector in (4.1) we take:

$$V = (u_0 \tau_0 + u_j \tau_j, w_0 \sigma_0 + w_j \sigma_j) \in (H_{\theta,0}^1)^2,$$

whereby $divV = 0$, thus:

$$\sqrt{\lambda_j} w_j = -\partial_x u_j \text{ and } \sqrt{\lambda_0} w_0 = -\partial_x u_0. \tag{4.2}$$

Here $u_0 \in H_\theta^2(Q_L)$ is arbitrarily fixed.

As in paper [7], we have $w_0 + w_j = 0$. From the divergence condition we deduce that since $\frac{1}{\sqrt{\lambda_0}} u_0 + \frac{1}{\sqrt{\lambda_j}} u_j$ is constant Θ-periodic, then $u_j = -\frac{\sqrt{\lambda_j}}{\sqrt{\lambda_0}} u_0$.

By exploiting the arbitrariness of U_0, ψ we reach certain equations for the Fourier coefficients $A_{j,\alpha}, B_{j,\alpha}, a_{j,\alpha}, b_{j,\alpha}$.
We note:

$$\hat{\lambda_j} = \lambda_j + \hat{\alpha}^2, j \geq 0, \ \ \alpha \in \mathbb{Z},$$

$$(A)_j(\alpha) = \hat{\lambda_j} A_{j,\alpha} - a_{j,\alpha}, j \geq 0, \ \ \alpha \in \mathbb{Z},$$

$$(B)_j(\alpha) = \hat{\lambda_j} B_{j,\alpha} - b_{j,\alpha}, j \geq 0, \ \ \alpha \in \mathbb{Z}.$$

We obtain:

$$-\frac{\sqrt{\lambda_j}}{\sqrt{\lambda_0}}(A)_j(\alpha) + (A)_0(\alpha) - \frac{i\hat{\alpha}}{\sqrt{\lambda_0}}(B)_j(\alpha) + \frac{i\hat{\alpha}}{\sqrt{\lambda_0}}(B)_0(\alpha) = 0, \quad , j \geq 0. \tag{4.3}$$

From the divergence condition for u, f we get:

$$(B)_j(\alpha) = -\frac{i\hat{\alpha}}{\sqrt{\lambda_j}}(A)_j(\alpha), j \geq 0. \tag{4.4}$$

From the condition $\theta \in \dot{M}_\epsilon$ we get $\hat{\alpha} \neq 0$ then:

$$(A)_j(\alpha) = \frac{i\sqrt{\lambda_j}}{\hat{\alpha}}(B)_j(\alpha). \tag{4.5}$$

So according to (4.3) and (4.5) we have:

$$\hat{\lambda_j}(B)_j(\alpha) = \hat{\lambda_0}(B)_0(\alpha). \tag{4.6}$$

By using Proposition 4.4 we have $\sum_j B_{j\alpha} = 0$, and then:

$$\begin{cases} B_{0,\alpha} = k\left(\hat{\lambda_0}\sum_{j\geq 1}(\hat{\lambda_j})^{-2} b_{0,\alpha} - \sum_{j\geq 1}(\hat{\lambda_j})^{-1} b_{j,\alpha}\right), \\ k = \left(1 + (\hat{\lambda_0})^2 \sum_{j\geq 1}(\hat{\lambda_j})^{-2}\right)^{-1} = k(\alpha). \end{cases} \tag{4.7}$$

Having $B_{0,\alpha}$, we can express $B_{j,\alpha}, j \geq 1$ via (4.7) and then $A_{j,\alpha}, j \geq 0$ via (4.5). Then (4.3) becomes:

$$\frac{\hat{\lambda_j}}{\sqrt{\lambda_j}}(A)_j = \frac{\hat{\lambda_0}}{\sqrt{\lambda_0}}(A)_0 \tag{4.8}$$

Equation (4.5) gives us (for $j = 0$):

$$(A)_0(\alpha) = \frac{i\sqrt{\lambda_0}}{\hat{\alpha}}(B)_0(\alpha).$$

Thus,

$$(A)_j(\alpha) = \frac{i\sqrt{\lambda_j}\hat{\lambda_0}}{\hat{\alpha}\hat{\lambda_j}}(B)_0(\alpha), \tag{4.9}$$

and from (4.7) we deduce:

$$(B)_0(\alpha) = -k(b_{0,\alpha} + \hat{\lambda_0}\sum_{j\geq 1}(\hat{\lambda_j})^{-1} b_{j,\alpha}). \tag{4.10}$$

By the divergence condition we replace $b_{j,\alpha}$ by $a_{j,\alpha}$ in (4.10). If we replace $(B)_0(\alpha)$ in (4.9) by its value we obtain:

$$(A)_j(\alpha) = \frac{-\sqrt{\lambda_j}\hat{\lambda_0}k}{\hat{\lambda_j}}\left(\frac{1}{\sqrt{\lambda_0}}a_{0,\alpha} + \hat{\lambda_0}\sum_{s\geq 1}(\lambda_s^{1/2}\hat{\lambda_s})^{-1}a_{s,\alpha}\right). \tag{4.11}$$

As can be seen from (4.11), the expression for $(A)_j(\alpha)$ does not contain any factor $\hat{\alpha}^{-1}$, that is no singularity, we may therefore assume from now on that $\theta \in M_\epsilon$.

By (4.11) we have:

$$(A)_j(\alpha) = I_j + II_j,$$

where

$$I_j = \frac{-\sqrt{\lambda_j}\hat{\lambda_0}k}{\hat{\lambda_j}}\hat{\lambda_0}\sum_{j\geq 1}(\lambda_j^{1/2}\hat{\lambda_j})^{-1}a_{j,\alpha}$$

and

$$II_j = \frac{-\sqrt{\lambda_j}\hat{\lambda_0}k}{\hat{\lambda_j}\sqrt{\lambda_0}}a_{0,\alpha}.$$

We note that by Proposition 4.4 (i) a Γ_2 is found such that,

$$k \leq \Gamma_2(1+s)^{-1}, \quad (s = |\hat{\alpha}|),$$

then:

$$\begin{aligned}
|I_j|^2 &\leq \frac{\lambda_j\hat{\lambda_0}^2k^2}{\hat{\lambda_j}^2}\left(\sum_{s\geq 1}(\hat{\lambda_s})^{-2}(\hat{\lambda_0})^2(\lambda_s)^{-1}\right)\left(\sum_{s\geq 1}|a_{s,\alpha}|^2\right)\\
&\leq \frac{\Gamma_2^2(1+s)^{-2}(\lambda_0+s)^2\lambda_j}{(\lambda_j+s^2)^2}\left(\sum_{s\geq 1}\lambda_s^{-1}\right)\left(\sum_{s\geq 1}|a_{s,\alpha}|^2\right)\\
&\leq \frac{C'}{\lambda_j}\sum_{s\geq 1}|a_{s,\alpha}|^2.
\end{aligned}$$

Thus,

$$\sum_\alpha\sum_{j\geq 1}|I_j|^2 \leq C\sum_\alpha\sum_{s\geq 1}|a_{s,\alpha}|^2$$

and for II_j we have:

$$|II_j|^2 = \frac{\lambda_j\hat{\lambda_0}^2k^2}{\hat{\lambda_j}^2\lambda_0}|a_{0,\alpha}|^2$$

then:

$$|II_j|^2 \leq \frac{\lambda_j(\lambda_0+s^2)^2\Gamma_2^2(1+s)^{-2}}{(\lambda_j+s^2)^2\lambda_0}|a_{0,\alpha}|^2$$

and

$$\sum_{j\geq 1}|II_j|^2 \leq C'(1+s)^{-2}|a_{0,\alpha}|^2.$$

Therefore

$$\sum_\alpha\sum_{j\geq 1}|II_j|^2 \leq C_1\sum_\alpha|a_{0,\alpha}|^2.$$

We still have to look at $(A)_0(\alpha)$. We recall (4.11) for $j = 0$ and we can estimate $k(\alpha)$ by Proposition 4.6.

For $(B)_j(\alpha)$: By (4.4) and (4.9) we can deduce by using Proposition 4.4 that there is a $\theta-$independent C_2 such that:

$$\sum_\alpha \sum_j |(B)_j(\alpha)|^2 \leq C_2 \sum_\alpha \sum_j |b_{j,\alpha}|^2.$$

The proof of (b) is very similar.

Conclusion:

$$\|U\|_{H^2} \leq C \|F\|_{\mathcal{L}^2}.$$

□

5. COMMENTS

As indicated, due to the fact that the singularity $\theta = 0$ resp. $\theta = 2\pi$ drops out in the computations presented in the previous sections, the spectral theory, carried out for dimension $d = 3$ in [6], [8] simplifies considerably. Partly for this reason and partly for reasons of space we concentrate here on briefly describing the final result which emerges from this simplification. In order to describe the manner in which the spectral formula (**) in [6] simplifies, we recall the objects which appear in it. Following Sections 2 and 3, we have the θ-periodic Sobolev spaces $H^p_\theta(Q)$, $H^1_{\theta,0}(Q)$, $\theta \in]-\varepsilon, 2\pi+\varepsilon[$, the orthogonal projection P_θ from $\mathcal{L}^2(Q)^2$ onto E_θ, with E_θ the $\mathcal{L}^2$-closure of the set of $f \in H^1_\theta(Q) \times H^1_{\theta,0}(Q)$ such that $\operatorname{div} f = 0$. The periodic Stokes operator $A_S(\theta)$ is now defined as follows:

$$\begin{aligned} &f \in \operatorname{dom}(A_S(\theta)) \quad \text{iff} \quad f \in (H^2_\theta(Q) \cap H^1_{\theta,0}(Q))^2 \\ &\text{and } \operatorname{div} f = 0, \text{ and for such } f,\ A_S(\theta)f = \nu P_\theta \Delta f. \end{aligned} \tag{5.1}$$

Next, we recall that, as stressed in the introduction, we are given a smooth velocity field $v = (v_1, v_3)$ on $R \times [\frac{1}{2}, \frac{1}{2}]$ which is L-periodic in the unbounded variable x, that gives rise to an operator T acting on elements $u = (u_1, u_3) \in \operatorname{dom}(A_S(\theta))$ according to

$$Tu = -(v_1\partial_x u_1 + v_3\partial_z u_1,\ v_1\partial_x u_3 + v_3\partial_z u_3). \tag{5.2}$$

We briefly digress on the periodic case which arises for $\theta = 0$ of $\theta = 2\pi$. In accordance with [6] we stress this case by the label 'per' rather than by $\theta = 0$ or $\theta = 2\pi$. Thus $A_S(\text{per}) = A_S(0) = A_S(2\pi)$, $H^p_{\text{per}}(Q) = H^p_0(Q) = H^p_{2\pi}(Q)$, etc. In order for the spectral formulas below to be valid, we have to define E_{per}, $A_S(\text{per})$, P_{per} as follows:

$$\begin{aligned} &E_{\text{per}} \text{ is the } \mathcal{L}^2-\text{closure of all vector fields } v = (f, h) \\ &\text{in } H^1_{\text{per}}(Q) \times H^1_{\text{per},0}(Q) \text{ such that div } f = 0 \text{ and } \int_Q f\, dxdz = 0 \end{aligned} \tag{5.3}$$

$$\begin{aligned} &v = (f, h) \text{ is in } \operatorname{dom}(A_S(\text{per})) \text{ if } v \in (H^2_{\text{per}}(Q) \cap H^1_{\text{per},0}(Q))^2, \\ &\operatorname{div} v = 0 \text{ and } \int_Q f\, dxdz = 0; \text{ for such } v \text{ we set} \\ &A_S(\text{per})v = \nu P_{\text{per}}\Delta v, \text{ where } P_{\text{per}} \text{ is the orthogonal projection} \\ &\text{from } \mathcal{L}^2(Q)^2 \text{ onto } E_{\text{per}}. \end{aligned} \tag{5.4}$$

With this definition, $A_s(\text{per})$ is selfadjoint on E_{per}.

Finally we need corresponding objects defined on the whole strip $\Omega = R \times \left(-\frac{1}{2}, \frac{1}{2}\right)$. Thus

$$\begin{aligned} &E \text{ is the } \mathcal{L}^2-\text{closure of } f \in H^1(\Omega) \times H^1_0(\Omega) \\ &\text{such that } \operatorname{div} f = 0, \end{aligned} \tag{5.5}$$

$$f \in \operatorname{dom}(A_S) \text{ iff } f \in (H^2(\Omega) \cap H_0^1(\Omega))^2 \text{ and } \operatorname{div} f = 0, \quad \text{and for such } f \text{ we set } A_S f = \nu P \Delta f. \tag{5.6}$$

For elements $f \in \operatorname{dom}(A_S)$, the operator T acts again via (5.2). Under these stipulations, the operators

$$G = A_S + PT, \quad G_\theta = A_S(\theta) + P_\theta T, \quad G_{\text{per}} = A_S(\text{per}) + P_{\text{per}} T$$

all become holomorphic semigroup generators on E, E_θ, E_{per} respectively. The spectral formulas, announced above now are:

$$\sigma(A_S + PT) = \text{closure}\left(\bigcup_{\theta \in (0,2\pi)} (A_S(\theta) + P_\theta T)\right), \tag{$(22)_1$}$$

$$\sigma(A_S + PT) = \bigcup_{\theta \in [0,2\pi]} (A_S(\theta) + P_\theta T). \tag{$(22)_2$}$$

These formulas correspond to formulas (*), (**) in [6, p. 169]. While $(22)_1$ looks the same as (*) in [6], $(22)_2$ is definitely simpler; it implies in particular that if $\lambda \in \sigma(A_S(\text{per}) + P_{\text{per}} T)$ then $\lambda \in \sigma(A_S + PT)$, a statement which cannot be asserted in dimension $d = 3$ as can be seen from formula (**) in [6]. The proof of $(22)_2$ is based on the computations in the present Section 4, which entail that the singularities which arise in dimension $d = 3$ in [6], drop out. The detailed verification of this claim is by a careful examination of the arguments in [6], a task within the scope of this paper.

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