Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 4, Issue 5, Article 83, 2003

RATIONAL IDENTITIES AND INEQUALITIES INVOLVING FIBONACCI AND LUCAS NUMBERS

JOSÉ LUIS DÍAZ-BARRERO

APPLIED MATHEMATICS III
UNIVERSITAT POLITÈCNICA DE CATALUNYA
JORDI GIRONA 1-3, C2,
08034 BARCELONA, SPAIN.
jose.luis.diaz@upc.es

Received 21 May, 2003; accepted 21 August, 2003
Communicated by J. Sándor

ABSTRACT. In this paper we use integral calculus, complex variable techniques and some classical inequalities to establish rational identities and inequalities involving Fibonacci and Lucas numbers.

Key words and phrases: Rational Identities and Inequalities, Sequences of Integers, Fibonacci and Lucas numbers.

2000 *Mathematics Subject Classification.* 05A19, 11B39.

1. INTRODUCTION

The Fibonacci sequence is a source of many nice and interesting identities and inequalities. A similar interpretation exists for Lucas numbers. Many of these identities have been documented in an extensive list that appears in the work of Vajda [1], where they are proved by algebraic means, even though combinatorial proofs of many of these interesting algebraic identities are also given (see [2]). However, rational identities and inequalities involving Fibonacci and Lucas numbers seldom have appeared (see [3]). In this paper, integral calculus, complex variable techniques and some classical inequalities are used to obtain several rational Fibonacci and Lucas identities and inequalities.

2. RATIONAL IDENTITIES

In what follows several rational identities are considered and proved by using results on contour integrals. We begin with:

ISSN (electronic): 1443-5756

The author would like to thank the anonymous referee for the suggestions and comments that have improved the final version of the paper.
067-03

Theorem 2.1. *Let F_n denote the n^{th} Fibonacci number. That is, $F_0 = 0$, $F_1 = 1$ and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Then, for all positive integers r,*

$$\sum_{k=1}^{n} \frac{1 + F_{r+k}^{\ell}}{F_{r+k}} \left\{ \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{1}{F_{r+k} - F_{r+j}} \right\} = \frac{(-1)^{n+1}}{F_{r+1}F_{r+2}\cdots F_{r+n}} \tag{2.1}$$

holds, with $0 \leq \ell \leq n-1$.

Proof. To prove the preceding identity we consider the integral

$$I = \frac{1}{2\pi i} \oint_{\gamma} \frac{1 + z^{\ell}}{A_n(z)} \frac{dz}{z}, \tag{2.2}$$

where $A_n(z) = \prod_{j=1}^{n}(z - F_{r+j})$.

Let γ be the curve defined by $\gamma = \{z \in \mathbb{C} : |z| < F_{r+1}\}$. Evaluating the preceding integral in the exterior of the γ contour, we obtain

$$I_1 = \frac{1}{2\pi i} \oint_{\gamma} \left\{ \frac{1 + z^{\ell}}{z} \prod_{j=1}^{n} \frac{1}{(z - F_{r+j})} \right\} dz = \sum_{k=1}^{n} R_k,$$

where

$$R_k = \lim_{z \to F_{r+k}} \left\{ \frac{1 + z^{\ell}}{z} \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{1}{(z - F_{r+j})} \right\} = \frac{1 + F_{r+k}^{\ell}}{F_{r+k}} \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{1}{(F_{r+k} - F_{r+j})}.$$

Then, I_1 becomes

$$I_1 = \sum_{k=1}^{n} \left\{ \frac{1 + F_{r+k}^{\ell}}{F_{r+k}} \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{1}{(F_{r+k} - F_{r+j})} \right\}.$$

Evaluating (2.2) in the interior of the γ contour, we get

$$\begin{aligned} I_2 &= \frac{1}{2\pi i} \oint_{\gamma} \left\{ \frac{1 + z^{\ell}}{z} \prod_{j=1}^{n} \frac{1}{(z - F_{r+j})} \right\} dz \\ &= \lim_{z \to 0} \left\{ \frac{1 + z^{\ell}}{A_n(z)} \right\} \\ &= \frac{1}{A_n(0)} = \frac{(-1)^n}{F_{r+1}F_{r+2}\cdots F_{r+n}}. \end{aligned}$$

By Cauchy's theorem on contour integrals we have that $I_1 + I_2 = 0$ and the proof is complete. □

A similar identity also holds for Lucas numbers. It can be stated as:

Corollary 2.2. *Let L_n denote the n^{th} Lucas number. That is, $L_0 = 2, L_1 = 1$ and for $n \geq 2$, $L_n = L_{n-1} + L_{n-2}$. Then, for all positive integers r,*

$$\sum_{k=1}^{n} \frac{1 + L_{r+k}^{\ell}}{L_{r+k}} \left\{ \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{1}{L_{r+k} - L_{r+j}} \right\} = \frac{(-1)^{n+1}}{L_{r+1}L_{r+2}\cdots + L_{r+n}} \tag{2.3}$$

holds, with $0 \leq \ell \leq n-1$.

In particular (2.1) and (2.3) can be used (see [3]) to obtain

Corollary 2.3. *For* $n \geq 2$,

$$\frac{(F_n^2+1)F_{n+1}F_{n+2}}{(F_{n+1}-F_n)(F_{n+2}-F_n)} + \frac{F_n(F_{n+1}^2+1)F_{n+2}}{(F_n-F_{n+1})(F_{n+2}-F_{n+1})} + \frac{F_nF_{n+1}(F_{n+2}^2+1)}{(F_n-F_{n+2})(F_{n+1}-F_{n+2})} = 1.$$

Corollary 2.4. *For* $n \geq 2$,

$$\frac{L_{n+1}L_{n+2}}{(L_{n+1}-L_n)(L_{n+2}-L_n)} + \frac{L_{n+2}L_n}{(L_n-L_{n+1})(L_{n+2}-L_{n+1})} + \frac{L_nL_{n+1}}{(L_n-L_{n+2})(L_{n+1}-L_{n+2})} = 1.$$

In the sequel F_n and L_n denote the n^{th} Fibonacci and Lucas numbers, respectively.

Theorem 2.5. *If* $n \geq 3$, *then we have*

$$\sum_{i=1}^{n} \frac{1}{L_i^{n-2}} \left[\prod_{\substack{j=1 \\ j\neq i}}^{n} \left(1 - \frac{L_j}{L_i}\right)^{-1} + L_i^{n-1} \right] = L_{n+2} - 3.$$

Proof. First, we observe that the given statement can be written as

$$\sum_{i=1}^{n} \left[\frac{1}{L_i^{n-2}} \prod_{\substack{j=1 \\ j\neq i}}^{n} \left(1 - \frac{L_j}{L_i}\right)^{-1} \right] + \sum_{i=1}^{n} L_i = L_{n+2} - 3.$$

Since $\sum_{i=1}^{n} L_i = L_{n+2} - 3$, as can be easily established by mathematical induction, then it will suffice to prove

$$\sum_{i=1}^{n} \left[\frac{1}{L_i^{n-2}} \prod_{\substack{j=1 \\ j\neq i}}^{n} \left(1 - \frac{L_j}{L_i}\right)^{-1} \right] = 0. \tag{2.4}$$

We will argue by using residue techniques. We consider the monic complex polynomial $A(z) = \prod_{k=1}^{n}(z - L_k)$ and we evaluate the integral

$$I = \frac{1}{2\pi i} \oint_\gamma \frac{z}{A(z)} dz$$

over the interior and exterior domains limited by γ, a circle centered at the origin and radius L_{n+1}, i.e., $\gamma = \{z \in \mathbb{C} : |z| < L_{n+1}\}$.

Integrating in the region inside the γ contour we have

$$\begin{aligned}
I_1 &= \frac{1}{2\pi i}\oint_\gamma \frac{z}{A(z)}\,dz \\
&= \sum_{i=1}^{n} \operatorname{Res}\left\{\frac{z}{A(z)}, z = L_i\right\} \\
&= \sum_{i=1}^{n}\left(\prod_{\substack{j=1\\ j\neq i}}^{n} \frac{L_i}{L_i - L_j}\right) \\
&= \sum_{i=1}^{n}\left[\frac{1}{L_i^{n-2}}\prod_{\substack{j=1\\ j\neq i}}^{n}\left(1-\frac{L_j}{L_i}\right)^{-1}\right].
\end{aligned}$$

Integrating in the region outside of the γ contour we get

$$I_2 = \frac{1}{2\pi i}\oint_\gamma \frac{z}{A(z)}\,dz = 0.$$

Again, by Cauchy's theorem on contour integrals we have $I_1 + I_2 = 0$. This completes the proof of (2.4). $\square$

Note that (2.4) can also be established by using routine algebra.

3. Inequalities

Next, several inequalities are considered and proved with the aid of integral calculus and the use of classical inequalities. First, we state and prove some nice inequalities involving circular powers of Lucas numbers similar to those obtained for Fibonacci numbers in [4].

Theorem 3.1. *Let n be a positive integer, then the following inequalities hold*

(a) $$L_n^{L_{n+1}} + L_{n+1}^{L_n} < L_n^{L_n} + L_{n+1}^{L_{n+1}},$$

(b) $$L_{n+1}^{L_{n+2}} - L_{n+1}^{L_n} < L_{n+2}^{L_{n+2}} - L_{n+2}^{L_n}.$$

Proof. To prove part (a) we consider the integral

$$I_1 = \int_{L_n}^{L_{n+1}} \left(L_{n+1}^x \log L_{n+1} - L_n^x \log L_n\right) dx.$$

Since $L_n < L_{n+1}$ if $n \geq 1$, then for $L_n \leq x \leq L_{n+1}$ we have

$$L_n^x \log L_n < L_{n+1}^x \log L_n < L_{n+1}^x \log L_{n+1}$$

and $I_1 > 0$. On the other hand, evaluating the integral, we obtain

$$\begin{aligned}
I_1 &= \int_{L_n}^{L_{n+1}} \left(L_{n+1}^x \log L_{n+1} - L_n^x \log L_n\right) dx \\
&= \left[L_{n+1}^x - L_n^x\right]_{L_n}^{L_{n+1}} \\
&= \left(L_n^{L_n} + L_{n+1}^{L_{n+1}}\right) - \left(L_n^{L_{n+1}} + L_{n+1}^{L_n}\right)
\end{aligned}$$

and (a) is proved. To prove (b), we consider the integral

$$I_2 = \int_{L_n}^{L_{n+2}} \left(L_{n+2}^x \log L_{n+2} - L_{n+1}^x \log L_{n+1}\right) dx.$$

Since $L_{n+1} < L_{n+2}$, then for $L_n \leq x \leq L_{n+2}$ we have

$$L_{n+1}^x \log L_{n+1} < L_{n+2}^x \log L_{n+2}$$

and $I_2 > 0$. On the other hand, evaluating I_2, we get

$$\begin{aligned} I_2 &= \int_{L_n}^{L_{n+2}} \left(L_{n+2}^x \log L_{n+2} - L_{n+1}^x \log L_{n+1} \right) dx \\ &= \left[L_{n+2}^x - L_{n+1}^x \right]_{L_n}^{L_{n+2}} \\ &= \left(L_{n+2}^{L_{n+2}} - L_{n+2}^{L_n} \right) - \left(L_{n+1}^{L_{n+2}} - L_{n+1}^{L_n} \right). \end{aligned}$$

This completes the proof. □

Corollary 3.2. *For $n \geq 1$, we have*

$$L_n^{L_{n+1}} + L_{n+1}^{L_{n+2}} + L_{n+2}^{L_n} < L_n^{L_n} + L_{n+1}^{L_{n+1}} + L_{n+2}^{L_{n+2}}.$$

Proof. The statement immediately follows from the fact that

$$\begin{aligned} &\left(L_n^{L_n} + L_{n+1}^{L_{n+1}} + L_{n+2}^{L_{n+2}} \right) - \left(L_n^{L_{n+1}} + L_{n+1}^{L_{n+2}} + L_{n+2}^{L_n} \right) \\ &\qquad = \left[\left(L_n^{L_n} + L_{n+1}^{L_{n+1}} \right) - \left(L_n^{L_{n+1}} + L_{n+1}^{L_n} \right) \right] \\ &\qquad\qquad + \left[\left(L_{n+2}^{L_{n+2}} - L_{n+2}^{L_n} \right) - \left(L_{n+1}^{L_{n+2}} - L_{n+1}^{L_n} \right) \right] \end{aligned}$$

and Theorem 3.1. □

Theorem 3.3. *Let n be a positive integer, then the following inequality*

$$L_n^{L_{n+1}} L_{n+1}^{L_{n+2}} L_{n+2}^{L_n} < L_n^{L_n} L_{n+1}^{L_{n+1}} L_{n+2}^{L_{n+2}}$$

holds.

Proof. We will argue by using the weighted AM-GM-HM inequality (see [5]). The proof will be done in two steps. First, we will prove

$$L_n^{L_{n+1}} L_{n+1}^{L_{n+2}} L_{n+2}^{L_n} < \left(\frac{L_n + L_{n+1} + L_{n+2}}{3} \right)^{L_n + L_{n+1} + L_{n+2}}. \tag{3.1}$$

In fact, setting $x_1 = L_n$, $x_2 = L_{n+1}$, $x_3 = L_{n+2}$ and

$$\begin{aligned} w_1 &= \frac{L_{n+1}}{L_n + L_{n+1} + L_{n+2}}, \\ w_2 &= \frac{L_{n+2}}{L_n + L_{n+1} + L_{n+2}}, \\ w_3 &= \frac{L_n}{L_n + L_{n+1} + L_{n+2}} \end{aligned}$$

and applying the AM-GM inequality, we have

$$\begin{aligned} &L_n^{L_{n+1}/(L_n + L_{n+1} + L_{n+2})} L_{n+1}^{L_{n+2}/(L_n + L_{n+1} + L_{n+2})} L_{n+2}^{L_n/(L_n + L_{n+1} + L_{n+2})} \\ &\qquad < \frac{L_n L_{n+1}}{L_n + L_{n+1} + L_{n+2}} + \frac{L_{n+1} L_{n+2}}{L_n + L_{n+1} + L_{n+2}} + \frac{L_{n+2} L_n}{L_n + L_{n+1} + L_{n+2}} \end{aligned}$$

or

$$L_n^{L_{n+1}} L_{n+1}^{L_{n+2}} L_{n+2}^{L_n} < \left(\frac{L_n L_{n+1} + L_{n+1} L_{n+2} + L_{n+2} L_n}{L_n + L_{n+1} + L_{n+2}} \right)^{L_n + L_{n+1} + L_{n+2}}.$$

Inequality (3.1) will be established if we prove that

$$\left(\frac{L_nL_{n+1}+L_{n+1}L_{n+2}+L_{n+2}L_n}{L_n+L_{n+1}+L_{n+2}}\right)^{L_n+L_{n+1}+L_{n+2}} < \left(\frac{L_n+L_{n+1}+L_{n+2}}{3}\right)^{L_n+L_{n+1}+L_{n+2}}$$

or, equivalently,

$$\frac{L_nL_{n+1}+L_{n+1}L_{n+2}+L_{n+2}L_n}{L_n+L_{n+1}+L_{n+2}} < \frac{L_n+L_{n+1}+L_{n+2}}{3}.$$

That is,

$$L_n^2+L_{n+1}^2+L_{n+2}^2 > L_nL_{n+1}+L_{n+1}L_{n+2}+L_{n+2}L_n.$$

The last inequality immediately follows by adding up the inequalities

$$\begin{aligned} L_n^2+L_{n+1}^2 &\geq 2L_nL_{n+1}, \\ L_{n+1}^2+L_{n+2}^2 &> 2L_{n+1}L_{n+2}, \\ L_{n+2}^2+L_n^2 &> 2L_{n+2}L_n \end{aligned}$$

and the result is proved.

Finally, we will prove

$$\left(\frac{L_n+L_{n+1}+L_{n+2}}{3}\right)^{L_n+L_{n+1}+L_{n+2}} < L_n^{L_n}L_{n+1}^{L_{n+1}}L_{n+2}^{L_{n+2}}. \tag{3.2}$$

In fact, setting

$$\begin{aligned} &x_1 = L_n, \quad x_2 = L_{n+1}, \quad x_3 = L_{n+2}, \\ &w_1 = L_n/(L_n+L_{n+1}+L_{n+2}), \\ &w_2 = L_{n+1}/(L_n+L_{n+1}+L_{n+2}), \quad \text{and} \\ &w_3 = L_{n+2}/(L_n+L_{n+1}+L_{n+2}) \end{aligned}$$

and using the GM-HM inequality, we have

$$\begin{aligned} \frac{L_n+L_{n+1}+L_{n+2}}{3} &= \left(\frac{3}{L_n+L_{n+1}+L_{n+2}}\right)^{-1} \\ &= \frac{1}{\frac{1}{L_n+L_{n+1}+L_{n+2}}+\frac{1}{L_n+L_{n+1}+L_{n+2}}+\frac{1}{L_n+L_{n+1}+L_{n+2}}} \\ &< L_n^{L_n/(L_n+L_{n+1}+L_{n+2})}L_{n+1}^{L_{n+1}/(L_n+L_{n+1}+L_{n+2})}L_{n+2}^{L_{n+2}/(L_n+L_{n+1}+L_{n+2})}. \end{aligned}$$

Hence,

$$\left(\frac{L_n+L_{n+1}+L_{n+2}}{3}\right)^{L_n+L_{n+1}+L_{n+2}} < L_n^{L_n}L_{n+1}^{L_{n+1}}L_{n+2}^{L_{n+2}}$$

and (3.2) is proved. This completes the proof of the theorem. □

Stronger inequalities for second order recurrence sequences, generalizing the ones given in [4] have been obtained by Stanica in [6].

Finally, we state and prove an inequality involving Fibonacci and Lucas numbers.

Theorem 3.4. *Let n be a positive integer, then the following inequality*

$$\sum_{k=1}^{n}\frac{F_{k+2}}{F_{2k+2}} \geq \frac{n^{n+1}}{(n+1)^n}\prod_{k=1}^{n}\left\{\frac{F_{k+1}^{-\frac{n+1}{n}}-L_{k+1}^{-\frac{n+1}{n}}}{F_{k+1}^{-1}-L_{k+1}^{-1}}\right\}$$

holds.

Proof. From the AM-GM inequality, namely,

$$\frac{1}{n}\sum_{k=1}^{n} x_k \geq \prod_{k=1}^{n} x_k^{\frac{1}{n}}, \quad \text{where} \quad x_k > 0,\ k = 1, 2, \dots, n,$$

and taking into account that for all $j \geq 2$, $0 < L_j^{-1} < F_j^{-1}$, we get

$$\text{(3.3)} \quad \int_{L_2^{-1}}^{F_2^{-1}} \int_{L_3^{-1}}^{F_3^{-1}} \cdots \int_{L_{n+1}^{-1}}^{F_{n+1}^{-1}} \left(\frac{1}{n} \sum_{\ell=2}^{n+1} x_\ell \right) dx_2 dx_3 \dots dx_{n+1} \geq \int_{L_2^{-1}}^{F_2^{-1}} \int_{L_3^{-1}}^{F_3^{-1}} \cdots \int_{L_{n+1}^{-1}}^{F_{n+1}^{-1}} \left(\prod_{\ell=1}^{n+1} x_\ell^{\frac{1}{n}} \right) dx_2 dx_3 \dots dx_{n+1}.$$

Evaluating the preceding integrals (3.3) becomes

$$\text{(3.4)} \quad \sum_{\ell=2}^{n+1} (F_2^{-1} - L_2^{-1}) \cdots (F_{\ell-1}^{-1} - L_{\ell-1}^{-1})(F_\ell^{-2} - L_\ell^{-2})(F_{\ell+1}^{-1} - L_{\ell+1}^{-1}) \cdots (F_{n+1}^{-1} - L_{n+1}^{-1}) \geq \frac{2n^{n+1}}{(n+1)^n} \prod_{\ell=2}^{n+1} \left(F_\ell^{-\frac{n+1}{n}} - L_\ell^{-\frac{n+1}{n}} \right)$$

or, equivalently,

$$\prod_{\ell=2}^{n+1} (F_\ell^{-1} - L_\ell^{-1}) \sum_{\ell=2}^{n+1} (F_\ell^{-1} + L_\ell^{-1}) \geq \frac{2n^{n+1}}{(n+1)^n} \prod_{\ell=2}^{n+1} \left(F_\ell^{-\frac{n+1}{n}} - L_\ell^{-\frac{n+1}{n}} \right).$$

Setting $k = \ell - 1$ in the preceding inequality, taking into account that $F_k + L_k = 2F_{k+1}$, $F_k L_k = F_{2k}$ and after simplification, we obtain

$$\sum_{k=1}^{n} \frac{F_{k+2}}{F_{2k+2}} \geq \frac{n^{n+1}}{(n+1)^n} \prod_{k=1}^{n} \left\{ \frac{F_{k+1}^{-\frac{n+1}{n}} - L_{k+1}^{-\frac{n+1}{n}}}{F_{k+1}^{-1} - L_{k+1}^{-1}} \right\}$$

and the proof is completed. □

REFERENCES

[1] S. VAJDA, *Fibonacci and Lucas Numbers and the Golden Section*, New York, Wiley, 1989.

[2] A.T. BENJAMIN AND J. J. QUINN, Recounting Fibonacci and Lucas identities, *College Math. J.*, **30**(5) (1999), 359–366.

[3] J.L. DÍAZ-BARRERO, Problem B–905, *The Fibonaci Quarterly,* **38**(4) (2000), 373.

[4] J.L. DÍAZ-BARRERO, Advanced Problem H–581, *The Fibonaci Quarterly,* **40**(1) (2002), 91.

[5] G. HARDY, J.E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, Cambridge, 1997.

[6] P. STANICA, Inequalities on linear functions and circular powers, *J. Ineq. Pure and Appl. Math.*, **3**(3) (2002), Article 43. [ONLINE: `http://jipam.vu.edu.au/v3n3/015_02.html`]