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OSTROWSKI INEQUALITIES ON TIME SCALES

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ABSTRACT. We prove Ostrowski inequalities (regular and weighted cases) on time scales and thus unify and extend corresponding continuous and discrete versions from the literature. We also apply our results to the quantum calculus case.

Key words and phrases: Ostrowski inequality, Montgomery identity, Time scales.

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1. INTRODUCTION

In 1938, Ostrowski derived a formula to estimate the absolute deviation of a differentiable function from its integral mean. As shown in [6], the so-called Ostrowski inequality

$$\left| f(t) - \frac{1}{b-a} \int_a^b f(s)ds \right| \leq \sup_{a<t<b} |f'(t)| \, (b-a) \left[\frac{\left(t - \frac{a+b}{2}\right)^2}{(b-a)^2} + \frac{1}{4} \right] \tag{1.1}$$

holds and can be shown by using the Montgomery identity [5]. These two properties will be proved for general time scales, which unify discrete, continuous and many other cases. The setup of this paper is as follows. In Section 2 we first give some preliminary results on time scales that are needed in the remainder of this paper. Next, in Section 3 we prove time scales versions of the Montgomery identity and of the Ostrowski inequality (1.1) (the question of sharpness is also addressed), while in Section 4 we offer several weighted time scales versions of the Ostrowski inequality. Throughout, we apply our results to the special cases of continuous, discrete, and quantum time scales.

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2. TIME SCALES ESSENTIALS

Definition 2.1. A *time scale* is an arbitrary nonempty closed subset of the real numbers.

The most important examples of time scales are $\mathbb{R}$, $\mathbb{Z}$ and $q^{\mathbb{N}_0} := \{q^k |\ k \in \mathbb{N}_0\}$.

Definition 2.2. If $\mathbb{T}$ is a time scale, then we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T}|\ s > t\}$ for all $t \in \mathbb{T}$, the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) := \sup\{s \in \mathbb{T}|\ s < t\}$ for all $t \in \mathbb{T}$, and the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ by $\mu(t) := \sigma(t) - t$ for all $t \in \mathbb{T}$. Furthermore for a function $f : \mathbb{T} \to \mathbb{R}$, we define $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$ and $f^\rho(t) = f(\rho(t))$ for all $t \in \mathbb{T}$. In this definition we use $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if t is the maximum of $\mathbb{T}$) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if t is the minimum of $\mathbb{T}$).

These definitions allow us to characterize every point in a time scale as displayed in Table 2.1.

t right-scattered	$t < \sigma(t)$
t right-dense	$t = \sigma(t)$
t left-scattered	$\rho(t) < t$
t left-dense	$\rho(t) = t$
t isolated	$\rho(t) < t < \sigma(t)$
t dense	$\rho(t) = t = \sigma(t)$

Table 2.1: Classification of Points

Definition 2.3. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* (denoted by $C_{\rm rd}$) if it is continuous at right-dense points of $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points of $\mathbb{T}$.

Theorem 2.1 (Existence of Antiderivatives). *Let f be rd-continuous. Then f has an antiderivative F satisfying $F^\Delta = f$.*

Proof. See [1, Theorem 1.74]. ☐

Definition 2.4. If f is rd-continuous and $t_0 \in \mathbb{T}$, then we define the *integral*

$$F(t) = \int_{t_0}^{t} f(\tau)\Delta\tau \quad \text{for} \quad t \in \mathbb{T}. \tag{2.1}$$

Therefore for $f \in C_{\rm rd}$ we have $\int_a^b f(\tau)\Delta\tau = F(b) - F(a)$, where $F^\Delta = f$.

Theorem 2.2. *Let f, g be rd-continuous, $a, b, c \in \mathbb{T}$ and $\alpha, \beta \in \mathbb{R}$. Then*

(1) $\int_a^b [\alpha f(t) + \beta g(t)]\Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t$,
(2) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$,
(3) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$,
(4) $\int_a^b f(t)g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t$,
(5) $\int_a^a f(t)\Delta t = 0$.

Proof. See [1, Theorem 1.77]. ☐

Definition 2.5. Let $g_k, h_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$ be defined by

$$g_0(t,s) = h_0(t,s) = 1 \quad \text{for all} \quad s,t \in \mathbb{T}$$

and then recursively by

$$g_{k+1}(t,s) = \int_s^t g_k(\sigma(\tau),s)\Delta\tau \quad \text{for all} \quad s,t \in \mathbb{T}$$

and

$$h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\Delta\tau \quad \text{for all} \quad s,t \in \mathbb{T}.$$

Theorem 2.3 (Hölder's Inequality)**.** *Let $a, b \in \mathbb{T}$ and $f, g : [a,b] \to \mathbb{R}$ be rd-continuous. Then*

$$\int_a^b |f(t)g(t)|\,\Delta t \le \left\{\int_a^b |f(t)|^p\,\Delta t\right\}^{\frac{1}{p}} \left\{\int_a^b |g(t)|^q\,\Delta t\right\}^{\frac{1}{q}}, \tag{2.2}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. See [1, Theorem 6.13]. □

3. THE OSTROWSKI INEQUALITY ON TIME SCALES

Lemma 3.1 (Montgomery Identity)**.** *Let $a, b, s, t \in \mathbb{T}$, $a < b$ and $f : [a,b] \to \mathbb{R}$ be differentiable. Then*

$$f(t) = \frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s + \frac{1}{b-a}\int_a^b p(t,s) f^\Delta(s)\Delta s, \tag{3.1}$$

where

$$p(t,s) = \begin{cases} s-a, & a \le s < t, \\ s-b, & t \le s \le b. \end{cases} \tag{3.2}$$

Proof. Using Theorem 2.2 (4), we have

$$\int_a^t (s-a)f^\Delta(s)\Delta s = (t-a)f(t) - \int_a^t f^\sigma(s)\Delta s$$

and similarily

$$\int_t^b (s-b)f^\Delta(s)\Delta s = (b-t)f(t) - \int_t^b f^\sigma(s)\Delta s.$$

Therefore

$$\begin{aligned}\frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s &+ \frac{1}{b-a}\int_a^b p(t,s)f^\Delta(s)\Delta s \\ &= \frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s + \frac{1}{b-a}\left[(b-a)f(t) - \int_a^b f^\sigma(s)\Delta s\right] \\ &= f(t),\end{aligned}$$

i.e., (3.1) holds. □

If we apply Lemma 3.1 to the discrete and continuous cases, we have the following results.

Corollary 3.2 (discrete case). *We let* $\mathbb{T} = \mathbb{Z}$. *Let* $a = 0$, $b = n$, $s = j$, $t = i$ *and* $f(k) = x_k$. *Then*

$$x_i = \frac{1}{n}\sum_{j=1}^{n} x_j + \frac{1}{n}\sum_{j=0}^{n-1} p(i,j)\Delta x_j,$$

where

$$\begin{gathered} p(i,0) = 0, \\ p(1,j) = j - n \quad \textit{for} \quad 1 \le j \le n-1, \\ p(n,j) = j \quad \textit{for} \quad 0 \le j \le n-1, \\ p(i,j) = \begin{cases} j, & 0 \le j < i, \\ j - n, & i \le j \le n-1 \end{cases} \end{gathered}$$

as we just need $1 \le i \le n$ *and* $0 \le j \le n-1$. *This result is the same as in* [2, Theorem 2.1].

Corollary 3.3 (continuous case). *We let* $\mathbb{T} = \mathbb{R}$. *Then*

$$f(t) = \frac{1}{b-a}\int_a^b f(s)ds + \frac{1}{b-a}\int_a^b p(t,s)f'(s)ds.$$

This is the Montgomery identity in the continuous case, which can be found in [5, p. 565].

Corollary 3.4 (quantum calculus case). *We let* $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, $a = q^m$ *and* $b = q^n$ *with* $m < n$. *Then*

$$f(t) = \frac{\sum\limits_{k=m}^{n-1} q^k f(q^{k+1})}{\sum\limits_{k=m}^{n-1} q^k} + \frac{1}{q^n - q^m}\sum_{k=m}^{n-1}\left[f(q^{k+1}) - f(q^k)\right] p(t,q^k),$$

where

$$p(t,q^k) = \begin{cases} q^k - q^m, & q^m \le q^k < t, \\ q^k - q^n, & t \le q^k \le q^n. \end{cases}$$

Theorem 3.5 (Ostrowski Inequality). *Let* $a, b, s, t \in \mathbb{T}$, $a < b$ *and* $f : [a,b] \to \mathbb{R}$ *be differentiable. Then*

$$\left| f(t) - \frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s \right| \le \frac{M}{b-a}\left(h_2(t,a) + h_2(t,b)\right), \tag{3.3}$$

where

$$M = \sup_{a<t<b} |f^\Delta(t)|.$$

This inequality is sharp in the sense that the right-hand side of (3.3) *cannot be replaced by a smaller one.*

Proof. With Lemma 3.1 and $p(t,s)$ defined as in (3.2), we have

$$\begin{aligned} \left| f(t) - \frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s \right| &= \left| \frac{1}{b-a}\int_a^b p(t,s) f^\Delta(s)\Delta s \right| \\ &\le \frac{M}{b-a}\left(\int_a^t |s-a|\,\Delta s + \int_t^b |s-b|\,\Delta s \right) \\ &= \frac{M}{b-a}\left(\int_a^t (s-a)\,\Delta s + \int_t^b (b-s)\,\Delta s \right) \\ &= \frac{M}{b-a}\left(h_2(t,a) + h_2(t,b)\right). \end{aligned}$$

Note that, since $p(t,a)=0$, the smallest value attaining the supremum in M is greater than a. To prove the sharpness of this inequality, let $f(t)=t$, $a=t_1$, $b=t_2$ and $t=t_2$. It follows that $f^\Delta(t)=1$ and $M=1$. Starting with the left-hand side of (3.3), we have

$$\begin{aligned}\left|f(t)-\frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s\right| &= \left|t_2-\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\sigma(s)\Delta s\right| \\ &= \left|t_2-\frac{1}{t_2-t_1}\left(\int_{t_1}^{t_2}(\sigma(s)+s)\Delta s-\int_{t_1}^{t_2}s\Delta s\right)\right| \\ &= \left|t_2-\frac{1}{t_2-t_1}\left(\int_{t_1}^{t_2}(s^2)^\Delta\Delta s-\int_{t_1}^{t_2}s\Delta s\right)\right| \\ &= \left|-t_1+\frac{1}{t_2-t_1}\int_{t_1}^{t_2}s\Delta s\right|.\end{aligned}$$

Starting with the right-hand side of (3.3), we have

$$\begin{aligned}\frac{M}{b-a}\left(h_2(t,a)+h_2(t,b)\right) &= \frac{1}{t_2-t_1}\left(\int_{t_1}^{t_2}(s-t_1)\Delta s+\int_{t_2}^{t_2}(s-t_2)\Delta s\right) \\ &= \frac{1}{t_2-t_1}\left(-t_1t_2+t_1^2+\int_{t_1}^{t_2}s\Delta s\right) \\ &= -t_1+\frac{1}{t_2-t_1}\int_{t_1}^{t_2}s\Delta s.\end{aligned}$$

Therefore in this particular case

$$\left|f(t)-\frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s\right|\ge\frac{M}{b-a}\left(h_2(t,a)+h_2(t,b)\right)$$

and by (3.3) also

$$\left|f(t)-\frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s\right|\le\frac{M}{b-a}\left(h_2(t,a)+h_2(t,b)\right).$$

So the sharpness of the Ostrowski inequality is shown. □

Corollary 3.6 (discrete case). *Let $\mathbb{T}=\mathbb{Z}$. Let $a=0$, $b=n$, $s=j$, $t=i$ and $f(k)=x_k$. Then*

(3.4) $$\left|x_i-\frac{1}{n}\sum_{j=1}^n x_j\right|\le\frac{M}{n}\left[\left|i-\frac{n+1}{2}\right|^2+\frac{n^2-1}{4}\right],$$

where

$$M=\max_{1\le i\le n-1}|\Delta x_i|.$$

This is the discrete Ostrowski inequality from [2, Theorem 3.1], *where the constant $\frac{1}{4}$ in the right-hand side of* (3.4) *is the best possible in the sense that it cannot be replaced by a smaller one.*

Corollary 3.7 (continuous case). *If $\mathbb{T}=\mathbb{R}$, then*

$$\left|f(t)-\frac{1}{b-a}\int_a^b f(s)ds\right|\le M(b-a)\left[\frac{\left(t-\frac{a+b}{2}\right)^2}{(b-a)^2}+\frac{1}{4}\right],$$

where

$$M=\sup_{a<t<b}|f'(t)|.$$

This is the Ostrowski inequality in the continuous case [6, p. 226–227], *where again the constant* $\frac{1}{4}$ *in the right-hand side is the best possible.*

Corollary 3.8 (quantum calculus case). *Let* $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, $a = q^m$ *and* $b = q^n$ *with* $m < n$. *Then*

$$\left|f(t) - \frac{1}{q^n - q^m}\int_{q^m}^{q^n} f^\sigma(s)\Delta s\right| \leq \frac{M}{q^n - q^m}\left[\frac{2}{1+q}\left(\left(t - \frac{\frac{1+q}{2}(q^m + q^n)}{2}\right)^2 + \frac{-\left(\frac{1+q}{2}\right)^2(q^m+q^n)^2 + (2(1+q)-2)\left(q^{2m}+q^{2n}\right)}{4}\right)\right],$$

where

$$M = \sup_{q^m < t < q^n}\left|\frac{f(qt) - f(t)}{(q-1)t}\right|,$$

and the constant $\frac{1}{4}$ *in the right-hand side is the best possible.*

4. THE WEIGHTED CASE

The following weighted Ostrowski inequality on time scales holds.

Theorem 4.1. *Let* $a, b, s, t, \tau \in \mathbb{T}$, $a < b$ *and* $f : [a, b] \to \mathbb{R}$ *be differentiable,* $q \in C_{\rm rd}$. *Then*

$$\left|f(t) - \int_a^b q^\sigma(s) f^\sigma(s)\Delta s\right|$$

$$\leq \int_a^b q^\sigma(s)\left|\sigma(s) - t\right| M\Delta s \tag{4.1}$$

$$\leq M\begin{cases}\left(\int_a^b |\sigma(s)-t|^p\,\Delta s\right)^{\frac{1}{p}}\left(\int_a^b (q^\sigma(s))^q\,\Delta s\right)^{\frac{1}{q}}, & \frac{1}{p}+\frac{1}{q}=1, \quad p>1;\\ \sup\limits_{a\le s<b} q^\sigma(s)\left[g_2(a,t)+g_2(b,t)\right]; & \\ \frac{b-\sigma(a)}{2} + \left|t - \frac{b+\sigma(a)}{2}\right|, & \end{cases} \tag{4.2}$$

where

$$M = \sup_{\sigma(a)\le\tau<b}\left|f^\Delta(\tau)\right|$$

and

$$\int_a^b q^\sigma(s)\Delta s = 1, \quad q(s) \geq 0.$$

Proof. We have

$$\begin{aligned}\left|f(t) - \int_a^b q^\sigma(s)f^\sigma(s)\Delta s\right| &= \left|\int_a^b q^\sigma(s)\left(f(t) - f^\sigma(s)\right)\Delta s\right|\\ &\leq \int_a^t q^\sigma(s)\left|f(t)-f^\sigma(s)\right|\Delta s + \int_t^b q^\sigma(s)\left|f(t)-f^\sigma(s)\right|\Delta s\\ &\leq \int_a^t q^\sigma(s)\int_{\sigma(s)}^t \left|f^\Delta(\tau)\right|\Delta\tau\Delta s + \int_t^b q^\sigma(s)\int_t^{\sigma(s)}\left|f^\Delta(\tau)\right|\Delta\tau\Delta s\\ &\leq M\int_a^b q^\sigma(s)\left|\sigma(s)-t\right|\Delta s,\end{aligned}$$

and therefore (4.1) is shown.

The first part of (4.2) can be done easily by applying (2.2). By factoring $\sup_{a\le s<b} q^\sigma(s)$, we have

$$\begin{aligned}\int_a^b q^\sigma(s)\,|\sigma(s)-t|\,\Delta s &\le \sup_{a\le s<b} q^\sigma(s)\left(\int_a^t (t-\sigma(s))\Delta s + \int_t^b (\sigma(s)-t)\Delta s\right)\\ &= \sup_{a\le s<b} q^\sigma(s)\left(\int_t^a (\sigma(s)-t)\Delta s + \int_t^b (\sigma(s)-t)\Delta s\right)\\ &= \sup_{a\le s<b} q^\sigma(s)\left[g_2(a,t)+g_2(b,t)\right],\end{aligned}$$

and therefore the second part of (4.2) holds. Finally for proving the third inequality, we use the fact that

$$\sup_{a\le s<b}\{|\sigma(s)-t|\} = \max\{b-t, t-\sigma(a)\} = \frac{b-\sigma(a)}{2} + \left|t-\frac{b+\sigma(a)}{2}\right|.$$

Thus (4.2) is shown. □

Remark 4.2. Theorem 4.1 states a similar result as shown in [3, Theorem 3.1], if we consider the normalized isotonic functional $A(f) = \int_a^b q^\sigma(s) f^\sigma(s)\Delta s$. Moreover the second inequality of (4.2) is comparable to the achievement in [4, Theorem 3.1] for the continuous case (see [4, Corollary 3.3]).

Corollary 4.3 (discrete case). *Let* $\mathbb{T}=\mathbb{Z}$. *Let* $a=0$, $b=n$, $s=j$, $t=i$, $\tau=k$ *and* $f(k)=x_k$. *Then* $\sum_{i=1}^n q_i = 1$, $q_i \le 0$ *and*

$$\begin{aligned}\left|x_i - \sum_{j=1}^n q_j x_j\right| &\le \sum_{j=1}^n q_j\,|j-i|\,M\\ &\le M\begin{cases}\left(\sum\limits_{j=1}^n |j-i|^p\right)^{\frac1p}\left(\sum\limits_{j=1}^n q_j^q\right)^{\frac1q}, & \frac1p+\frac1q=1,\quad p>1;\\ \left[\frac{n^2-1}{4}+\left(i-\frac{n+1}{2}\right)^2\right]\max\limits_{j=1..n} q(j);\\ \frac{n-1}{2}+\left|i-\frac{n+1}{2}\right|,\end{cases}\end{aligned}$$

where

$$M = \max_{k=1..n-1}|\Delta x_k|.$$

This is the result given in [2, Theorem 4.1].

Corollary 4.4 (continuous case). *If* $\mathbb{T}=\mathbb{R}$, *then* $\int_a^b q(s)ds = 1$, $q(s)\ge 0$ *and*

$$\begin{aligned}\left|f(t)-\int_a^b q(s)f(s)ds\right| &\le \int_a^b q(s)\,|s-t|\,M ds\\ &\le M\begin{cases}\left(\int_a^b |s-t|^p\,ds\right)^{\frac1p}\left(\int_a^b (q(s))^q\,ds\right)^{\frac1q}, & \frac1p+\frac1q=1,\quad p>1;\\ \sup\limits_{a\le s<b} q(s)(b-a)^2\left[\frac{\left(t-\frac{a+b}{2}\right)^2}{(b-a)^2}+\frac14\right];\\ \frac{b-a}{2}+\left|t-\frac{b+a}{2}\right|,\end{cases}\end{aligned}$$

where

$$M = \sup_{a \le \tau < b} |f'(\tau)| .$$

Another interesting conclusion of Theorem 4.1 is the following corollary.

Corollary 4.5. *Let* $a, b, s, t \in \mathbb{T}$ *and* f *differentiable. Then*

$$\left| f(t) - \frac{1}{b-a} \int_a^b f^\sigma(s) \Delta s \right| \le \frac{M}{b-a} \left(h_2(t,a) + h_2(t,b) \right), \tag{4.3}$$

where

$$M = \sup_{\sigma(a) \le t < b} |f^\Delta(t)|.$$

Note that this was shown in a different manner in Theorem 3.5. In (4.3) we use the fact that the functions g_2 and h_2 satisfy $g_2(s,t) = (-1)^2 h_2(t,s)$ for all $t \in \mathbb{T}$ and all $s \in \mathbb{T}^\kappa$ (see [1, Theorem 1.112]).

Remark 4.6. Moreover note that there is a small difference of (4.2) in comparison to Theorem 3.5, as we have $\sup\limits_{\sigma(a) \le t < b}$ instead of $\sup\limits_{a < t < b}$. This is just important if a is right-dense, i.e., $\sigma(a) = a$. But in those cases the inequality does not change and is still sharp. Furthermore in the proof of Theorem 3.5 we could have picked $\sup\limits_{a \le t < b}$ as explained before.

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