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# REGULARITY FOR VECTOR VALUED MINIMIZERS OF SOME ANISOTROPIC INTEGRAL FUNCTIONALS

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ABSTRACT. We deal with anisotropic integral functionals $\int_\Omega f(x, Du(x))dx$ defined on vector valued mappings $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$. We show that a suitable "monotonicity" inequality, on the density $f$, guarantees global pointwise bounds for minimizers $u$.

*Key words and phrases:* Anisotropic, Integral, Functional, Regularity, Minimizer.

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## 1. INTRODUCTION

We consider the integral functional

$$\mathcal{F}(u) = \int_\Omega f(x, Du(x))dx \tag{1.1}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ and $\Omega$ is a bounded open set. When $N = 1$ we are dealing with scalar functions $u : \Omega \to \mathbb{R}$; on the contrary, vector valued mappings $u : \Omega \to \mathbb{R}^N$ appear when $N \geq 2$. Local and global pointwise bounds for scalar minimizers of (1.1) have been proved in [2], [7], [5], [4]. A model functional for these results is

$$\int_\Omega \left( \sum_{i,j=1}^n a_{ij}(x) D_j u(x) D_i u(x) \right)^{\frac{p}{2}} dx, \tag{1.2}$$

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where coefficients $a_{ij}$ are measurable, bounded and elliptic. Previous results for scalar minimizers are no longer true in the vector valued case $N \geq 2$ as De Giorgi's counterexample shows, [3]. Some years later, attention has been paid to anisotropic functionals whose model is

$$\int_\Omega \left(|D_1 u(x)|^{p_1} + |D_2 u(x)|^{p_2} + \cdots + |D_n u(x)|^{p_n}\right) dx, \tag{1.3}$$

where each component $D_i u$ of the gradient $Du = (D_1 u, D_2 u, \dots, D_n u)$ may have a (possibly) different exponent $p_i$: this seems useful when dealing with some reinforced materials, [9]; see also [6, Example 1.7.1, page 169]. In the framework of anisotropic functionals, global pointwise bounds have been proved for scalar minimizers in [1] and [8]. If no additional conditions are assumed, these bounds are false in the vectorial case, as the above mentioned counterexample shows, [3]. The aim of this paper is to present a "monotonicity" assumption ensuring boundedness of vector valued minimizers. In order to do that, we recall that $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ thus $Du(x)$ is a matrix with $N$ rows and $n$ columns; the density $f(x, A)$ in (1.1) is assumed to be measurable with respect to $x$, continuous with respect to $A$ and $f : \Omega \times \mathbb{R}^{N\times n} \to [0, +\infty)$. Every matrix $A = \{A_i^\alpha\} \in \mathbb{R}^{N\times n}$ will have $N$ rows $A^1, \dots, A^N$ and $n$ columns $A_1, \dots, A_n$. In this paper we will show that the following "monotonicity" inequality guarantees global pointwise bounds for vector valued minimizers of (1.1):

$$f(x, \tilde{A}) + \mu \sum_{i=1}^n \left|\tilde{A}_i - A_i\right|^{p_i} \leq f(x, A) + M(x) \tag{1.4}$$

for every pair of matrices $\tilde{A}, A \in \mathbb{R}^{N\times n}$ such that there exists a row $\beta$ with $\tilde{A}^\beta = 0$ and for every remaining row $\alpha \neq \beta$ we have $\tilde{A}^\alpha = A^\alpha$. In (1.4) $\mu, p_1, \dots, p_n$ are positive constants with $p_i > 1$ and $M : \Omega \to [0, +\infty)$ with $M \in L^r(\Omega)$, $r \geq 1$. If we keep in mind that $A = Du(x)$, then the left hand side of (1.4) shows $\sum_{i=1}^n |\tilde{A}_i - D_i u(x)|^{p_i}$, thus each component $D_i u$ of the gradient $Du$ may have a possibly different exponent $p_i$, so we are in the anisotropic framework: $u \in W^{1,1}(\Omega, \mathbb{R}^N)$ with $D_i u \in L^{p_i}(\Omega, \mathbb{R}^N)$. In this case the harmonic mean $\overline{p} = \left(\frac{1}{n}\sum_{i=1}^n \frac{1}{p_i}\right)^{-1}$ comes into play. In Section 2 we will prove the following

**Theorem 1.1.** *We consider the functional (1.1) under the "monotonicity" inequality (1.4) with*

$$\frac{\overline{p}^*}{\overline{p}}\left(1 - \frac{1}{r}\right) > 1 \tag{1.5}$$

*where $\overline{p}^*$ is the Sobolev exponent of $\overline{p} < n$. We consider $u = (u^1, \dots, u^N) \in W^{1,1}(\Omega, \mathbb{R}^N)$, with $D_i u \in L^{p_i}(\Omega, \mathbb{R}^N)$ $\forall i \in \{1, \dots, n\}$, such that*

$$\mathcal{F}(u) < +\infty$$

*and*

$$\mathcal{F}(u) \leq \mathcal{F}(v) \tag{1.6}$$

*for every $v \in u + W_0^{1,1}(\Omega, \mathbb{R}^N)$ with $D_i v \in L^{p_i}(\Omega, \mathbb{R}^N)$ $\forall i \in \{1, \dots, n\}$. Then, for every component $u^\beta$, we have*

$$\inf_{\partial\Omega} u^\beta - c_* \leq u^\beta(x) \leq \sup_{\partial\Omega} u^\beta + c_* \tag{1.7}$$

*for almost every $x \in \Omega$, where*

$$c_* = c\left(\frac{\|M\|_{L^r(\Omega)}}{\mu}\right)^{\frac{1}{\overline{p}}} |\Omega|^{\left[\left(1-\frac{1}{r}\right)\frac{\overline{p}^*}{\overline{p}}-1\right]\frac{1}{\overline{p}^*}} 2^{\left(1-\frac{1}{r}\right)\frac{\overline{p}^*}{\overline{p}}\left[\left(1-\frac{1}{r}\right)\frac{\overline{p}^*}{\overline{p}}-1\right]^{-1}},$$

*$c = c(n, p_1, \dots, p_n) > 0$ and $|\Omega|$ is the Lebesgue measure of $\Omega$.*

A model density $f$ for the "monotonicity" inequality (1.4) is given in the following.

**Lemma 1.2.** *For every $i = 1, \dots, n$, let us consider $p_i \in [2, +\infty)$ and $a_i \in (0, +\infty)$; we take $m : \Omega \to [0, +\infty)$ and $h : \mathbb{R} \to \mathbb{R}$ with $-\infty < \inf_{\mathbb{R}} h$. Let us consider $f : \Omega \times \mathbb{R}^{n\times n} \to \mathbb{R}$ defined as follows:*

$$f(x, A) = \sum_{i=1}^{n} a_i |A_i|^{p_i} + m(x)h(\det A). \tag{1.8}$$

*Then the "monotonicity" inequality (1.4) holds true with $\mu = \min_j a_j$ and $M(x) = m(x)[h(0) - \inf_{\mathbb{R}} h]$. Moreover, if $h \geq 0$, then $f \geq 0$ too.*

## 2. PROOFS

In order to prove Theorem 1.1, we need the following

**Lemma 2.1.** *Let us consider the functional (1.1) under the "monotonicity" assumption (1.4). Then, for every $v = (v^1, \dots, v^N) \in W^{1,1}(\Omega, \mathbb{R}^N)$ with $D_i v \in L^{p_i}(\Omega, \mathbb{R}^N)$ $\forall i \in \{1, \dots, n\}$, for any $\beta \in \{1, \dots, N\}$, for all $t \in \mathbb{R}$, it results that*

$$\mathcal{F}(I_{\beta,t}(v)) + \mu \sum_{i=1}^{n} \int_\Omega |D_i(I_{\beta,t}(v(x))) - D_i v(x)|^{p_i} dx \leq \mathcal{F}(v) + \int_{\{v^\beta > t\}} M(x) dx \tag{2.1}$$

*where $I_{\beta,t} : \mathbb{R}^N \to \mathbb{R}^N$ is defined as follows:*

$$\forall y = (y^1, \dots, y^N) \in \mathbb{R}^N, \qquad I_{\beta,t}(y) = (I^1_{\beta,t}(y), \dots, I^N_{\beta,t}(y))$$

*with*

$$I^\alpha_{\beta,t}(y) = \begin{cases} y^\alpha & \text{if } \alpha \neq \beta \\ y^\beta \wedge t = \min\{y^\beta, t\} & \text{if } \alpha = \beta. \end{cases} \tag{2.2}$$

*Proof.* For every $v \in W^{1,1}(\Omega, \mathbb{R}^N)$, with $D_i v \in L^{p_i}(\Omega, \mathbb{R}^N)$ $\forall i \in \{1, \dots, n\}$, it results that $I_{\beta,t}(v) \in W^{1,1}(\Omega, \mathbb{R}^N)$; moreover

$$D_i(I^\alpha_{\beta,t}(v)) = \begin{cases} D_i v^\alpha & \text{if } \alpha \neq \beta \\ 1_{\{v^\beta \leq t\}} D_i v^\beta & \text{if } \alpha = \beta, \end{cases} \tag{2.3}$$

where $1_B$ is the characteristic function of the set $B$, that is, $1_B(x) = 1$ if $x \in B$ and $1_B(x) = 0$ if $x \notin B$. Therefore $D_i(I_{\beta,t}(v)) \in L^{p_i}(\Omega, \mathbb{R}^N)$ $\forall i \in \{1, \dots, n\}$. On $\{x \in \Omega : v^\beta(x) > t\}$ we have $D(I^\beta_{\beta,t}(v)) = 0$ and, for $\alpha \neq \beta$, $D(I^\alpha_{\beta,t}(v)) = Dv^\alpha$; so we can apply (1.4) with $\tilde{A} = D(I_{\beta,t}(v))$ and $A = Dv$; we obtain

$$f(x, D(I_{\beta,t}(v(x)))) + \mu \sum_{i=1}^{n} |D_i(I_{\beta,t}(v(x))) - D_i v(x)|^{p_i} \leq f(x, Dv(x)) + M(x) \tag{2.4}$$

for $x \in \{v^\beta > t\}$. On $\{x \in \Omega : v^\beta(x) \leq t\}$ $D(I_{\beta,t}(v)) = Dv$, thus

$$f(x, D(I_{\beta,t}(v(x)))) + \mu \sum_{i=1}^{n} |D_i(I_{\beta,t}(v(x))) - D_i v(x)|^{p_i} = f(x, Dv(x)) \tag{2.5}$$

for $x \in \{v^\beta \leq t\}$. From (2.4) and (2.5) we have

$$f(x, D(I_{\beta,t}(v(x)))) + \mu \sum_{i=1}^{n} |D_i(I_{\beta,t}(v(x))) - D_i v(x)|^{p_i} \leq f(x, Dv(x)) + M(x) 1_{\{v^\beta > t\}}(x)$$

for $x \in \Omega$. If $x \to f(x, Dv(x)) \in L^1(\Omega)$, then $x \to f(x, D(I_{\beta,t}(v(x)))) \in L^1(\Omega)$ too and, integrating the last inequality with respect to $x$, we get (2.1). When $x \to f(x, Dv(x)) \notin L^1(\Omega)$, we have $\mathcal{F}(v) = +\infty$ and (2.1) holds true. This ends the proof of Lemma 2.1. □

Now we are ready to prove Theorem 1.1.

*Proof.* Let us fix $\beta \in \{1, \dots, N\}$. If $\sup_{\partial\Omega} u^\beta = +\infty$ then the right hand side of (1.7) is satisfied. Thus we assume $\sup_{\partial\Omega} u^\beta < t_0 \le t < +\infty$ and we note that under this assumption $I_{\beta,t}(u) \in u + W_0^{1,1}(\Omega, \mathbb{R}^N)$ and $D_i(I_{\beta,t}(u)) \in L^{p_i}(\Omega, \mathbb{R}^N)$ $\forall i \in \{1, \dots, n\}$ since

$$u^\beta \wedge t = \min\{u^\beta, t\} = u^\beta - [\max\{u^\beta - t, 0\}] = u^\beta - [(u^\beta - t) \vee 0],$$

where $(u^\beta - t) \vee 0 \in W_0^{1,1}(\Omega)$ and $D_i((u^\beta - t) \vee 0) = D_i u^\beta 1_{\{u^\beta > t\}} \in L^{p_i}(\Omega)$ $\forall i \in \{1, \dots, n\}$. From (1.6) and (2.1) it results that

$$\begin{aligned} \mathcal{F}(u) &\le \mathcal{F}(I_{\beta,t}(u)) \\ &\le \mathcal{F}(u) - \mu \sum_{i=1}^n \int_\Omega |D_i(I_{\beta,t}(u(x))) - D_i u(x)|^{p_i} dx + \int_{\{u^\beta > t\}} M(x) dx, \end{aligned}$$

that is

$$\mu \sum_{i=1}^n \int_\Omega |D_i(I_{\beta,t}(u(x))) - D_i u(x)|^{p_i} dx \le \int_{\{u^\beta > t\}} M(x) dx. \tag{2.6}$$

If we define $\phi = (u^\beta - t) \vee 0$, then we can write (2.6) as follows:

$$\mu \sum_{i=1}^n \int_\Omega |D_i \phi(x)|^{p_i} dx \le \int_{\{u^\beta > t\}} M(x) dx. \tag{2.7}$$

If $r < +\infty$, we apply Hölder's inequality to $\int_{\{u^\beta > t\}} M(x) dx$ and we obtain

$$\int_{\{u^\beta > t\}} M(x) dx \le \|M\|_{L^r(\Omega)} |\{u^\beta > t\}|^{\left(1 - \frac{1}{r}\right)}.$$

If $r = +\infty$, then

$$\int_{\{u^\beta > t\}} M(x) dx \le \|M\|_{L^\infty(\Omega)} |\{u^\beta > t\}| = \|M\|_{L^r(\Omega)} |\{u^\beta > t\}|^{\left(1 - \frac{1}{r}\right)}.$$

In both cases, from (2.7) it results that

$$\sum_{i=1}^n \int_\Omega |D_i \phi(x)|^{p_i} dx \le \frac{\|M\|_{L^r(\Omega)}}{\mu} |\{u^\beta > t\}|^{\left(1 - \frac{1}{r}\right)}$$

in particular, $\forall i \in \{1, \dots, n\}$

$$\int_\Omega |D_i \phi(x)|^{p_i} dx \le \frac{\|M\|_{L^r(\Omega)}}{\mu} |\{u^\beta > t\}|^{\left(1 - \frac{1}{r}\right)}$$

from which

$$\left( \int_\Omega |D_i \phi(x)|^{p_i} dx \right)^{\frac{1}{p_i}} \le \left[ \frac{\|M\|_{L^r(\Omega)}}{\mu} |\{u^\beta > t\}|^{\left(1 - \frac{1}{r}\right)} \right]^{\frac{1}{p_i}}$$

and finally

$$\left[ \prod_{i=1}^n \left( \int_\Omega |D_i \phi(x)|^{p_i} dx \right)^{\frac{1}{p_i}} \right]^{\frac{1}{n}} \le \left[ \frac{\|M\|_{L^r(\Omega)}}{\mu} |\{u^\beta > t\}|^{\left(1 - \frac{1}{r}\right)} \right]^{\frac{1}{\overline{p}}}. \tag{2.8}$$

We apply the anisotropic imbedding theorem [10] and we use (2.8):

$$
\begin{aligned}
(2.9)\qquad 0 &\leq \left(\int_{\{u^\beta>t\}} \left[u^\beta(x)-t\right]^{\overline{p}^*} dx\right)^{\frac{1}{\overline{p}^*}} \\
&= \|\phi\|_{L^{\overline{p}^*}(\Omega)} \\
&\leq c\left[\prod_{i=1}^{n}\left(\int_\Omega |D_i\phi(x)|^{p_i} dx\right)^{\frac{1}{p_i}}\right]^{\frac{1}{n}} \\
&\leq c\left[\frac{\|M\|_{L^r(\Omega)}}{\mu}|\{u^\beta>t\}|^{\left(1-\frac{1}{r}\right)}\right]^{\frac{1}{\overline{p}}},
\end{aligned}
$$

where $c=c(n,p_1,\ldots,p_n)>0$. If $\|M\|_{L^r(\Omega)}=0$, then from (2.9) it results that $u^\beta\leq t$ almost everywhere in $\Omega$ and we are done. If $\|M\|_{L^r(\Omega)}>0$, then for $T>t$ we have

$$
\begin{aligned}
(2.10)\qquad (T-t)^{\overline{p}^*}|\{u^\beta>T\}| &= \int_{\{u^\beta>T\}}(T-t)^{\overline{p}^*}dx \\
&\leq \int_{\{u^\beta>T\}}\left[u^\beta(x)-t\right]^{\overline{p}^*}dx \\
&\leq \int_{\{u^\beta>t\}}\left[u^\beta(x)-t\right]^{\overline{p}^*}dx
\end{aligned}
$$

and from (2.9) and (2.10) we get

$$
(2.11)\qquad |\{u^\beta>T\}| \leq c^{\overline{p}^*}\left(\frac{\|M\|_{L^r(\Omega)}}{\mu}\right)^{\frac{\overline{p}^*}{\overline{p}}}\frac{1}{(T-t)^{\overline{p}^*}}|\{u^\beta>t\}|^{\left(1-\frac{1}{r}\right)\frac{\overline{p}^*}{\overline{p}}}
$$

for every $T,t$ with $T>t\geq t_0$. We set $\chi(t)=|\{u^\beta>t\}|$ and we use [7, Lemma 4.1, p. 93], that we provide below for the convenience of the reader.

**Lemma 2.2.** *Let* $\chi:[t_0,+\infty)\to[0,+\infty)$ *be decreasing. We assume that there exist* $k,a\in(0,+\infty)$ *and* $b\in(1,+\infty)$ *such that*

$$
(2.12)\qquad T>t\geq t_0 \Longrightarrow \chi(T)\leq\frac{k}{(T-t)^a}(\chi(t))^b.
$$

*Then it results that*

$$
(2.13)\qquad \chi(t_0+d)=0 \quad \textit{where} \quad d=\left[k(\chi(t_0))^{b-1}2^{\frac{ab}{(b-1)}}\right]^{\frac{1}{a}}.
$$

We use the previous Lemma 2.2 and we have

$$
(2.14)\qquad |\{u^\beta>t_0+d\}|=0
$$

that is

$$
(2.15)\qquad u^\beta\leq t_0+d
$$

almost everywhere in $\Omega$, where

$$
d=c\left(\frac{\|M\|_{L^r(\Omega)}}{\mu}\right)^{\frac{1}{\overline{p}}}\left|\{u^\beta>t_0\}\right|^{\left[\left(1-\frac{1}{r}\right)\frac{\overline{p}^*}{\overline{p}}-1\right]\frac{1}{\overline{p}^*}}2^{\left(1-\frac{1}{r}\right)\frac{\overline{p}^*}{\overline{p}}\left[\left(1-\frac{1}{r}\right)\frac{\overline{p}^*}{\overline{p}}-1\right]^{-1}}.
$$

In order to get the right hand side of (1.7), we control $|\{u^\beta > t_0\}|$ by means of $|\Omega|$ and we take a sequence $\{(t_0)_m\}_m$ with $(t_0)_m \to \sup_{\partial\Omega} u^\beta$. Let us show how we obtain the left hand side of (1.7): we apply the right hand side of (1.7) to $-u$. This ends the proof of Theorem 1.1. $\square$

Now we are going to prove Lemma 1.2.

*Proof.* We assume that $\tilde{A}, A \in \mathbb{R}^{n\times n}$ with $\tilde{A}^\beta = 0$ and $\tilde{A}^\alpha = A^\alpha$ for $\alpha \neq \beta$. Then

$$\begin{aligned}\sum_\alpha |A_i^\alpha|^2 &= |A_i^\beta|^2 + \sum_{\alpha\neq\beta} |A_i^\alpha|^2 \\ &= |A_i^\beta - \tilde{A}_i^\beta|^2 + \sum_{\alpha\neq\beta} |\tilde{A}_i^\alpha|^2 \\ &= \sum_\alpha |A_i^\alpha - \tilde{A}_i^\alpha|^2 + \sum_\alpha |\tilde{A}_i^\alpha|^2\end{aligned} \tag{2.16}$$

so

$$|A_i|^2 = |A_i - \tilde{A}_i|^2 + |\tilde{A}_i|^2. \tag{2.17}$$

Since $p_i \geq 2$, the previous equality gives

$$|A_i|^{p_i} \geq |A_i - \tilde{A}_i|^{p_i} + |\tilde{A}_i|^{p_i}. \tag{2.18}$$

Moreover

$$h(\det A) \geq \inf_{\mathbb{R}} h = h(0) - [h(0) - \inf_{\mathbb{R}} h] = h(\det \tilde{A}) - [h(0) - \inf_{\mathbb{R}} h]. \tag{2.19}$$

Now we are able to estimate $f(x, A)$ and $f(x, \tilde{A})$ by means of (2.18) and (2.19) as follows:

$$\begin{aligned} & f(x, \tilde{A}) + (\min_j a_j) \sum_{i=1}^n |A_i - \tilde{A}_i|^{p_i} \\ &\leq \sum_{i=1}^n a_i |\tilde{A}_i|^{p_i} + m(x) h(\det \tilde{A}) + \sum_{i=1}^n a_i |A_i - \tilde{A}_i|^{p_i} \\ &\leq \sum_{i=1}^n a_i |A_i|^{p_i} + m(x) h(\det A) + m(x)[h(0) - \inf_{\mathbb{R}} h] \\ &= f(x, A) + m(x)[h(0) - \inf_{\mathbb{R}} h]\end{aligned} \tag{2.20}$$

thus the "monotonicity" inequality (1.4) holds true with $\mu = \min_j a_j$ and $M(x) = m(x)[h(0) - \inf_{\mathbb{R}} h]$. This ends the proof of Lemma 1.2. $\square$

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