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OSTROWSKI TYPE INEQUALITIES OVER BALLS AND SHELLS VIA A TAYLOR-WIDDER FORMULA

GEORGE A. ANASTASSIOU

DEPARTMENT OF MATHEMATICAL SCIENCES
THE UNIVERSITY OF MEMPHIS
MEMPHIS, TN 38152, U.S.A.
ganastss@memphis.edu

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ABSTRACT. The classical Ostrowski inequality for functions on intervals estimates the value of the function minus its average in terms of the maximum of its first derivative. This result is extended to higher order over shells and balls of $\mathbb{R}^N$, $N \geq 1$, with respect to an *extended complete Tschebyshev system* and the *generalized radial derivatives of Widder type*. We treat radial and non-radial functions.

Key words and phrases: Ostrowski inequality, multivariate inequality, ball and shell, Taylor-Widder formula, extended complete Tschebyshev system.

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1. INTRODUCTION

The classical Ostrowski inequality (of 1938, see [12]) is

$$\left| \frac{1}{b-a} \int_a^b f(y)dy - f(x) \right| \leq \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right) (b-a) \|f'\|_\infty,$$

for $f \in C^1([a,b])$, $x \in [a,b]$, and it is a sharp inequality. This was extended to $\mathbb{R}^N$, $N \geq 1$, over balls and shells in [5], [6], [4]. Earlier this extension was done over boxes and rectangles, see [2, p. 507-520], and [3], see also [1]. The produced Ostrowski type inequalities, in the above mentioned references, were mostly sharp and they involved the first and higher order derivatives of the engaged function f.

Here we derive a set of very general higher order Ostrowski type inequalities over shells and balls with respect to an *extended complete Tschebyshev system* (see [10]) *and generalized derivatives of Widder type* (see [15]). The proofs are based on the *polar method* and the *general Taylor-Widder formula* (see [15], 1928). Our results generalize the higher order Ostrowski type inequalities established in the above mentioned references.

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2. BACKGROUND

The following are taken from [15]. Let $f, u_0, u_1, \dots, u_n \in C^{n+1}([a,b])$,$n \geq 0$, and the Wronskians

$$W_i(x) := W[u_0(x), u_1(x), \dots, u_i(x)]$$
$$:= \begin{vmatrix} u_0(x) & u_1(x) & \dots & u_i(x) \\ u_0'(x) & u_1'(x) & \dots & u_i'(x) \\ \vdots & & & \\ u_0^{(i)}(x) & u_1^{(i)}(x) & \dots & u_i^{(i)}(x) \end{vmatrix}, \quad i = 0, 1, \dots, n.$$

Assume $W_i(x) > 0$ over $[a,b]$. Clearly then

$$\phi_0(x) := W_0(x) = u_0(x),\ \phi_1(x) := \frac{W_1(x)}{(W_0(x))^2}, \dots, \phi_i(x) := \frac{W_i(x) W_{i-2}(x)}{(W_{i-1}(x))^2}, \quad i = 2, 3, \dots, n$$

are positive on $[a,b]$.

For $i \geq 0$, the linear differentiable operator of order i:

$$L_i f(x) := \frac{W[u_0(x), u_1(x), \dots, u_{i-1}(x), f(x)]}{W_{i-1}(x)}, \quad i = 1, \dots, n+1;$$
$$L_0 f(x) := f(x),\ \forall\, x \in [a,b].$$

Then for $i = 1, \dots, n+1$ we have

$$L_i f(x) = \phi_0(x)\phi_1(x)\cdots\phi_{i-1}(x) \frac{d}{dx}\frac{1}{\phi_{i-1}(x)}\frac{d}{dx}\frac{1}{\phi_{i-2}(x)}\frac{d}{dx}\cdots\frac{d}{dx}\frac{1}{\phi_1(x)}\frac{d}{dx}\frac{f(x)}{\phi_0(x)}.$$

Consider also

$$g_i(x,t) := \frac{1}{W_i(t)} \begin{vmatrix} u_0(t) & u_1(t) & \cdots & u_i(t) \\ u_0'(t) & u_1'(t) & \cdots & u_i'(t) \\ \cdots & \cdots & \cdots & \cdots \\ u_0^{(i-1)}(t) & u_1^{(i-1)}(t) & \cdots & u_i^{(i-1)}(t) \\ u_0(x) & u_1(x) & \cdots & u_i(x) \end{vmatrix},$$

$i = 1, 2, \dots, n$; $g_0(x,t) := \frac{u_0(x)}{u_0(t)}$, $\forall x, t \in [a,b]$.

Note that $g_i(x,t)$ as a function of x is a linear combination of $u_0(x), u_1(x), \dots, u_i(x)$ and it holds

$$g_i(x,t) = \frac{\phi_0(x)}{\phi_0(t)\cdots\phi_i(t)} \int_t^x \phi_1(x) \int_t^{x_1} \cdots \int_t^{x_{i-2}} \phi_{i-1}(x_{i-1}) \int_t^{x_{i-1}} \phi_i(x_i) dx_i dx_{i-1} \dots dx_1$$
$$= \frac{1}{\phi_0(t)\cdots\phi_i(t)} \int_t^x \phi_0(s)\cdots\phi_i(s) g_{i-1}(x,s) ds, \qquad i = 1, 2, \dots, n.$$

Example 2.1 ([15])**.** The sets

$$\{1, x, x^2, \dots, x^n\}, \quad \{1, \sin x, -\cos x, -\sin 2x, \cos 2x, \dots, (-1)^{n-1}\sin nx, (-1)^n \cos nx\}$$

fulfill the above theory.

We mention

Theorem 2.1 (Karlin and Studden (1966), see [10, p. 376])**.** *Let $u_0, u_1, \dots, u_n \in C^n([a,b])$, $n \geq 0$. Then $\{u_i\}_{i=0}^n$ is an extended complete Tschebyshev system on $[a,b]$ iff $W_i(x) > 0$ on $[a,b]$, $i = 0, 1, \dots, n$.*

We also mention

Theorem 2.2 (D. Widder, [15, p. 138]). *Let the functions*

$$f(x), u_0(x), u_1(x), \dots, u_n(x) \in C^{n+1}([a,b]),$$

and the Wronskians $W_0(x), W_1(x), \dots, W_n(x) > 0$ *on* $[a,b]$, $x \in [a,b]$. *Then for* $t \in [a,b]$ *we have*

$$f(x) = f(t)\frac{u_0(x)}{u_0(t)} + L_1 f(t) g_1(x,t) + \dots + L_n f(t) g_n(x,t) + R_n(x),$$

where

$$R_n(x) := \int_t^x g_n(x,s) L_{n+1} f(s) ds.$$

For example, one could take $u_0(x) = c > 0$. If $u_i(x) = x^i$, $i = 0, 1, \dots, n$, defined on $[a,b]$, then

$$L_i f(t) = f^{(i)}(t) \quad \text{and} \quad g_i(x,t) = \frac{(x-t)^i}{i!}, \quad t \in [a,b].$$

So under the assumptions of Theorem 2.2, we have

$$f(x) = f(y)\frac{u_0(x)}{u_0(y)} + \sum_{i=1}^{n} L_i f(y) g_i(x,y) + \int_y^x g_n(x,t) L_{n+1} f(t) dt, \quad \forall\, x, y \in [a,b]. \tag{2.1}$$

If $u_0(x) = c > 0$, then

$$f(x) = f(y) + \sum_{i=1}^{n} L_i f(y) g_i(x,y) + \int_y^x g_n(x,t) L_{n+1} f(t) dt, \quad \forall\, x, y \in [a,b]. \tag{2.2}$$

We call L_i the *generalized Widder-type derivative.*

We need

Notation 2.1. Let A be a *spherical shell* $\subseteq \mathbb{R}^N$, $N \geq 1$, i.e. $A := B(0,R_2) - \overline{B(0,R_1)}$, $0 < R_1 < R_2$.

Here the ball $B(0,R) := \{x \in \mathbb{R}^N : |x| < R\}$, $R > 0$, where $|\cdot|$ is the Euclidean norm, also $S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ is the unit sphere in $\mathbb{R}^N$ with surface area $\omega_N := \frac{2\pi^{N/2}}{\Gamma(N/2)}$. For $x \in \mathbb{R}^N - \{0\}$ one can write uniquely $x = r\omega$, where $r > 0$, $\omega \in S^{N-1}$.

Let $f \in C^{n+1}(\overline{A})$, $n \geq 0$. If f is radial, i.e., $f(x) = g(r)$, where $r = |x|$, $R_1 \leq r \leq R_2$, then $g \in C^{n+1}([R_1, R_2])$.

For radial f define

$$\theta_i f(x) := L_i g(r), \quad \text{all} \quad i = 1, \dots, n+1, \quad \forall\, x \in \overline{A}. \tag{2.3}$$

Here

$$g^{(i)}(r) = \frac{\partial^i f(x)}{\partial r^i}, \quad i = 1, \dots, n+1.$$

For $F \in C(\overline{A})$ we have

$$\int_A F(x) dx = \int_{S^{N-1}} \left(\int_{R_1}^{R_2} F(r\omega) r^{N-1} dr \right) d\omega. \tag{2.4}$$

We notice that

$$\frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} s^{N-1} ds = 1, \tag{2.5}$$

and

$$Vol(A) = \frac{\omega_N (R_2^N - R_1^N)}{N}.$$

3. RESULTS ON THE SHELL

Remark 3.1. Here, let $u_0, u_1, \dots, u_n \in C^{n+1}([R_1, R_2])$, and $W_0, W_1, \dots, W_n > 0$ on $[R_1, R_2]$, $0 < R_1 < R_2$, $n \geq 0$ integer, with $u_0(r) = c > 0$.

Let also $f \in C^{n+1}(\overline{A})$. We assume first that f is radial, i.e. there exists g such that $f(x) = g(r)$, $r = |x|$, $R_1 \leq r \leq R_2$. Clearly $g \in C^{n+1}([R_1, R_2])$.

Let $x \in \overline{A}$. Then by using the polar method (2.4) we obtain

$$E(x) := \left| f(x) - \frac{\int_A f(y)dy}{Vol(A)} \right|$$

$$= \left| f(x) - \frac{N \int_{S^{N-1}} \left(\int_{R_1}^{R_2} f(s\omega) s^{N-1} ds \right) d\omega}{\omega_N (R_2^N - R_1^N)} \right| \tag{3.1}$$

$$= \left| g(r) - \frac{N \int_{S^{N-1}} \left(\int_{R_1}^{R_2} g(s) s^{N-1} ds \right) d\omega}{\omega_N (R_2^N - R_1^N)} \right| \tag{3.2}$$

$$= \left| g(r) - \left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} g(s) s^{N-1} ds \right| \tag{3.3}$$

$$= \left| \left(\frac{N}{R_2^N - R_1^N} \right) \left(\int_{R_1}^{R_2} g(r) s^{N-1} ds - \int_{R_1}^{R_2} g(s) s^{N-1} ds \right) \right| \tag{3.4}$$

$$= \left(\frac{N}{R_2^N - R_1^N} \right) \left| \int_{R_1}^{R_2} (g(r) - g(s)) s^{N-1} ds \right| =: (*). \tag{3.5}$$

Let $s, r \in [R_1, R_2]$, then by generalized Taylor's formula (2.2) we get

$$g(s) - g(r) = \sum_{i=1}^{n} L_i g(r) g_i(s, r) + R_n(r, s), \tag{3.6}$$

where

$$R_n(r, s) := \int_r^s g_n(s, t) L_{n+1} g(t) dt. \tag{3.7}$$

But it holds

$$|R_n(r, s)| \leq \left| \int_r^s |g_n(s, t)| dt \right| \|L_{n+1} g\|_{\infty, [R_1, R_2]}. \tag{3.8}$$

By calling

$$N_n(r, s) := \left| \int_r^s |g_n(s, t)| dt \right|, \quad \forall\, s, r \in [R_1, R_2], \tag{3.9}$$

we get

$$|R_n(r, s)| \leq N_n(r, s) \|L_{n+1} g\|_{\infty, [R_1, R_2]}, \quad \forall\, s, r \in [R_1, R_2]. \tag{3.10}$$

Therefore by (3.5), (3.6), we have

$$(*) = \left(\frac{N}{R_2^N - R_1^N}\right)\left|\int_{R_1}^{R_2}\left[\sum_{i=1}^{n} L_i g(r) g_i(s,r) + R_n(r,s)\right] s^{N-1} ds\right| \tag{3.11}$$

$$\le \left(\frac{N}{R_2^N - R_1^N}\right)\left[\sum_{i=1}^{n}\left|\int_{R_1}^{R_2} L_i g(r) g_i(s,r) s^{N-1} ds\right| + \int_{R_1}^{R_2} |R_n(r,s)| s^{N-1} ds\right]$$

$$\overset{\text{by (3.10)}}{\le} \left(\frac{N}{R_2^N - R_1^N}\right)\left[\sum_{i=1}^{n} |L_i g(r)| \left|\int_{R_1}^{R_2} g_i(s,r) s^{N-1} ds\right|\right.$$

$$\left. + (\|L_{n+1} g\|_{\infty,[R_1,R_2]}) \int_{R_1}^{R_2} N_n(r,s) s^{N-1} ds\right]. \tag{3.12}$$

We have established the following result.

Theorem 3.2. *Let* $u_0, u_1, \ldots, u_n \in C^{n+1}([R_1, R_2])$, $W_0, W_1, \ldots, W_n > 0$ *on* $[R_1, R_2]$, $0 < R_1 < R_2$, $n \ge 0$ *integer, with* $u_0(r) = c > 0$. *Let* $f \in C^{n+1}(\overline{A})$ *be radial, i.e. there exists* g *such that* $f(x) = g(r)$, $r = |x|$, $R_1 \le r \le R_2$, $x \in \overline{A}$.

Then

$$E(x) := \left|f(x) - \frac{\int_A f(y)dy}{Vol(A)}\right|$$

$$= \left|g(r) - \left(\frac{N}{R_2^N - R_1^N}\right)\int_{R_1}^{R_2} g(s) s^{N-1} ds\right|$$

$$\le \left(\frac{N}{R_2^N - R_1^N}\right)\left[\sum_{i=1}^{n} |L_i g(r)| \left|\int_{R_1}^{R_2} g_i(s,r) s^{N-1} ds\right|\right.$$

$$\left. + (\|L_{n+1} g\|_{\infty,[R_1,R_2]}) \int_{R_1}^{R_2} N_n(r,s) s^{N-1} ds\right] \tag{3.13}$$

$$= \left(\frac{N}{R_2^N - R_1^N}\right)\left[\sum_{i=1}^{n} |\theta_i f(x)| \left|\int_{R_1}^{R_2} g_i(s,|x|) s^{N-1} ds\right|\right.$$

$$\left. + (\|\theta_{n+1} f\|_{\infty,\overline{A}}) \int_{R_1}^{R_2} N_n(|x|,s) s^{N-1} ds\right].$$

Corollary 3.3. *Let the conditions of Theorem 3.2 hold. Assume further that* $L_i g(r_0) = 0$, $i = 1, \ldots, n$, *for a fixed* $r_0 \in [R_1, R_2]$. *For all* $x_0 = r_0\omega \in \overline{A}$, $\omega \in S^{N-1}$, *we have*

$$E(x_0) = \left|f(x_0) - \frac{\int_A f(y)dy}{Vol(A)}\right|$$

$$= \left|g(r_0) - \left(\frac{N}{R_2^N - R_1^N}\right)\int_{R_1}^{R_2} g(s) s^{N-1} ds\right|$$

$$\le \left(\frac{N}{R_2^N - R_1^N}\right)(\|L_{n+1} g\|_{\infty,[R_1,R_2]})\left(\int_{R_1}^{R_2} N_n(r_0,s) s^{N-1} ds\right)$$

$$= \left(\frac{N}{R_2^N - R_1^N}\right)(\|\theta_{n+1} f\|_{\infty,\overline{A}})\left(\int_{R_1}^{R_2} N_n(|x_0|,s) s^{N-1} ds\right). \tag{3.14}$$

Interesting cases also arise when $r_0 = R_1$ or R_2.

We continue Remark 3.1 with

Remark 3.4. Now let $f \in C^{n+1}(\overline{A})$, $n \geq 0$, $x \in \overline{A}$, $x = r\omega$, $r > 0$. Clearly for fixed $\omega \in S^{N-1}$, since the function $f(r\omega)$, $r \in [R_1, R_2]$ is radial, it also belongs to $C^{n+1}([R_1, R_2])$.

By applying the internal inequality (3.13) we get

$$\left|f(r\omega) - \left(\frac{N}{R_2^N - R_1^N}\right) \int_{R_1}^{R_2} f(s\omega)s^{N-1}ds\right| \leq \left(\frac{N}{R_2^N - R_1^N}\right) \left[\sum_{i=1}^{n} |(L_i f(\cdot\omega))(r)| \left|\int_{R_1}^{R_2} g_i(s,r)s^{N-1}ds\right| + (\|L_{n+1}f(\cdot\omega)\|_{\infty,[R_1,R_2]}) \int_{R_1}^{R_2} N_n(r,s)s^{N-1}ds\right]. \tag{3.15}$$

For non-radial f we define again

$$\theta_i f(x) = \theta_i f(r\omega) := (L_i f(\cdot\omega))(r), \quad \text{all} \quad i = 1, \dots, n+1, \ \forall\, x \in \overline{A}. \tag{3.16}$$

Here the involved

$$\frac{\partial^i f(r\omega)}{\partial r^i} = \frac{\partial^i f(x)}{\partial r^i}, \quad i = 1, \dots, n+1,$$

are the radial derivatives. In a sense θ_i is a generalized radial derivative of Widder-type.

Hence

$$R.H.S.(3.15) \leq \left(\frac{N}{R_2^N - R_1^N}\right) \left[\sum_{i=1}^{n} |\theta_i f(r\omega)| \left|\int_{R_1}^{R_2} g_i(s,r)s^{N-1}ds\right| + \|\theta_{n+1}f\|_{\infty,\overline{A}} \int_{R_1}^{R_2} N_n(|x|,s)s^{N-1}ds\right]. \tag{3.17}$$

Therefore, by (3.15) and (3.17) we have

$$\left|\frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{N/2}} \int_{S^{N-1}} f(r\omega)d\omega - \frac{1}{Vol(A)} \int_A f(y)dy\right| \leq \left(\frac{N}{R_2^N - R_1^N}\right) \left[\frac{\Gamma(N/2)}{2\pi^{N/2}} \left\{\sum_{i=1}^{n} \left(\int_{S^{N-1}} |\theta_i f(r\omega)|d\omega\right) \left|\int_{R_1}^{R_2} g_i(s,r)s^{N-1}ds\right|\right\} + \|\theta_{n+1}f\|_{\infty,\overline{A}} \int_{R_1}^{R_2} N_n(|x|,s)s^{N-1}ds\right]. \tag{3.18}$$

We have established the following.

Theorem 3.5. *Let* $u_0, u_1, \dots, u_n \in C^{n+1}([R_1, R_2])$*;* $W_0, W_1, \dots, W_n > 0$ *on* $[R_1, R_2]$, $0 < R_1 < R_2$, $n \geq 0$ *an integer, with* $u_0(r) = c > 0$*. Let* $f \in C^{n+1}(\overline{A})$, $x \in \overline{A}$*;* $x = r\omega$, $r \in [R_1, R_2]$, $\omega \in S^{N-1}$.

Then

$$E(x) = \left|f(x) - \frac{\int_A f(y)dy}{Vol(A)}\right| \leq \left|f(x) - \frac{\Gamma\left(\frac{N}{2}\right) \int_{S^{N-1}} f(r\omega)d\omega}{2\pi^{N/2}}\right| + \left(\frac{N}{R_2^N - R_1^N}\right) \left[\frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{N/2}} \left\{\sum_{i=1}^{n} \left(\int_{S^{N-1}} |\theta_i f(r\omega)|d\omega\right) \left|\int_{R_1}^{R_2} g_i(s,r)s^{N-1}ds\right|\right\} + \|\theta_{n+1}f\|_{\infty,\overline{A}} \int_{R_1}^{R_2} N_n(|x|,s)s^{N-1}ds\right], \quad \forall\, x \in \overline{A}. \tag{3.19}$$

Corollary 3.6. *Let the conditions of Theorem 3.5 hold. Assume that*

$$\theta_i f(r_0\omega) = 0,\ i = 1,\dots,n,\ \textit{for a fixed } r_0 \in [R_1, R_2], \quad \forall\, \omega \in S^{N-1};$$

also consider all $x_0 = r_0\omega \in \overline{A}$ *for any* $\omega \in S^{N-1}$. *Then*

$$\begin{aligned} E(x_0) &= \left| f(x_0) - \frac{\int_A f(y)dy}{Vol(A)} \right| \\ &\le \left| f(x_0) - \frac{\Gamma\left(\frac{N}{2}\right)\int_{S^{N-1}} f(r_0\omega)d\omega)}{2\pi^{N/2}} \right| \\ &\quad + \left(\frac{N}{R_2^N - R_1^N}\right) \|\theta_{n+1} f\|_{\infty,\overline{A}} \left(\int_{R_1}^{R_2} N_n(|x_0|, s)s^{N-1}ds\right). \end{aligned} \tag{3.20}$$

When $r_0 = R_1$ or R_2 is of special interest.

4. RESULTS ON THE SPHERE

Notation 4.1. Let $N \ge 1$, $B(0,R) := \{x \in \mathbb{R}^N : |x| < R\}$ be the ball in $\mathbb{R}^N$ centered at the origin and of radius $R > 0$. Note that $Vol(B(0,R)) = \frac{\omega_N R^N}{N}$.

Let f from $\overline{B(0,R)}$ into $\mathbb{R}$ and consider f to be radial, i.e. $f(x) = g(r)$, where $r = |x|$, $0 \le r \le R$. We assume $g \in C^{n+1}([0,R])$, $n \ge 0$. Clearly then $f \in C(\overline{B(0,R)})$.

For $F \in C(\overline{B(0,R)})$ we have

$$\int_{B(0,R)} F(x)dx = \int_{S^{N-1}} \left(\int_0^R F(r\omega)r^{N-1}dr\right) d\omega. \tag{4.1}$$

We notice that

$$\frac{N}{R^N}\int_0^R s^{N-1}ds = 1. \tag{4.2}$$

The operator θ_i in the radial case, is as defined in (2.3), now $\forall x \in \overline{B(0,R)}$. In the non-radial case, for $f \in C^{n+1}(\overline{B(0,R)})$, θ_i is defined as in (3.16), $\forall\, x \in \overline{B(0,R)} - \{0\}$.

We make the following remark.

Remark 4.1. Here, let $u_0, u_1, \dots, u_n \in C^{n+1}([0,R])$ and $W_0, W_1, \dots, W_n > 0$ on $[0,R]$, $R > 0$, $n \ge 0$ an integer, with $u_0(r) = c > 0$.

We again first assume that f is radial on $\overline{B(0,R)}$, i.e. there exists g such that $f(x) = g(r)$, $r = |x|$, $0 \le r \le R$. Assume further that $g \in C^{n+1}([0,R])$. Let $x \in \overline{B(0,R)}$. Then by using the polar method (4.1) we obtain

$$E(x) := \left| f(x) - \frac{\int_{B(0,R)} f(y)dy}{Vol(B(0,R))} \right| = \left| g(r) - \frac{N\int_{S^{N-1}}\left(\int_0^R g(s)s^{n-1}ds\right)d\omega}{\omega_N R^N} \right| \tag{4.3}$$

$$= \left| g(r) - \frac{N}{R^N}\int_0^R g(s)s^{n-1}ds \right| \overset{\text{by (4.2)}}{=} \frac{N}{R^N}\left| \int_0^R (g(r) - g(s))s^{N-1}ds \right| =: (*). \tag{4.4}$$

Let $s, r \in [0,R]$, then by the generalized Taylor's formula (2.2) we get

$$g(s) - g(r) = \sum_{i=1}^n L_i g(r) g_i(s,r) + R_n(r,s), \tag{4.5}$$

where

$$R_n(r,s) := \int_r^s g_n(s,t) L_{n+1} g(t) dt. \tag{4.6}$$

But it holds that

$$|R_n(r,s)| \le \left| \int_r^s |g_n(s,t)| dt \right| \|L_{n+1} g\|_{\infty,[0,R]}. \tag{4.7}$$

By calling

$$N_n(r,s) := \left| \int_r^s |g_n(s,t)| dt \right|, \quad \forall\, s,r \in [0,R], \tag{4.8}$$

we get

$$|R_n(r,s)| \le N_n(r,s) \|L_{n+1} g\|_{\infty,[0,R]}, \quad \forall\, s,r \in [0,R]. \tag{4.9}$$

Therefore by (4.4) and (4.5), we have

$$\begin{aligned}
(*) &= \frac{N}{R^N} \left| \int_0^R \left[\sum_{i=1}^n L_i g(r) g_i(s,r) + R_n(r,s) \right] s^{N-1} ds \right| \\
&\le \frac{N}{R^N} \left[\sum_{i=1}^n \left| \int_0^R L_i g(r) g_i(s,r) s^{N-1} ds \right| + \int_0^R |R_n(r,s)| s^{N-1} ds \right] \qquad (4.10) \\
&\overset{\text{by (4.9)}}{\le} \frac{N}{R^N} \left[\sum_{i=1}^n |L_i g(r)| \left| \int_0^R g_i(s,r) s^{N-1} ds \right| \right. \\
&\qquad \left. + (\|L_{n+1} g\|_{\infty,[0,R]}) \int_0^R N_n(r,s) s^{N-1} ds \right]. \qquad (4.11)
\end{aligned}$$

We have established the next result.

Theorem 4.2. *Let* $u_0, u_1, \ldots, u_n \in C^{n+1}([0,R])$, $W_0, W_1, \ldots, W_n > 0$ *on* $[0,R]$, $R > 0$, $n \ge 0$ *an integer, with* $u_0(r) = c > 0$*. Let* f *from* $\overline{B(0,R)}$ *into* $\mathbb{R}$ *be radial, i.e. there exists* g *such that* $f(x) = g(r)$, $r = |x|$, $0 \le r \le R$, $\forall\, x \in \overline{B(0,R)}$*; further assume that* $g \in C^{n+1}([0,R])$.

Then

$$\begin{aligned}
E(x) &:= \left| f(x) - \frac{\int_{B(0,R)} f(y) dy}{Vol(B(0,R))} \right| = \left| g(r) - \frac{N}{R^N} \int_0^R g(s) s^{n-1} ds \right| \\
&\le \frac{N}{R^N} \left[\sum_{i=1}^n |L_i g(r)| \left| \int_0^R g_i(s,r) s^{N-1} ds \right| + (\|L_{n+1} g\|_{\infty,[0,R]}) \int_0^R N_n(r,s) s^{N-1} ds \right] \\
&= \frac{N}{R^N} \left[\sum_{i=1}^n |\theta_i f(x)| \left| \int_0^R g_i(s,|x|) s^{N-1} ds \right| \right. \\
&\qquad \left. + (\|\theta_{n+1} f\|_{\infty,\overline{B(0,R)}}) \int_0^R N_n(|x|,s) s^{N-1} ds \right]. \qquad (4.12)
\end{aligned}$$

The following corollary holds.

Corollary 4.3. *Let the conditions of Theorem 4.2 hold. Assume further that* $L_i g(r_0) = 0$, $i = 1, \ldots, n$, *for a fixed* $r_0 \in [0,R]$*; consider all* $x_0 = r_0 \omega \in \overline{B(0,R)}$ *and any* $\omega \in S^{N-1}$.

Then

$$E(x_0) := \left| f(x_0) - \frac{\int_{B(0,R)} f(y)dy}{Vol(B(0,R))} \right|$$
$$= \left| g(r_0) - \frac{N}{R^N} \int_0^R g(s)s^{n-1}ds \right|$$
$$\leq \frac{N}{R^N} (\|L_{n+1}g\|_{\infty,[0,R]}) \left(\int_0^R N_n(r_0,s)s^{N-1}ds \right)$$
$$= \frac{N}{R^N} (\|\theta_{n+1}f\|_{\infty,\overline{B(0,R)}}) \left(\int_0^R N_n(|x_0|,s)s^{N-1}ds \right). \tag{4.13}$$

Interesting cases especially arise when $r_0 = 0$ or R.

We continue Remark 4.1 with

Remark 4.4. Let f be non-radial. Here assume that $f \in C^{n+1}(\overline{B(0,R)})$. Consider $x \in \overline{B(0,R)} - \{0\}$, which is written uniquely as $x = r\omega$, $r \in (0,R]$, $\omega \in S^{N-1}$. Then by using again the polar method (4.1) we obtain

$$\left| \frac{\int_{S^{N-1}} f(r\omega)d\omega}{\omega_N} - \frac{\int_{B(0,R)} f(y)dy}{Vol(B(0,R))} \right| \tag{4.14}$$

$$= \left| \frac{\int_{S^{N-1}} f(r\omega)d\omega}{\omega_N} - \frac{N \int_{S^{N-1}} \left(\int_0^R f(s\omega)s^{N-1}ds \right) d\omega}{\omega_N R^N} \right| \tag{4.15}$$

$$\text{by} \overset{(4.2)}{=} \left| \frac{N}{\omega_N R^N} \left(\int_{S^{N-1}} \left(\int_0^R f(r\omega)s^{N-1}ds \right) d\omega \right) - \frac{N}{\omega_N R^N} \left(\int_{S^{N-1}} \left(\int_0^R f(s\omega)s^{N-1}ds \right) d\omega \right) \right|$$

$$= \frac{N}{\omega_N R^N} \left| \int_{S^{N-1}} \left(\int_0^R (f(r\omega) - f(s\omega))s^{N-1}ds \right) d\omega \right| =: (*). \tag{4.16}$$

Clearly here $f(\cdot\omega) \in C^{n+1}((0,R])$.

Let $\rho \in (0,R]$, then by (2.2) we get

$$f(\rho\omega) - f(r\omega) = \sum_{i=1}^n ((L_i(f(\cdot\omega)))(r))g_i(\rho,r) + R_n(r,\rho), \tag{4.17}$$

where

$$R_n(r,\rho) := \int_r^\rho g_n(\rho,t)(L_{n+1}(f(\cdot\omega)))(t)dt. \tag{4.18}$$

That is

$$f(\rho\omega) - f(r\omega) = \sum_{i=1}^n (\theta_i f(r\omega))g_i(\rho,r) + R_n(r,\rho), \tag{4.19}$$

with

$$R_n(r,\rho) = \int_r^\rho g_n(\rho,t)\theta_{n+1}f(t\omega)dt. \tag{4.20}$$

We further assume that

$$\theta := \|\theta_{n+1} f\|_{\infty, \overline{B(0,R)} - \{0\}} < +\infty. \tag{4.21}$$

Therefore we obtain

$$|R_n(r,\rho)| \leq \left| \int_r^\rho |g_n(\rho,t)| dt \right| \theta. \tag{4.22}$$

By calling

$$N_n(r,\rho) := \left| \int_r^\rho |g_n(\rho,t)| dt \right|, \qquad \forall\, \rho, r \in [0,R], \tag{4.23}$$

we get

$$|R_n(r,\rho)| \leq N_n(r,\rho)\theta. \tag{4.24}$$

Hence by (4.19) we obtain

$$\begin{aligned} |f(\rho\omega) - f(r\omega)| &\leq \sum_{i=1}^{n} |\theta_i f(r\omega)||g_i(\rho,r)| + |R_n(r,\rho)| \\ &\leq \sum_{i=1}^{n} |\theta_i f(r\omega)||g_i(\rho,r)| + N_n(r,\rho)\theta. \end{aligned} \tag{4.25}$$

By the continuity of f and g_i, $i = 1, \ldots, n$,and by taking the limit as $\rho \to 0$ in the external inequality (4.25), we obtain

$$|f(0) - f(r\omega)| \leq \sum_{i=1}^{n} |\theta_i f(r\omega)||g_i(0,r)| + N_n(r,0)\theta. \tag{4.26}$$

Notice here that $g_n(\rho,t)$ is jointly continuous in $(\rho,t) \in [0,R]^2$, hence $N_n(r,\rho)$ is continuous in $\rho \in [0,R]$.

That is, $\forall\, s \in [0,R]$ we get

$$|f(s\omega) - f(r\omega)| \leq \sum_{i=1}^{n} |\theta_i f(r\omega)||g_i(s,r)| + N_n(r,s)\theta. \tag{4.27}$$

Consequently by (4.16) and (4.27) we find

$$(*) \leq \frac{N}{\omega_N R^N} \int_{S^{N-1}} \left(\int_0^R |f(s\omega) - f(r\omega)| s^{N-1} ds \right) d\omega \tag{4.28}$$

$$\leq \frac{N}{\omega_N R^N} \left[\sum_{i=1}^{n} \left(\int_{S^{N-1}} \left(\int_0^R |\theta_i f(r\omega)||g_i(s,r)| s^{N-1} ds \right) d\omega \right) \right. \tag{4.29}$$

$$\left. + \theta \int_{S^{N-1}} \left(\int_0^R N_n(r,s) s^{N-1} ds \right) d\omega \right]$$

$$= \frac{\sum_{i=1}^{n} \left(\int_{S^{N-1}} \left(\int_0^R |\theta_i f(r\omega)||g_i(s,r)| s^{N-1} ds \right) d\omega \right)}{Vol(B(0,R))}$$

$$+ \frac{\theta N \left(\int_0^R N_n(r,s) s^{N-1} ds \right)}{R^N} \tag{4.30}$$

$$= \sum_{i=1}^{N} \left(\frac{\left(\int_{S^{N-1}} |\theta_i f(r\omega)| d\omega\right) \left(\int_0^R |g_i(s,r)| s^{N-1} ds\right)}{Vol(B(0,R))} \right)$$

$$+ \frac{\theta N \left(\int_0^R N_n(r,s) s^{N-1} ds\right)}{R^N}. \tag{4.31}$$

That is

$$\Delta := \left| \frac{\int_{S^{N-1}} f(r\omega) d\omega}{\omega_N} - \frac{\int_{B(0,R)} f(y) dy}{Vol(B(0,R))} \right|$$

$$\leq \sum_{i=1}^{n} \left[\frac{\left(\int_{S^{N-1}} |\theta_i f(r\omega)| d\omega\right) \left(\int_0^R |g_i(s,r)| s^{N-1} ds\right)}{Vol(V(0,R))} \right]$$

$$+ \frac{\theta N \left(\int_0^R N_n(r,s) s^{N-1} ds\right)}{R^N}. \tag{4.32}$$

The following theorem holds.

Theorem 4.5. *Let $u_0, u_1, \ldots, u_n \in C^{n+1}([0,R])$, $W_0, W_1, \ldots, W_n > 0$ on $[0,R]$, $R > 0$, $n \geq 0$ an integer, with $u_0(r) = c > 0$. Let $f \in C^{n+1}(\overline{B(0,R)})$. Assume that*

$$\theta := \|\theta_{n+1} f\|_{\infty, \overline{B(0,R)} - \{0\}} < +\infty. \tag{4.33}$$

Let $x \in \overline{B(0,R)} - \{0\}$, which is written uniquely as $x = r\omega$, $r \in (0,R]$, $\omega \in S^{N-1}$.
Then

$$E(x) := \left| f(x) - \frac{\int_{B(0,R)} f(y) dy}{Vol(B(0,R))} \right|$$

$$\leq \left| f(x) - \frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{S^{N-1}} f(r\omega') d\omega' \right|$$

$$+ \sum_{i=1}^{n} \left[\frac{\left(\int_{S^{N-1}} |\theta_i f(r\omega') d\omega'\right) \left(\int_0^R |g_i(s,r)| s^{N-1} ds\right)}{Vol(B(0,R))} \right]$$

$$+ \frac{\theta N \left(\int_0^R N_n(r,s) s^{N-1} ds\right)}{R^N}. \tag{4.34}$$

We finally give the following corollary.

Corollary 4.6. *Let the conditions of Theorem 4.5 hold. Assume further that $\theta_i f(r_0 \omega') = 0$, $\forall\, \omega'^{N-1}$, for some $r_0 \in (0,R]$, for all $i = 1, \ldots, n$. Consider all $x_0 = r_0 \omega \in \overline{B(0,R)} - \{0\}$, and any $\omega \in S^{N-1}$.*

Then

$$E(x_0) := \left| f(x_0) - \frac{\int_{B(0,r)} f(y)dy}{Vol(B(0,R))} \right|$$
$$\le \left| f(x_0) - \frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{S^{N-1}} f(|x_0|\omega')d\omega' \right|$$
$$+ \frac{\theta N \left(\int_0^R N_n(|x_0|, s)s^{N-1}ds \right)}{R^N}. \tag{4.35}$$

An interesting case is when $r_0 = R$.

5. Addendum

We give

Proposition 5.1. *Let* $f \in C^1(\overline{B(0,R)})$ *such that* f *is radial, i.e.* $f(x) = g(r)$, $r = |x|$, $0 \le r \le R$, $\forall\, x \in \overline{B(0,R)}$. *Then*

(i) $\exists\, g' \in C((0,R])$,

(ii) $\exists\, g'(0) = 0$.

Proof. (i) is obvious.

(ii) Let u be a unit vector, $h > 0$, then

$$\frac{g(h) - g(0)}{h} = \frac{f(hu) - f(0)}{h}$$
$$\text{(by first order multivariate Taylor's formula)} = \frac{\nabla f(0) \cdot hu + o(h)}{h}$$
$$= \frac{\nabla f(0) \cdot hu}{h} + \frac{o(h)}{h} = \nabla f(0) \cdot u + \frac{o(h)}{h}$$
$$\xrightarrow{h \to 0} \nabla f(0) \cdot u, \text{ by } \frac{o(h)}{h} \to 0.$$

The last is true for any unit vector u. Thus $\nabla f(0) = 0$, proving the claim. □

By Proposition 5.1, we see that, it may well be that g' is discontinuous at zero, if only $f \in C^1(\overline{B(0,R)})$.

Therefore the assumption that $g \in C^{n+1}([0,R])$ in Theorem 4.2 seems to be the best.

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