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ON THE DETERMINANTAL INEQUALITIES

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ABSTRACT. In this paper, we discuss the determinantal inequalities over arbitrary complex matrices, and give some sufficient conditions for

$$d[A+B]^t \geq d[A]^t + d[B]^t,$$

where $t \in \mathrm{R}$ and $t \geq \frac{2}{n}$. If B is nonsingular and $\operatorname{Re}\lambda(B^{-1}A) \geq 0$, the sufficient and necessary condition is given for the above equality at $t = \frac{2}{n}$. The famous Minkowski inequality and many recent results about determinantal inequalities are extended.

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1. PRELIMINARIES

We use conventional notions and notations, as in [2]. Let $A \in M_n(C)$, $d[A]$ stands for the modulus of $\det(A)$ (or $|A|$), where $\det(A)$ is the determinant of A. $\sigma(A)$ is the spectrum of A, namely the set of eigenvalues of matrix A. A matrix $X \in M_n(C)$ is called complex (semi-) positive definite if $\operatorname{Re}(x^*Ax) > 0$ ($\operatorname{Re}(x^*Ax) \geq 0$) for all nonzero $x \in C^n$ or if $\frac{1}{2}(X + X^*)$ is a complex (semi-)positive definite matrix (see [4, 7, 8, 2]). Throughout this paper, we denote $C = B^{-1}A$ for $A, B \in M_n(C)$ and B is invertible.

The famous Minkowski inequality states:

If $A, B \in M_n(R)$ are real positive definite symmetric matrices, then

$$|A+B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}. \tag{1.1}$$

It is a very interesting work to generalize the Minkowski inequality. Obviously, (1.1) holds if $A, B \in M_n(C)$ are positive definite Hermitian matrices. Recently, (1.1) has been generalized for $A, B \in M_n(C)$ positive definite matrices (see [8], [9], [10], [3]).

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In this paper, we discuss determinantal inequalities over arbitrary complex matrices, and give some sufficient conditions for

$$d[A+B]^t \geq d[A]^t + d[B]^t, \tag{1.2}$$

where $t \in \mathrm{R}$.

If B is nonsingular and $\operatorname{Re}\lambda(B^{-1}A) \geq 0$, a sufficient and necessary condition has been given for equality as $t = \frac{2}{n}$ in (1.2). The famous Minkowski inequality and many results about determinantal inequalities are extended.

For $c \in C$, $\operatorname{Re}(c)$ denotes the real part of c and $|c|$ denotes the modulus of c. Let $t > 0$ be fixed, we have

Lemma 1.1. *If $A, B \in M_n(C)$ and B is invertible, $\sigma(C) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$, then inequality (1.2) is true if and only if*

$$\prod_{i=1}^{n} |\lambda_i + 1|^t \geq \prod_{i=1}^{n} |\lambda_i|^t + 1, \tag{1.3}$$

with equality holding in (1.2) if and only if it holds in (1.3).

Proof. Since $d[A+B]^t = d[B]^t d[C+I]^t$ and $d[A]^t + d[B]^t = d[B]^t(1 + d[C]^t)$, formula (1.2) is equivalent to

$$d[C+I]^t \geq 1 + d[C]^t. \tag{1.4}$$

Notice $\sigma(C+I) = \{\lambda_k + 1 : k = 1, 2, \dots, n\}$,

$$d[C+I]^t = \prod_{i=1}^{n} |\lambda_i + 1|^t \quad \text{and} \quad d[C]^t = \prod_{i=1}^{n} |\lambda_i|^t,$$

we obtain that formula (1.4) is equivalent to (1.3). Similarly, it is easy to see that the case of equality is true. Thus the lemma is proved. □

Lemma 1.2 (see [6]). *If $x_t, y_t \geq 0$ $(t = 1, 2, \dots, n)$, then*

$$\prod_{t=1}^{n} (x_t + y_t)^{\frac{1}{n}} \geq \prod_{t=1}^{n} x_t^{\frac{1}{n}} + \prod_{t=1}^{n} y_t^{\frac{1}{n}},$$

with equality if and only if there is linear dependence between $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_n)$ or $x_t + y_t = 0$ for a certain number t.

Lemma 1.3 (Jensen's inequality). *If a_1, a_2,$\dots$, a_m are positive numbers, then*

$$\left(\sum_{i=1}^{n} a_i^s\right)^{\frac{1}{s}} \leq \left(\sum_{i=1}^{n} a_i^r\right)^{\frac{1}{r}} \quad \textit{for} \quad 0 < r \leq s,\ n \geq 2.$$

Lemma 1.4. *If P_1, P_2,$\dots$, P_m are positive numbers and $T \geq \frac{1}{m}$, then*

$$\prod_{k=1}^{m} (P_k + 1)^T \geq \prod_{k=1}^{m} P_k^T + 1, \tag{1.5}$$

with equality if and only if P_k $(k = 1, 2, \dots, n)$ is constant as $T = \frac{1}{m}$.

Proof. By Lemma 1.2, we have

$$\prod_{k=1}^{m} (P_k + 1)^T = \left[\prod_{k=1}^{m} (P_k + 1)^{\frac{1}{m}}\right]^{mT} \geq \left[\prod_{k=1}^{m} \left(P_k^T\right)^{\frac{1}{mT}} + 1\right]^{mT}.$$

On noting that $0 < \frac{1}{mT} \le 1$, by Lemma 1.3, we obtain

$$\left[\prod_{k=1}^{m}\left(P_k^T\right)^{\frac{1}{mT}} + 1\right]^{mT} \ge \prod_{k=1}^{m} P_k^T + 1,$$

and inequality (1.5) is demonstrated. By Lemma 1.2, it is easy to see that equality holds if and only if P_k $(k = 1, 2, \dots, n)$ is constant as $T = \frac{1}{m}$. □

Remark 1.5. Apparently, Lemma 1.3 is tenable for $a_i \ge 0$ $(i = 1, 2, \dots, n)$, and Lemma 1.4 is tenable for $P_i \ge 0$ $(i = 1, 2, \dots, n)$.

2. MAIN RESULTS

Theorem 2.1. *Let* $A, B \in M_n(C)$*. If* B *is nonsingular and* $\operatorname{Re}\lambda_k \ge 0$ $(k = 1, 2, \dots, n)$*, where* $\sigma(C) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$*, then for* $t \ge \frac{2}{n}$

$$d[A+B]^t \ge d[A]^t + d[B]^t, \tag{2.1}$$

Proof. By Lemma 1.1, we need to prove inequality (1.3). Note that $\operatorname{Re}\lambda_k \ge 0$ $(k = 1, 2, \dots, n)$ and $|\lambda_k + 1|^2 \ge 1 + |\lambda_k|^2$,

$$\prod_{k=1}^{n} |\lambda_k + 1|^t = \left(\prod_{k=1}^{n} |\lambda_k + 1|^2\right)^{\frac{t}{2}} \ge \prod_{k=1}^{n}\left(|\lambda_k|^2 + 1\right)^{\frac{t}{2}}.$$

Applying Lemma 1.4, we can show that

$$\prod_{k=1}^{n} (|\lambda_k|^2 + 1)^{\frac{t}{2}} \ge \prod_{k=1}^{n} |\lambda_k|^t + 1 \quad \text{for } t \ge \frac{2}{n},$$

with equality if and only if $|\lambda_k|^2$ $(k = 1, 2, \dots, n)$ is constant as $t = \frac{2}{n}$. The above two inequalities imply formula (1.3). □

When $t = 1$, we have

Corollary 2.2. *Let* $A, B \in M_n(C)$ $(n \ge 2)$*. If* B *is invertible and* $\operatorname{Re}\lambda_k \ge 0$ $(k = 1, 2, \dots, n)$*, where* $\sigma(C) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$*, then*

$$d[A+B] \ge d[A] + d[B]. \tag{2.2}$$

Corollary 2.3. *Let* A *be an* n*-by-*n *complex positive definite matrix, and* B *be an* n*-by-*n *positive definite Hermitian matrix* $(n \ge 2)$*. Then for* $t \ge \frac{2}{n}$

$$d[A+B]^t \ge d[A]^t + [\det(B)]^t. \tag{2.3}$$

Proof. Observing $C = B^{-1}A$ is similar to $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ and $\operatorname{Re}\lambda(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}) > 0$, where $\lambda(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})$ is an arbitrary eigenvalue of $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$. Therefore, $\operatorname{Re}\lambda_k \ge 0$ and $\sigma(C) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$. Hence, Theorem 2.1 yields Corollary 2.3. □

When $t = \frac{2}{n}$, inequality (2.3) gives Theorem 4 of [3]. When $t = 1$, inequality (2.3) gives Theorem 1 of [3]. To merit attention, Theorem 2 in [8] proves that if A is real positive definite and B is real positive definite symmetric, then (2.3) holds for $t = \frac{1}{n}$. It is untenable for example: $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Corollary 2.7 and Corollary 2.8 in this paper have been given correction.

Theorem 2.4. *Let A, $B \in M_n(C)$. If B is nonsingular, and* $\operatorname{Re}\lambda_k \geq 0$ $(k = 1, 2, \ldots, n)$, *where* $\sigma(C) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, *then n eigenvalues of C are pure imaginary complex numbers with the same modulus if and only if*

$$d[A+B]^{\frac{2}{n}} = d[A]^{\frac{2}{n}} + d[B]^{\frac{2}{n}}, \tag{2.4}$$

Proof. If n eigenvalues of C are $\pm id$ $(i = \sqrt{-1}, d > o, d \in R)$, then

$$\prod_{i=1}^{n} |\lambda_i + 1|^{\frac{2}{n}} = \prod_{i=1}^{n} \left(1 + d^2\right)^{\frac{1}{n}} = 1 + d^2 = \prod_{i=1}^{n} |\lambda_i|^{\frac{2}{n}} + 1.$$

Hence equality (2.4) holds by Lemma 1.1.

Conversely, suppose (2.4) holds, then

$$\prod_{i=1}^{n} |\lambda_i + 1|^{\frac{2}{n}} = \prod_{i=1}^{n} |\lambda_i|^{\frac{2}{n}} + 1.$$

So

$$\prod_{i=1}^{n} (1 + 2\operatorname{Re}\lambda_i + |\lambda_i|^2)^{\frac{1}{n}} = \prod_{i=1}^{n} (|\lambda_i|^2)^{\frac{1}{n}} + 1.$$

Obviously, $\operatorname{Re}\lambda_k = 0$ $(k = 1, 2, \ldots, n)$, otherwise

$$\prod_{i=1}^{n} (1 + 2\operatorname{Re}\lambda_i + |\lambda_i|^2)^{\frac{1}{n}} > \prod_{i=1}^{n} \left(1 + |\lambda_i|^2\right)^{\frac{1}{n}} \geq \prod_{i=1}^{n} (|\lambda_i|^2)^{\frac{1}{n}} + 1,$$

with illogicality. Therefore

$$\prod_{i=1}^{n} \left[1 + (\operatorname{Im}\lambda_i)^2\right]^{\frac{1}{n}} = \prod_{i=1}^{n} \left[(\operatorname{Im}\lambda_i)^2\right]^{\frac{1}{n}} + 1.$$

By Lemma 1.2 we obtain $(\mathrm{Im}\lambda_k)^2 = d^2$ and $\lambda_k = \pm id$ $(k = 1, 2, \ldots, n)$. This completes the proof. □

Corollary 2.5. *If A,$B \in M_n(C)$ with B is nonsingular and $C = B^{-1}A$ is skew–Hermitian, then formula* (2.4) *holds if and only if $A = idBUEU^*$, where $i^2 = -1$, $d > 0$, U is a unitary matrix, $E = \operatorname{diag}(e_1, e_2, \ldots, e_n)$ with $e_i = \pm 1$, $i = 1, 2, \ldots, n$.*

Proof. Since C is skew–Hermitian and its real parts of n eigenvalues are zero, then Theorem 2.4 implies that (2.4) holds if and only if

$$C = B^{-1}A = U \operatorname{diag}(\pm id, \pm id, \ldots, \pm id)U^*,$$

where $\sigma(C) = \{\pm id, \pm id, \ldots, \pm id\}$, $d > 0$ and U is unitary. Hence $A = idBUEU^*$, where $i^2 = -1$, $d > 0$, U is a unitary matrix, $E = \operatorname{diag}(e_1, e_2, \ldots, e_n)$ and $e_i = \pm 1$, $i = 1, 2, \ldots, n$. □

Theorem 2.6. *Suppose A, $B \in M_n(C)$ with B nonsingular and* $\mathrm{Re}\lambda_k \geq 0$ $(k = 1, 2, \ldots, n)$, *where $\sigma(C) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. If the number of the real eigenvalues of C is r, and the non-real eigenvalues of C are pair wise conjugate, then inequality* (1.2) *holds for $t \geq \frac{2}{n+r}$.*

Proof. By Lemma 1.1, we need to prove (1.3) for $t \geq \frac{2}{n+r}$. Without loss of generality, suppose $\lambda_j \geq 0$ $(j = 1, 2, \ldots, r)$ are the real eigenvalues of C and λ_k, $\overline{\lambda_k}$ $(k = r+1, r+2, \ldots, r+s)$

are s pairs of non-real eigenvalues of C, where $n = r + 2s$. Then the right-hand side of (1.3) becomes

$$\prod_{i=1}^{r} \lambda_i^t \prod_{j=r+1}^{r+s} \left(|\lambda_j|^2\right)^t + 1, \tag{2.5}$$

and the left-hand side of (1.3) is

$$\prod_{i=1}^{r} (\lambda_i + 1)^t \prod_{j=r+1}^{r+s} \left(|1 + \lambda_j|^2\right)^t. \tag{2.6}$$

Given $\operatorname{Re}\lambda_k \geq 0$ $(k = 1, 2, \ldots, r + s)$, so $|1 + \lambda_j|^2 \geq 1 + |\lambda_j|^2$, then

$$\prod_{i=1}^{r} (1 + \lambda_i)^t \prod_{j=r+1}^{r+s} \left(|1 + \lambda_j|^2\right)^t \geq \prod_{i=1}^{r} (1 + \lambda_i)^t \prod_{j=r+1}^{r+s} \left(1 + |\lambda_j|^2\right)^t. \tag{2.7}$$

By Lemma 1.2 and (2.7), we obtain that

$$\prod_{i=1}^{r} (\lambda_i + 1)^t \prod_{j=r+1}^{r+s} \left(|1 + \lambda_j|^2\right)^t \geq \prod_{i=1}^{r} \lambda_i^t \prod_{j=r+1}^{r+s} \left(|\lambda_j|^2\right)^t + 1, \text{ for } t \geq \frac{1}{r+s} = \frac{2}{n+r}.$$

This completes the proof. □

In the following, we present some generalizations of the Minkowski inequality. By Theorem 2.6, it is easy to show:

Corollary 2.7. *Let $A, B \in M_n(C)$. If B is nonsingular and n eigenvalues of C are positive numbers, then for $t \geq \frac{1}{n}$*

$$d[A + B]^{\frac{1}{n}} \geq d[A]^{\frac{1}{n}} + d[B]^{\frac{1}{n}}. \tag{2.8}$$

If A is an n-by-n complex positive definite matrix and B is an n-by-n positive definite Hermitian matrix, with n eigenvalues of C being real numbers, then $\sigma(C) = \sigma(B^{\frac{1}{2}}CB^{-\frac{1}{2}})$, and $B^{\frac{1}{2}}CB^{-\frac{1}{2}} = B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ is positive definite, so any eigenvalue of C has a positive real part. Thus n eigenvalues of C are positive numbers. By Corollary 2.7 we have

Corollary 2.8. *Suppose $A, B \in M_n(C)$, where A is a complex positive definite matrix and B is a positive definite Hermitian matrix. If n eigenvalues of C are real numbers, then inequality (2.8) holds for $t \geq \frac{1}{n}$.*

Corollary 2.9 (Minkowski inequality)**.** *Suppose $A, B \in M_n(C)$ are positive definite Hermitian matrices, then inequality (1.1) holds.*

Proof. Note that $C = B^{-1}A$ is similar to a real diagonal matrix, and its eigenvalues are real numbers, using Corollary 2.8 and letting $t = 1$, the proof is completed. □

Corollary 2.10. *Suppose $A, B \in M_n(C)$, where A is a complex positive definite matrix and B is a positive definite Hermitian matrix. If the non-real eigenvalues of C are m pairs conjugate complex numbers, then inequality (1.2) holds for $t \geq \frac{1}{n-m}$.*

Proof. Obviously $\operatorname{Re}\lambda_k \geq 0$ $(k = 1, 2, \ldots, n)$, where $\sigma(C) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Applying Theorem 2.6 completes the proof. □

Let $A = H + K \in M_n(C)$, where $H = \frac{1}{2}(A + A^*)$, and $K = \frac{1}{2}(A - A^*)$, then we have

Theorem 2.11. *Let $A = H + K$ be an n-by-n complex positive definite matrix, then for $t \geq \frac{2}{n}$*

$$d[A]^t \geq d[H]^t + d[K]^t, \tag{2.9}$$

with equality if and only if $K = idHQ^*EQ$ as $t = \frac{2}{n}$, where $i^2 = -1$, $d > 0$, Q is a unitary matrix,$E = \operatorname{diag}(e_1, e_2, \dots, e_n)$ with $e_i = \pm 1$, $i = 1, 2, \dots, n$.

Proof. Since $H^{-\frac{1}{2}}KH^{-\frac{1}{2}}$ is a skew-Hermitian matrix and is similar to $H^{-1}K$, $\operatorname{Re}\lambda(H^{-1}K) = \operatorname{Re}\lambda(H^{-\frac{1}{2}}KH^{-\frac{1}{2}}) = 0$. By Theorem 2.1 and Corollary 2.5, we get the desired result. □

Let $t = 1$, we have the following interesting result.

Corollary 2.12. *If $A = H + K$ is an n-by-n complex positive definite matrix $(n \geq 2)$, then*

$$d[A] \geq d[H] + d[K]. \tag{2.10}$$

Corollary 2.13 (Ostrowski-Taussky Inequality)**.** *If $A = H + K$ is an n-by-n positive definite matrix $(n \geq 2)$, then $\det H \leq d[A]$ with equality if and only if A is Hermitian.*

Theorem 2.14. *Let A, B be two n-by-n complex positive definite matrices, and n eigenvalues of B be real numbers. Suppose A, B are simultaneously upper triangularizable, namely, there exists a nonsingular matrix P, such that $P^{-1}AP$ and $P^{-1}BP$ are upper triangular matrices, then inequality (1.2) holds for any $t \geq \frac{2}{n}$.*

Proof. If $P^{-1}AP$ and $P^{-1}BP$ are upper triangular matrices, then

$$P^{-1}B^{-1}AP = (P^{-1}BP)^{-1}(P^{-1}AP)$$

is an upper triangular matrix, with the product of the eigenvalues of B^{-1} and A on its diagonal. We denote the eigenvalue of X by $\lambda(X)$. Notice that positive definiteness of A and B^{-1}, $\operatorname{Re}\lambda(A)$ and $\lambda(B^{-1})$ are positive numbers by hypothesis, it is easy to see that $\operatorname{Re}\lambda(B^{-1}A) \geq 0$. By Theorem 2.1, we get the desired result. □

Corollary 2.15. *Let A, B be two n-by-n complex positive definite matrices, and all the eigenvalues of B be real numbers. If $r([A, B]) \leq 1$, then inequality (1.2) holds for $t \geq \frac{2}{n}$, where $[A, B] = AB - BA$, $r([A, B])$ is the rank of $[A, B]$.*

Proof. It is easy to see that B^{-1} is a complex positive definite matrix and n eigenvalues of B^{-1} are real numbers. By the hypothesis and $r[B^{-1}, A] = r[A, B]$, we have $r([B^{-1}, A]) \leq 1$. By the Laffey-Choi Theorem (see [5], [1]), there exists a non-singular matrix P, such that $P^{-1}AP$ and $P^{-1}BP$ are upper triangular matrices. The result holds by Theorem 2.14. □

Corollary 2.16. *Let A, B be two n-by-n complex positive definite matrices $(n \geq 2)$. Suppose $AB = BA$ and n eigenvalues of B are real numbers, then inequality (1.2) holds for $t \geq \frac{2}{n}$.*

Proof. Follows from Corollary 2.15 and the fact that $r([A, B]) = 0$. □

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