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ON RANK SUBTRACTIVITY BETWEEN NORMAL MATRICES

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ABSTRACT. The rank subtractivity partial ordering is defined on $\mathbb{C}^{n\times n}$ $(n \geq 2)$ by $\mathbf{A} \leq^- \mathbf{B} \Leftrightarrow \operatorname{rank}(\mathbf{B}-\mathbf{A}) = \operatorname{rank}\mathbf{B} - \operatorname{rank}\mathbf{A}$, and the star partial ordering by $\mathbf{A} \leq^* \mathbf{B} \Leftrightarrow \mathbf{A}^*\mathbf{A} = \mathbf{A}^*\mathbf{B} \wedge \mathbf{A}\mathbf{A}^* = \mathbf{B}\mathbf{A}^*$. If $\mathbf{A}$ and $\mathbf{B}$ are normal, we characterize $\mathbf{A} \leq^- \mathbf{B}$. We also show that then $\mathbf{A} \leq^- \mathbf{B} \wedge \mathbf{AB} = \mathbf{BA} \Leftrightarrow \mathbf{A} \leq^* \mathbf{B} \Leftrightarrow \mathbf{A} \leq^- \mathbf{B} \wedge \mathbf{A}^2 \leq^- \mathbf{B}^2$. Finally, we remark that some of our results follow from well-known results on EP matrices.

Key words and phrases: Rank subtractivity, Minus partial ordering, Star partial ordering, Sharp partial ordering, Normal matrices, EP matrices.

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1. INTRODUCTION

The rank subtractivity partial ordering (also called the minus partial ordering) is defined on $\mathbb{C}^{n\times n}$ $(n \geq 2)$ by

$$\mathbf{A} \leq^- \mathbf{B} \Leftrightarrow \operatorname{rank}(\mathbf{B}-\mathbf{A}) = \operatorname{rank}\mathbf{B} - \operatorname{rank}\mathbf{A}.$$

The star partial ordering is defined by

$$\mathbf{A} \leq^* \mathbf{B} \Leftrightarrow \mathbf{A}^*\mathbf{A} = \mathbf{A}^*\mathbf{B} \wedge \mathbf{A}\mathbf{A}^* = \mathbf{B}\mathbf{A}^*.$$

(Actually these partial orderings can also be defined on $\mathbb{C}^{m\times n}$, $m \neq n$, but square matrices are enough for us.)

There is a great deal of research about characterizations of $\leq^*$ and $\leq^-$, see, e.g., [8] and its references. Hartwig and Styan [8] applied singular value decompositions to this purpose. In the case of normal matrices, the present authors [10] did some parallel work and further developments by applying spectral decompostitions in characterizing $\leq^*$. As a sequel to [10], we will now do similar work with $\leq^-$.

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In Section 2, we will present two well-known results. The first is a lemma about a matrix whose rank is equal to the rank of its submatrix. The second is a characterization of $\leq^-$ for general matrices from [8].

In Section 3, we will characterize $\leq^-$ for normal matrices.

Since $\leq^*$ implies $\leq^-$, it is natural to ask for an additional condition, which, together with $\leq^-$, is equivalent to $\leq^*$. Hartwig and Styan ([8, Theorem 2c]), presented ten such conditions for general matrices. In Sections 4 and 5, we will find two such conditions for normal matrices.

Finally, in Section 6, we will remark that some of our results follow from well-known results on EP matrices.

In [10], we proved characterizations of $\leq^*$ for normal matrices independently of general results from [8]. In dealing with the characterization of $\leq^-$ for normal matrices, an independent approach seems too complicated, and so we will apply [8].

2. PRELIMINARIES

If $1 \leq \operatorname{rank} \mathbf{A} = r < n$, then $\mathbf{A}$ can be constructed by starting from a nonsingular $r \times r$ submatrix according to the following lemma. Since this lemma is of independent interest, we present it more broadly than we would actually need.

Lemma 2.1. *Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $1 \leq r < n$, $s = n - r$. Then the following conditions are equivalent:*

(a) $\operatorname{rank} \mathbf{A} = r$.

(b) *If $\mathbf{E} \in \mathbb{C}^{r\times r}$ is a nonsingular submatrix of $\mathbf{A}$, then there are permutation matrices $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{n\times n}$ and matrices $\mathbf{R} \in \mathbb{C}^{s\times r}$, $\mathbf{S} \in \mathbb{C}^{r\times s}$ such that*

$$\mathbf{A} = \mathbf{P} \begin{pmatrix} \mathbf{RES} & \mathbf{RE} \\ \mathbf{ES} & \mathbf{E} \end{pmatrix} \mathbf{Q}.$$

Proof. If (a) holds, then proceeding as Ben-Israel and Greville ([3, p. 178]) gives (b). Conversely, if (b) holds, then

$$\mathbf{A} = \mathbf{P} \begin{pmatrix} \mathbf{R} \\ \mathbf{I} \end{pmatrix} \mathbf{E} \begin{pmatrix} \mathbf{S} & \mathbf{I} \end{pmatrix} \mathbf{Q}$$

(cf. (22) on [3, p. 178]), and (a) follows. □

Next, we recall a characterization of $\leq^-$ for general matrices, due to Hartwig and Styan [8] (and actually stated also for non-square matrices).

Theorem 2.2 ([8, Theorem 1])**.** *Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n\times n}$. If $a = \operatorname{rank} \mathbf{A}$, $b = \operatorname{rank} \mathbf{B}$, $1 \leq a < b \leq n$, and $p = b - a$, then the following conditions are equivalent:*

(a) $\mathbf{A} \leq^- \mathbf{B}$.

(b) *There are unitary matrices $\mathbf{U}, \mathbf{V} \in \mathbb{C}^{n\times n}$ such that*

$$\mathbf{U}^*\mathbf{A}\mathbf{V} = \begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}$$

and

$$\mathbf{U}^*\mathbf{B}\mathbf{V} = \begin{pmatrix} \boldsymbol{\Sigma} + \mathbf{RES} & \mathbf{RE} & \mathbf{O} \\ \mathbf{ES} & \mathbf{E} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{pmatrix},$$

where $\boldsymbol{\Sigma} \in \mathbb{R}^{a\times a}$, $\mathbf{E} \in \mathbb{R}^{p\times p}$ are diagonal matrices with positive diagonal elements, $\mathbf{R} \in \mathbb{C}^{a\times p}$, and $\mathbf{S} \in \mathbb{C}^{p\times a}$.

In fact, $\mathbf{U}^*\mathbf{A}\mathbf{V}$ is a singular value decomposition of $\mathbf{A}$. (If $b = n$, then omit the zero blocks in the representation of $\mathbf{U}^*\mathbf{B}\mathbf{V}$.)

3. CHARACTERIZATIONS OF $\mathbf{A} \leq^- \mathbf{B}$

Now we characterize $\leq^-$ for normal matrices.

Theorem 3.1. *Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n\times n}$ be normal. If $a = \operatorname{rank} \mathbf{A}$, $b = \operatorname{rank} \mathbf{B}$, $1 \leq a < b \leq n$, and $p = b - a$, then the following conditions are equivalent:*

(a) $\mathbf{A} \leq^- \mathbf{B}$.

(b) *There is a unitary matrix $\mathbf{U} \in \mathbb{C}^{n\times n}$ such that*

$$\mathbf{U}^*\mathbf{A}\mathbf{U} = \begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}$$

and

$$\mathbf{U}^*\mathbf{B}\mathbf{U} = \begin{pmatrix} \mathbf{D} + \mathbf{RES} & \mathbf{RE} & \mathbf{O} \\ \mathbf{ES} & \mathbf{E} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{pmatrix},$$

where $\mathbf{D} \in \mathbb{C}^{a\times a}$, $\mathbf{E} \in \mathbb{C}^{p\times p}$ are nonsingular diagonal matrices, $\mathbf{R} \in \mathbb{C}^{a\times p}$, and $\mathbf{S} \in \mathbb{C}^{p\times a}$.

(c) *There is a unitary matrix $\mathbf{U} \in \mathbb{C}^{n\times n}$ such that*

$$\mathbf{U}^*\mathbf{A}\mathbf{U} = \begin{pmatrix} \mathbf{G} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}$$

and

$$\mathbf{U}^*\mathbf{B}\mathbf{U} = \begin{pmatrix} \mathbf{G} + \mathbf{RFS} & \mathbf{RF} & \mathbf{O} \\ \mathbf{FS} & \mathbf{F} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{pmatrix},$$

where $\mathbf{G} \in \mathbb{C}^{a\times a}$, $\mathbf{F} \in \mathbb{C}^{p\times p}$ are nonsingular matrices, $\mathbf{R} \in \mathbb{C}^{a\times p}$, and $\mathbf{S} \in \mathbb{C}^{p\times a}$.

(*If $b = n$, then omit the zero blocks in the representations of* $\mathbf{U}^*\mathbf{B}\mathbf{U}$.)

Proof. We proceed via (b) $\Rightarrow$ (c) $\Rightarrow$ (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (c). Trivial.

(c) $\Rightarrow$ (a). Assume (c). Then

$$\mathbf{B} - \mathbf{A} = \mathbf{U}\mathbf{C}\mathbf{U}^*,$$

where

$$\mathbf{C} = \begin{pmatrix} \mathbf{RFS} & \mathbf{RF} & \mathbf{O} \\ \mathbf{FS} & \mathbf{F} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{pmatrix}$$

satisfies

$$\operatorname{rank} \mathbf{C} = \operatorname{rank}(\mathbf{B} - \mathbf{A}).$$

On the other hand, by Lemma 2.1,

$$\operatorname{rank} \mathbf{C} = \operatorname{rank} \mathbf{F} = p = b - a = \operatorname{rank} \mathbf{B} - \operatorname{rank} \mathbf{A},$$

and (a) follows.

(a) $\Rightarrow$ (b). Assume that $\mathbf{A}$ and $\mathbf{B}$ satisfy (a). Then, with the notations of Theorem 2.2,

$$\mathbf{U}^*\mathbf{A}\mathbf{V} = \begin{pmatrix} \mathbf{\Sigma} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix} = \mathbf{\Sigma}_0$$

and

$$\mathbf{U}^*\mathbf{B}\mathbf{V} = \begin{pmatrix} \mathbf{\Sigma} + \mathbf{RES} & \mathbf{RE} & \mathbf{O} \\ \mathbf{ES} & \mathbf{E} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{pmatrix}.$$

The singular values of a normal matrix are absolute values of its eigenvalues. Therefore the diagonal matrix of (appropriately ordered) eigenvalues of $\mathbf{A}$ is $\mathbf{D}_0 = \mathbf{\Sigma}_0\mathbf{J}$, where $\mathbf{J}$ is a diagonal matrix of elements with absolute value 1. Furthermore, $\mathbf{V} = \mathbf{U}\mathbf{J}^{-1}$, and

$$\mathbf{U}^*\mathbf{A}\mathbf{U} = \mathbf{D}_0 = \begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix},$$

where $\mathbf{D}$ is the diagonal matrix of nonzero eigenvalues of $\mathbf{A}$. For details, see, e.g., [9, p. 417].

To study $\mathbf{U}^*\mathbf{B}\mathbf{V}$, let us denote

$$\mathbf{J} = \begin{pmatrix} \mathbf{K} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{L} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{M} \end{pmatrix},$$

partitioned as $\mathbf{U}^*\mathbf{B}\mathbf{V}$ above. Now,

$$\begin{aligned} \mathbf{U}^*\mathbf{B}\mathbf{U} = \mathbf{U}^*\mathbf{B}\mathbf{V}\mathbf{J} &= \begin{pmatrix} \mathbf{\Sigma} + \mathbf{RES} & \mathbf{RE} & \mathbf{O} \\ \mathbf{ES} & \mathbf{E} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{K} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{L} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{M} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{\Sigma K} + \mathbf{RESK} & \mathbf{REL} & \mathbf{O} \\ \mathbf{ESK} & \mathbf{EL} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{pmatrix} = \begin{pmatrix} \mathbf{D} + \mathbf{RESK} & \mathbf{REL} & \mathbf{O} \\ \mathbf{ESK} & \mathbf{EL} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{pmatrix}. \end{aligned}$$

By (a),

$$b - a = \operatorname{rank}(\mathbf{B} - \mathbf{A}) = \operatorname{rank} \mathbf{U}^*(\mathbf{B} - \mathbf{A})\mathbf{U} = \operatorname{rank} \begin{pmatrix} \mathbf{RESK} & \mathbf{REL} \\ \mathbf{ESK} & \mathbf{EL} \end{pmatrix}.$$

Denote $\mathbf{E}' = \mathbf{EL}$. Because $\mathbf{E}$ and $\mathbf{L}$ are nonsingular, $\operatorname{rank} \mathbf{E}' = b - a$. Hence, by Lemma 2.1, there are matrices $\mathbf{R}' \in \mathbb{C}^{a \times p}$ and $\mathbf{S}' \in \mathbb{C}^{p \times a}$ such that

$$\begin{pmatrix} \mathbf{RESK} & \mathbf{REL} \\ \mathbf{ESK} & \mathbf{EL} \end{pmatrix} = \begin{pmatrix} \mathbf{R}'\mathbf{E}'\mathbf{S}' & \mathbf{R}'\mathbf{E}' \\ \mathbf{E}'\mathbf{S}' & \mathbf{E}' \end{pmatrix}.$$

Consequently,

$$\mathbf{U}^*\mathbf{B}\mathbf{U} = \begin{pmatrix} \mathbf{D} + \mathbf{R}'\mathbf{E}'\mathbf{S}' & \mathbf{R}'\mathbf{E}' & \mathbf{O} \\ \mathbf{E}'\mathbf{S}' & \mathbf{E}' & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{pmatrix},$$

and (b) follows. □

Corollary 3.2. *Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n \times n}$. If $\mathbf{A}$ is normal, $\mathbf{B}$ is Hermitian, and $\mathbf{A} \leq^- \mathbf{B}$, then $\mathbf{A}$ is Hermitian.*

Proof. If $\operatorname{rank} \mathbf{A} = 0$ or $\operatorname{rank} \mathbf{A} = \operatorname{rank} \mathbf{B}$, the claim is trivial. Otherwise, with the notations of Theorem 3.1,

$$\mathbf{A}' = \mathbf{U}^*\mathbf{A}\mathbf{U} = \begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \qquad \mathbf{B}' = \mathbf{U}^*\mathbf{B}\mathbf{U} = \begin{pmatrix} \mathbf{D} + \mathbf{RES} & \mathbf{RE} & \mathbf{O} \\ \mathbf{ES} & \mathbf{E} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{pmatrix}.$$

Since $\mathbf{B}$ is Hermitian, $\mathbf{B}'$ is also Hermitian. Therefore $\mathbf{E}^* = \mathbf{E}$ and $\mathbf{ES} = (\mathbf{RE})^* = \mathbf{ER}^*$, which implies $\mathbf{S} = \mathbf{R}^*$, since $\mathbf{E}$ is nonsingular. Now

$$\mathbf{A}' = \mathbf{B}' - \begin{pmatrix} \mathbf{RER}^* & \mathbf{RE} & \mathbf{O} \\ \mathbf{ER}^* & \mathbf{E} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{pmatrix}$$

is a difference of Hermitian matrices and so Hermitian. Hence also $\mathbf{A}$ is Hermitian. □

4. $\mathbf{A} \leq^- \mathbf{B} \wedge \mathbf{AB} = \mathbf{BA} \Leftrightarrow \mathbf{A} \leq^* \mathbf{B}$

The partial ordering $\leq^*$ implies $\leq^-$. For the proof, apply Theorem 2.2 and the corresponding characterization of $\leq^*$ ([8, Theorem 2]). In fact, this implication originates with Hartwig ([7, p. 4, (iii)]) on general star-semigoups.

We are therefore motivated to look for an additional condition, which, together with $\leq^-$, is equivalent to $\leq^*$. First we recall a characterization of $\leq^*$ from [10] but formulate it slightly differently.

Theorem 4.1 ([10, Theorem 2.1ab], cf. also [8, Theorem 2ab])**.** *Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n\times n}$ be normal. If $a = \operatorname{rank} \mathbf{A}$, $b = \operatorname{rank} \mathbf{B}$, $1 \leq a < b \leq n$, and $p = b - a$, then the following conditions are equivalent:*

(a) $\mathbf{A} \leq^* \mathbf{B}$.

(b) *There is a unitary matrix $\mathbf{U} \in \mathbb{C}^{n\times n}$ such that*

$$\mathbf{U}^*\mathbf{AU} = \begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}$$

and

$$\mathbf{U}^*\mathbf{BU} = \begin{pmatrix} \mathbf{D} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{E} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{pmatrix},$$

where $\mathbf{D} \in \mathbb{C}^{a\times a}$ and $\mathbf{E} \in \mathbb{C}^{p\times p}$ are nonsingular diagonal matrices. (*If $b = n$, then omit the third block-row and block-column of zeros in the expression of* $\mathbf{B}$.)

Hartwig and Styan [8] proved the following theorem assuming that $\mathbf{A}$ and $\mathbf{B}$ are Hermitian. We assume only normality.

Theorem 4.2 (cf. [8, Corollary 1ac])**.** *Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n\times n}$ be normal. The following conditions are equivalent:*

(a) $\mathbf{A} \leq^* \mathbf{B}$,

(b) $\mathbf{A} \leq^- \mathbf{B} \wedge \mathbf{AB} = \mathbf{BA}$.

Proof. If $a = \operatorname{rank} \mathbf{A}$ and $b = \operatorname{rank} \mathbf{B}$ satisfy $a = 0$ or $a = b$, then the claim is trivial. So we assume $1 \leq a < b \leq n$.

(a) $\Rightarrow$ (b). This follows immediately from Theorems 4.1 and 3.1.

(b) $\Rightarrow$ (a). Assume (b). Since $\mathbf{A} \leq^- \mathbf{B}$, we have with the notations of Theorem 3.1

$$\mathbf{U}^*\mathbf{AU} = \begin{pmatrix} \mathbf{D} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{pmatrix}, \qquad \mathbf{U}^*\mathbf{BU} = \begin{pmatrix} \mathbf{D} + \mathbf{RES} & \mathbf{RE} & \mathbf{O} \\ \mathbf{ES} & \mathbf{E} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{pmatrix}.$$

Thus

$$\mathbf{U}^*\mathbf{ABU} = \begin{pmatrix} \mathbf{D}^2 + \mathbf{DRES} & \mathbf{DRE} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{pmatrix}$$

and

$$\mathbf{U}^*\mathbf{BAU} = \begin{pmatrix} \mathbf{D}^2 + \mathbf{RESD} & \mathbf{O} & \mathbf{O} \\ \mathbf{ESD} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{pmatrix}.$$

Since $\mathbf{AB} = \mathbf{BA}$, also $\mathbf{U}^*\mathbf{ABU} = \mathbf{U}^*\mathbf{BAU}$, which implies $\mathbf{DRE} = \mathbf{O}$ and $\mathbf{ESD} = \mathbf{O}$. Because $\mathbf{D}$ and $\mathbf{E}$ are nonsingular, we therefore have $\mathbf{R} = \mathbf{O}$ and $\mathbf{S} = \mathbf{O}$. So

$$\mathbf{U}^*\mathbf{BU} = \begin{pmatrix} \mathbf{D} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{E} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{pmatrix},$$

and (a) follows from Theorem 4.1. □

5. $\mathbf{A} \leq^- \mathbf{B} \wedge \mathbf{A}^2 \leq^- \mathbf{B}^2 \Leftrightarrow \mathbf{A} \leq^* \mathbf{B}$

We first note that the conditions $\mathbf{A} \leq^- \mathbf{B}$ and $\mathbf{A}^2 \leq^- \mathbf{B}^2$ are independent, even if $\mathbf{A}$ and $\mathbf{B}$ are Hermitian.

Example 5.1. If

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix},$$

then

$$\operatorname{rank}(\mathbf{B} - \mathbf{A}) = \operatorname{rank} \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} = 1, \qquad \operatorname{rank} \mathbf{B} - \operatorname{rank} \mathbf{A} = 2 - 1 = 1,$$

and so $\mathbf{A} \leq^- \mathbf{B}$. However, $\mathbf{A}^2 \leq^- \mathbf{B}^2$ does not hold, since

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{B}^2 = \begin{pmatrix} 29 & 12 \\ 12 & 5 \end{pmatrix}, \qquad \mathbf{B}^2 - \mathbf{A}^2 = \begin{pmatrix} 28 & 12 \\ 12 & 5 \end{pmatrix},$$
$$\operatorname{rank}\left(\mathbf{B}^2 - \mathbf{A}^2\right) = 2, \qquad \operatorname{rank} \mathbf{B}^2 - \operatorname{rank} \mathbf{A}^2 = 2 - 1 = 1.$$

Example 5.2. If

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},$$

then $\mathbf{A}^2 \leq^- \mathbf{B}^2$ holds but $\mathbf{A} \leq^- \mathbf{B}$ does not hold.

Gross ([5, Theorem 5]) proved that, in the case of Hermitian nonnegative definite matrices, the conditions $\mathbf{A} \leq^- \mathbf{B}$ and $\mathbf{A}^2 \leq^- \mathbf{B}^2$ together are equivalent to $\mathbf{A} \leq^* \mathbf{B}$. Baksalary and Hauke ([1, Theorem 4]) proved it for all Hermitian matrices. We generalize this result.

Theorem 5.1. *Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n\times n}$ be normal. Assume that*

(i) $\mathbf{B}$ *is Hermitian*

or

(ii) $\mathbf{B} - \mathbf{A}$ *is Hermitian.*

Then the following conditions are equivalent:

(a) $\mathbf{A} \leq^* \mathbf{B}$,
(b) $\mathbf{A} \leq^- \mathbf{B} \wedge \mathbf{A}^2 \leq^- \mathbf{B}^2$.

Proof. First, assume (i). If $\mathbf{A} \leq^- \mathbf{B}$, then $\mathbf{A}$ is Hermitian by Corollary 3.2. If $\mathbf{A} \leq^* \mathbf{B}$, then $\mathbf{A} \leq^- \mathbf{B}$, and so $\mathbf{A}$ is Hermitian also in this case. Therefore, both (a) and (b) imply that $\mathbf{A}$ is actually Hermitian, and hence (a) $\Leftrightarrow$ (b) follows from [1, Theorem 4]. The following proof applies to an alternative.

Second, assume (ii). If $a = \operatorname{rank} \mathbf{A}$ and $b = \operatorname{rank} \mathbf{B}$ satisfy $a = 0$ or $a = b$, then the claim is trivial. So we let $1 \leq a < b \leq n$.

(a) $\Rightarrow$ (b). This is an immediate consequence of Theorems 4.1 and 3.1.

(b) ⇒ (a). Assume (b). Since $\mathbf{A} \leq^- \mathbf{B}$, we have with the notations of Theorem 3.1

$$\mathbf{A} = \mathbf{U}\begin{pmatrix}\mathbf{D} & \mathbf{O}\\ \mathbf{O} & \mathbf{O}\end{pmatrix}\mathbf{U}^*, \qquad \mathbf{B} = \mathbf{U}\begin{pmatrix}\mathbf{D}+\mathbf{RES} & \mathbf{RE} & \mathbf{O}\\ \mathbf{ES} & \mathbf{E} & \mathbf{O}\\ \mathbf{O} & \mathbf{O} & \mathbf{O}\end{pmatrix}\mathbf{U}^*.$$

Since $\mathbf{B}-\mathbf{A}$ is Hermitian, $\mathbf{U}^*(\mathbf{B}-\mathbf{A})\mathbf{U}$ is also Hermitian. Therefore $\mathbf{E}$ is Hermitian and $\mathbf{S} = \mathbf{R}^*$, and so

$$\mathbf{B} = \mathbf{U}\begin{pmatrix}\mathbf{D}+\mathbf{RER}^* & \mathbf{RE} & \mathbf{O}\\ \mathbf{ER}^* & \mathbf{E} & \mathbf{O}\\ \mathbf{O} & \mathbf{O} & \mathbf{O}\end{pmatrix}\mathbf{U}^*.$$

Furthermore,

$$\mathbf{A}^2 = \mathbf{U}\begin{pmatrix}\mathbf{D}^2 & \mathbf{O}\\ \mathbf{O} & \mathbf{O}\end{pmatrix}\mathbf{U}^*$$

and

$$\mathbf{B}^2 = \mathbf{U}\begin{pmatrix}(\mathbf{D}+\mathbf{RER}^*)^2+\mathbf{RE}^2\mathbf{R}^* & (\mathbf{D}+\mathbf{RER}^*)\mathbf{RE}+\mathbf{RE}^2 & \mathbf{O}\\ \mathbf{ER}^*(\mathbf{D}+\mathbf{RER}^*)+\mathbf{E}^2\mathbf{R}^* & \mathbf{ER}^*\mathbf{RE}+\mathbf{E}^2 & \mathbf{O}\\ \mathbf{O} & \mathbf{O} & \mathbf{O}\end{pmatrix}\mathbf{U}^*.$$

Now

$$\mathbf{B}^2 - \mathbf{A}^2 = \mathbf{U}\begin{pmatrix}\mathbf{H} & \mathbf{O}\\ \mathbf{O} & \mathbf{O}\end{pmatrix}\mathbf{U}^*,$$

where

$$\mathbf{H} = \begin{pmatrix}\mathbf{DRER}^*+\mathbf{RER}^*\mathbf{D}+(\mathbf{RER}^*)^2+\mathbf{RE}^2\mathbf{R}^* & \mathbf{DRE}+\mathbf{RER}^*\mathbf{RE}+\mathbf{RE}^2\\ \mathbf{ER}^*\mathbf{D}+\mathbf{ER}^*\mathbf{RER}^*+\mathbf{E}^2\mathbf{R}^* & \mathbf{ER}^*\mathbf{RE}+\mathbf{E}^2\end{pmatrix}.$$

Multiplying the second block-row of $\mathbf{H}$ by $-\mathbf{R}$ from the right and adding the result to the first block-row is a set of elementary row operations and so does not change the rank. Thus

$$\operatorname{rank}\mathbf{H} = \operatorname{rank}\begin{pmatrix}\mathbf{DRER}^* & \mathbf{DRE}\\ \mathbf{ER}^*\mathbf{D}+\mathbf{ER}^*\mathbf{RER}^*+\mathbf{E}^2\mathbf{R}^* & \mathbf{ER}^*\mathbf{RE}+\mathbf{E}^2\end{pmatrix} = \operatorname{rank}\mathbf{H}'.$$

Furthermore, multiplying the second block-column of $\mathbf{H}'$ by $-\mathbf{R}^*$ from the right and adding the result to the first block-column is a set of elementary column operations, and so

$$\operatorname{rank}\mathbf{H}' = \operatorname{rank}\begin{pmatrix}\mathbf{O} & \mathbf{DRE}\\ \mathbf{ER}^*\mathbf{D} & \mathbf{ER}^*\mathbf{RE}+\mathbf{E}^2\end{pmatrix} = \operatorname{rank}\mathbf{H}''.$$

Since $\mathbf{A}^2 \leq^- \mathbf{B}^2$, we therefore have

$$\operatorname{rank}\mathbf{H}'' = \operatorname{rank}(\mathbf{B}^2-\mathbf{A}^2) = \operatorname{rank}\mathbf{B}^2 - \operatorname{rank}\mathbf{A}^2 = b-a = p.$$

Because $\mathbf{ER}^*\mathbf{RE}$ is Hermitian nonnegative definite and $\mathbf{E}$ is Hermitian positive definite, their sum $\mathbf{E}' = \mathbf{ER}^*\mathbf{RE} + \mathbf{E}^2$ is Hermitian positive definite and hence nonsingular. Applying Lemma 2.1 to $\mathbf{H}''$, we see that there is a matrix $\mathbf{S} \in \mathbb{C}^{p\times a}$ such that (1) $\mathbf{S}^*\mathbf{E}' = \mathbf{DRE}$ and (2) $\mathbf{S}^*\mathbf{E}'\mathbf{S} = \mathbf{O}$. Since $\mathbf{E}'$ is positive definite, then (2) implies $\mathbf{S} = \mathbf{O}$, and so (1) reduces to $\mathbf{DRE} = \mathbf{O}$, which, in turn, implies $\mathbf{R} = \mathbf{O}$ by the nonsingularity of $\mathbf{D}$ and $\mathbf{E}$. Consequently,

$$\mathbf{B} = \mathbf{U}\begin{pmatrix}\mathbf{D} & \mathbf{O} & \mathbf{O}\\ \mathbf{O} & \mathbf{E} & \mathbf{O}\\ \mathbf{O} & \mathbf{O} & \mathbf{O}\end{pmatrix}\mathbf{U}^*,$$

and (a) follows from Theorem 4.1. □

6. REMARKS

A matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ is a group matrix if it belongs to a subset of $\mathbb{C}^{n\times n}$ which is a group under matrix multiplication. This happens if and only if $\operatorname{rank}\mathbf{A}^2 = \operatorname{rank}\mathbf{A}$ (see, e.g., [3, Theorem 4.2] or [11, Theorem 9.4.2]). A matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ is an EP matrix if $\mathcal{R}(\mathbf{A}^*) = \mathcal{R}(\mathbf{A})$ where $\mathcal{R}$ denotes the column space. There are plenty of characterizations for EP matrices, see Cheng and Tian [4] and its references. A normal matrix is EP, and an EP matrix is a group matrix (see, e.g., [3, p. 159]). The sharp partial ordering between group matrices $\mathbf{A}$ and $\mathbf{B}$ is defined by

$$\mathbf{A} \leq^{\#} \mathbf{B} \Leftrightarrow \mathbf{A}^2 = \mathbf{AB} = \mathbf{BA}.$$

Three of our results follow from well-known results on EP matrices.

First, Corollary 3.2 is a special case of Lemma 3.1 of Baksalary et al [2], where $\mathbf{A}$ is assumed only EP.

Second, let $\mathbf{A}$ and $\mathbf{B}$ be group matrices. Then

$$\mathbf{A} \leq^{\#} \mathbf{B} \Leftrightarrow \mathbf{A} \leq^{-} \mathbf{B} \wedge \mathbf{AB} = \mathbf{BA},$$

by Mitra ([12, Theorem 2.5]). On the other hand, if $\mathbf{A}$ is EP, then

$$\mathbf{A} \leq^{\#} \mathbf{B} \Leftrightarrow \mathbf{A} \leq^{*} \mathbf{B},$$

by Gross ([6, Remark 1]). Hence Theorem 4.2 follows assuming only that $\mathbf{A}$ is EP and $\mathbf{B}$ is a group matrix.

Third, Theorem 5.1 with assumption (i) is a special case of [2, Corollary 3.2], where $\mathbf{A}$ is assumed only EP.

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