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SEPARATION AND DISCONJUGACY

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ABSTRACT. We show that certain properties of positive solutions of disconjugate second order differential expressions $M[y] = -(py')' + qy$ imply the separation of the minimal and maximal operators determined by M in $L^2(I_a)$ where $I_a = [a, \infty)$, $a > -\infty$, i.e., the property that $M[y] \in L^2(I_a) \Rightarrow qy \in L^2(I_a)$. This result will allow the development of several new sufficient conditions for separation and various inequalities associated with separation. Some of these allow for rapidly oscillating q. It is shown in particular that expressions M with WKB solutions are separated, a property leading to a new proof and generalization of a 1971 separation criterion due to Everitt and Giertz. A final result shows that the disconjugacy of $M - \lambda q^2$ for some $\lambda > 0$ implies the separation of M.

Key words and phrases: Separation, Symmetric second order differential operator, Disconjugacy, Limit-point.

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1. INTRODUCTION

Consider the symmetric second order differential expression

$$M[y] := -(py')' + qy \tag{1.1}$$

where $p > 0$, p' and q are continuous on the interval $I_a = [a, \infty)$, $a > -\infty$. M is said to be disconjugate if every nontrivial real solution has at most one zero in I_a . A sufficient condition (from Sturm's comparison theorem) for disconjugacy is that $q \geq 0$, and in this case one can show existence of two positive solutions u_1 and u_2 of $M[y] = 0$ on I_a, called the *principal* and *nonprincipal* solution respectively, such that $u_1' \leq 0$ and $u_2' > 0$ on I_a. More generally, M is disconjugate on I_a if and only if there exists a positive solution u on the interior of I_a. For proofs of these facts and additional discussion see Hartman [15, Corollaries 6.1 and 6.4].

Recall also that M determines several differential operators in the Hilbert space $L^2(I_a)$. In particular the "preminimal" and "maximal" operators L_0' and L are given by $M[y]$ for y in the

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domains $\mathcal{D}_0' \equiv C_0^\infty(I_a)$, the space of infinitely differentiable functions with compact support in the interior of I_a and

$$\mathcal{D} = \{y \in L^2(I_a) \cap AC_{\text{loc}}(I_a) : py' \in AC_{\text{loc}}(I_a); M[y] \in L^2(I_a)\},$$

where AC_{loc} stands for the real locally absolutely continuous functions on I_a and $L^2(I_a)$ denotes the usual Hilbert space associated with equivalence classes of Lebesgue square integrable functions f, g having norm and inner product

$$\|f\| = \left(\int_{I_a} |f|^2\right)^{\frac{1}{2}}, \quad [f, g] := \int_{I_a} f\bar{g}.$$

The "minimal operator" L_0 with domain $\mathcal{D}_0$ is then defined as the closure of L_0'.

With the above definitions one can show that

(i) $C_0^\infty(I_a) \subset \mathcal{D}_0' \subset \mathcal{D}_0 \subset \mathcal{D}$,
(ii) $L_0'^* = L_0^* = L$,
(iii) $L^* = L_0$,
(iv) $\mathcal{D}_0', \mathcal{D}_0$, and $\mathcal{D}$ are dense in $L^2(I_a)$.

The regularity assumptions made in this paper on p and q are stronger than necessary to properly define L_0, L. In general one needs only to assume the so-called "minimal conditions" that p^{-1} and q are locally integrable on (a, ∞). In this case $C_0^\infty(I_a)$ may not be contained in $\mathcal{D}_0'$ but the properties (ii)–(iv) will still hold. The maximal and minimal operators L and L_0 can also be defined relative to an arbitrary interval (a, b) where $-\infty \le a < b \le \infty$. If p^{-1}, q are Lebesgue integrable on some interval (a, c) or (c, b) for $a < c < b$ then a or b are said to be "regular"; otherwise they are "singular". (Infinite endpoints however are considered singular even if p^{-1}, q are integrable on (a, b).) Thus in our setting a is regular and ∞ is singular–we signal this by writing $I_a = [a, \infty)$ rather than (a, b).

M is *limit-point* or LP at ∞ if there is at most one solution of $M[y] = 0$ which is in $L^2(I_a)$, and *limit-circle* or LC at the point if both solutions are so integrable. This can be shown equivalent to each of the following properties

(i) $\{y, z\}(\infty) := \lim_{x\to\infty}(yp\bar{z}' - py'\bar{z})(x) = 0$ for all $y, z \in \mathcal{D}$.
(ii) $\mathcal{D} = \mathcal{D}_0 \oplus \text{span}(\phi_1, \phi_2)$, where $\phi_1, \phi_2 \in \mathcal{D}$ and have compact support in I_a. Thus $\mathcal{D}$ is a two dimensional extension of $\mathcal{D}_0$.

It is clear that if M is disconjugate it is LP at ∞ since the nonprincipal solution $u_2 \notin L^2(I_a)$. A stronger condition at ∞ than LP is *strong limit-point* or SLP which means

$$\lim_{x\to\infty}(py\bar{z})(x) = 0$$

$\forall\, y, z \in \mathcal{D}$. For a thorough development of these operator theoretic ideas see Naimark, [17, §17]. Discussion of the SLP concept may be found in Everitt, [7].

We turn now to the central concern of this paper.

Definition 1.1. M is said to be *separated* on $\mathcal{D}_0$ or on $\mathcal{D}$—equivalently L_0 or L is separated—if $qy \in L^2(I_a)$. (Obviously also by application of the triangle inequality $(py')' \in L^2(I_a)$.)

The following is an exercise in the Closed Graph Theorem (see e.g. [16]).

Proposition 1.1. *Separation on $\mathcal{D}_0$ or $\mathcal{D}$ is equivalent to the inequality*

$$A\|(py')'\| + C\|qy\| \le K\|M[y]\| + L\|y\|. \tag{1.2}$$

for nonnegative constants A, C, K and L.

The next result shows some connections between LP or SLP at ∞ and separation. Its proof may be found in [2].

Proposition 1.2. *If M is separated on $\mathcal{D}_0$ then it is separated on $\mathcal{D}$ if M is LP at ∞. On the other hand, if M is separated on $\mathcal{D}$ then it is SLP at ∞.*

Remark 1.3. Two immediate consequences of Proposition 1.2 are (i) if M is LC at ∞ then it is not separated, (ii) if M is LP but not SLP at ∞ then M is not separated on $\mathcal{D}_0$.

Several criteria for separation of M given by Everitt and Giertz in a series of pioneering papers [8] – [12], also see Everitt, Giertz, and Weidmann [13], and Atkinson [1]. More recent results (that include weighted cases) may be found in Brown and Hinton [2],[3]. We quote three typical results.

Theorem A (Brown and Hinton [2]). *If p^{-1} is locally integrable on I_a, $pq \geq 0$, $q(x)$ is locally absolutely continuous, and*

$$\left|\frac{p^{1/2}q'(x)}{q^{3/2}(x)}\right| \leq \theta < 2, \tag{1.3}$$

on I_a then M is separated on $\mathcal{D}$.

Remark 1.4. The original version of Theorem A with $p = 1$ and $q > 0$ is due to Everitt and Giertz [11]. The case of nontrivial p but $\theta < 1$ is given in [9].

Theorem B (Brown and Hinton [3]). *Suppose that p^{-1} is locally integrable on I_a, $pq \geq 0$, and q, p are twice differentiable on I_a. Then M is separated on $\mathcal{D}_0$ if*

$$\limsup_{x\to\infty} \frac{(pq')'}{q^2} \leq \theta < 2. \tag{1.4}$$

Remark 1.5. Note that in the case $p = 1$ both Theorems A and B work for a wide class of increasing q such as $q(x) = \exp(x)$, $q(x) = \exp(x^n)$ for $n > 0$, $q(x) = \exp(\exp(\cdots\exp(x))\cdots)$, etc. On the other hand, both theorems fail if q is rapidly oscillatory, e.g., $q(x) = \exp(x)(1 + \sin(\exp(x))$. Note also that a consequence of Theorem B is that if $p = 1$ and $q'' \leq 0$ (i.e., q is concave down) then M is separated.

Theorem C (Brown and Hinton [2]). *Suppose $p^{-1} \in L^1_{loc}(I_a)$, $pq \geq 0$, q is differentiable. Then M is separated on $\mathcal{D}_0$ if either*

$$\sup_{x\in I_a}(x-a)\int_x^\infty \frac{q'}{q^2} = K_1 < \frac{1}{4} \tag{1.5}$$

or

$$\sup_{x\in I_a}(x-a)\int_x^\infty (q')^2 = K_2 < \infty. \tag{1.6}$$

Remark 1.6. In this theorem we see that separation holds for *any* p satisfying weak conditions provided that q is of slow enough growth. For example $q(x) = x^\beta$, $\beta < \frac{1}{2}$, satisfies (1.5) and $q(x) = K\log(x)$ satisfies (1.6). These facts should not be particularly surprising since if $q = 1$ then M would be separated for any p; consequently one can conjecture that the same ought to be true if q has slow enough growth.

Recently Chernyavskaya and Schuster,[4] have given necessary and sufficient conditions using averaging techniques due to Otelbaev for the inequalities

$$K\|M[y]\|_{p,\mathbb{R}} \geq \|y''\|_{p,\mathbb{R}} + \|qy\|_{p,\mathbb{R}} \tag{1.7}$$

$$\geq \|ry\|_{p,\mathbb{R}}, \tag{1.8}$$

where the norms are L^p norms on $\mathbb{R}$, $q \geq 1$ and is locally integrable, $r > 0$ is locally p integrable, $M[y] = -y'' + qy \in L^p(\mathbb{R})$, and $1 \leq p \leq \infty$. Note that (1.7) or (1.8) can hold on the L^p analog of $\mathcal{D}$ only if M has no L^p or r-weighted L^p solutions. Although the conditions in [4] seem challenging to implement they can be applied to rapidly oscillating potentials such as

$$q(x) = \exp(|x|) + \exp(|x|)(1 + \sin(\exp(|x|)) \tag{1.9}$$

for which both Theorems A and B fail.

In this paper we show that certain pointwise properties of a positive solution of a disconjugate expression M imply that M is separated on $\mathcal{D}$. This means in particular that separation occurs if M has a fundamental set of solutions, sometimes called WKB solutions, with a particular asymptotic behavior at ∞. Since the existence of WKB solutions follows from certain integral conditions satisfied by p and q, we are led to a test for separation that includes a well-known 1971 result of Everitt and Giertz as a special case. We also show that our approach leads to several other sufficient conditions for separation which do not require verification of properties of positive solutions of M. Some of these will work for rapidly oscillating potentials similar to (1.9). We look also at conditions that ensure that the mapping associated with the inequality

$$\left\|\sqrt{h}y\right\| \le K\|M[y]\| + L\|y\| \tag{1.10}$$

is compact where h is a weight, i.e., a positive locally integrable function, which in turn will lead to a more general inequality (see (2.17) below) than (1.10). We also investigate "perturbation" results: if $M_1[y] = -(py')' + q_1 y$ is separated, when is the same true of $M_2[y] = -(py')' + q_2 y$ when in some sense q_2 is "close" to q_1?

Although our tests for separation hold only in $L^2(I_a)$ and are sufficient but not necessary, they are easy to apply. Moreover we consider nontrivial p and on occasion allow q to be negative or even unbounded below which is a more general setting than in [4]. Finally, as already mentioned, the inequalities (such as (2.17) below) associated with separation may be more complicated than (1.7)–(1.8).

We use the following notational conventions in the paper. Positive constants will be denoted by capital letters with or without subscripts such as C, K, K_1, etc. The value of a constant may change from line to line without a change in the symbol denoting it. If f and g are functions $f \sim g$ denotes the asymptotic equivalence of f and g, i.e., $\lim_{x\to\infty} f/g = 1$. $L^2(w; I_a)$ is the standard w-weighted Hilbert space with norm and inner product

$$\|f\|_w = \left(\int_{I_a} w|f|^2\right)^{\frac{1}{2}}, \quad [f,g]_w = \int_{I_a} wf\bar{g},$$

where w is a weight. The class of Lebesgue integrable or locally Lebesgue integrable functions on I_a will be denoted by $L(I_a)$ or $L_{\text{loc}}(I_a)$.

Remark 1.7. The Hilbert space theory (see e.g. [17] of the operators L_0 and L is usually developed on complex domains. Thus $\mathcal{D}$ is the space of locally absolutely continuous complex valued functions f on I_a such that f and $M[f]$ belong to $L^2(I_a)$ with similar changes in the definitions of $\mathcal{D}_0'$ and $\mathcal{D}_0$. All the standard closure and adjoint properties of L_0 and L remain true in both cases. Since the chief tool in our development is the concept of disconjugacy which is defined only for real-valued solutions of M, we will derive conditions for the separation of M only for real $\mathcal{D}_0$ and $\mathcal{D}$. However all our results go over to the complex case. This is seen from observation that if $f = f_1 + if_2 \in \mathcal{D}$ then

$$\|M[f]\|^2 = \|M[f_1]\|^2 + \|M[f_2]\|^2,$$
$$\|qf\|^2 = \|qf_1\|^2 + \|qf_2\|^2.$$

Therefore $M(f),\ qf \in L^2(I_a) \leftrightarrow M[f_1], M[f_2], qf_1, qf_2 \in L^2(I_a)$.

2. MAIN RESULTS

Theorem 2.1. *Let $p > 0$, q be C^1 functions. Suppose $M[y] = -(py')' + qy$ has a positive solution on the interior of I_a such that*

$$(pu')'u \equiv qu^2 \leq 2p(u')^2, \tag{2.1}$$

$$(1-\delta)(u')^2 \leq u''u, \quad \delta \in [0, 1/3), \tag{2.2}$$

$$p'u' \geq 0. \tag{2.3}$$

Then $q \geq 0$ and M is separated on $\mathcal{D}$.

Proof. We need only show that M is separated on $\mathcal{D}_0$. Because M is disconjugate and as will be seen below (see (2.9)) $q \geq 0$, M is LP at ∞ and separation on $\mathcal{D}$ will follow by Proposition 1.2; in this case by Proposition 1.1 y will satisfy an inequality of the form

$$\|qy\|^2 \leq C\|y\|^2 + D\|M[y]\|^2$$

for certain positive constants C, D.

Let $z(t) = -u'/u$. Then z satisfies the Riccati-type equation

$$(pz)' = pz^2 - q. \tag{2.4}$$

Since

$$(pz)' = \frac{-u(pu')' + p(u')^2}{u^2} \tag{2.5}$$

(2.1) – (2.3) is equivalent to the properties

$$-pz^2 \leq (pz)', \tag{2.6}$$

$$z' \leq \delta z^2, \tag{2.7}$$

$$p'z \leq 0. \tag{2.8}$$

To see this, note that from the definition of z and (2.5)

$$\begin{aligned}(2.1) &\Leftrightarrow -2\frac{p(u')^2}{u^2} \leq \frac{-u(pu')'}{u^2} \\ &\Leftrightarrow -\frac{p(u')^2}{u^2} \leq \frac{-u(pu')' + p(u')^2}{u^2} \\ &\Leftrightarrow (pz)' \geq -pz^2.\end{aligned}$$

Also

$$\begin{aligned}(2.2) &\Leftrightarrow -(1-\delta)\frac{(u')^2}{u^2} \geq \frac{-uu''}{u^2} \\ &\Leftrightarrow \delta\frac{(u')^2}{u^2} \geq \frac{-uu'' + (u')^2}{u^2} \\ &\Leftrightarrow \delta z^2 \geq z'.\end{aligned}$$

Finally, the definition of z and (2.3) clearly implies that $p'z \leq 0$.

Next define the operators

$$L(y) = y' + zy,$$
$$L^*(y) = -y' + zy$$

where $y \in C_0^\infty(I_a)$.

We now derive sufficient conditions for the "separation" of L^*. We have

$$\begin{aligned} \|L^*(y)\|^2 &= [L^*(y), L^*(y)] \\ &= [LL^*(y), y] \\ &= [-y'' + (z^2+z')y, y] \\ &= \int_{I_a} (y')^2 + (z^2+z')y^2. \end{aligned}$$

Since $p'z \le 0$ we see that

$$\begin{aligned} (pz)' = p'z + pz' \Rightarrow pz' \ge (pz)' \ge -pz^2 \\ \Rightarrow z' \ge -z^2. \end{aligned}$$

Because $z' + z^2$ is nonnegative the inequality

$$\|L^*(y)\|^2 \ge \|y'\|^2$$

holds. By the triangle inequality it also follows that

$$\|zy\|^2 \le 4\|L^*(y)\|^2.$$

The remaining step is use the separation of L^* to show that M restricted to $C_0^\infty(I_a)$ is also separated. We first observe that

$$\begin{aligned} L^*(pL(y)) &= -(py' + pzy)' + z(py' + pzy) \\ &= -(py')' + [-(pz)' + pz^2]y \\ &= -(py')' + qy. \end{aligned}$$

A consequence of (2.7) – (2.8) is that

$$\begin{aligned} -(pz)' + pz^2 &= -pz' - p'z + pz^2 \\ &\ge -pz' + pz^2 \\ &\ge pz^2(1-\delta) \\ &\ge 0. \end{aligned}$$

Therefore both

$$q \ge 0 \quad \text{and} \quad (pz)' \le \delta pz^2. \tag{2.9}$$

Now also

$$\begin{aligned} \|M[y]\|^2 &= [L^*(pL)(y), L^*(pL)(y)] \\ &= \|L^*(pL(y))\|^2 \\ &\ge \frac{1}{4}\|z(pL(y))\|^2 \\ &= \frac{1}{4}[L^*((zp)^2L(y)), y] \\ &= \frac{1}{4}\left[-((zp)^2y')' + (z^4p^2 - (z^3p^2)'y, y\right] \\ &= \frac{1}{4}\int_{I_a}[(zp)^2(y')^2 + (z^4p^2 - (z^3p^2)')y^2]. \end{aligned} \tag{2.10}$$

Hence since $p'z \le 0$ and $z' \le \delta z^2$,

$$\begin{aligned} z^4p^2 - (z^3p^2)' &= z^4p^2 - 3z^2z'p^2 - 2z^2p(p'z) \\ &\ge (1-3\delta)z^4p^2. \end{aligned} \tag{2.11}$$

But $(pz)' \ge -pz^2$, so

$$pz^2 = q + (pz)' \ge q - pz^2.$$

Hence also $pz^2 \ge q/2$. Combining this with (2.10) and (2.11) gives the inequality

$$\|M[y]\|^2 \ge \frac{1}{8}\,\|\sqrt{pq}y'\|^2 + \frac{1-3\delta}{4}\|qy\|^2, \tag{2.12}$$

which immediately yields the separation inequality

$$\frac{16}{1-3\delta}\|M[y]\|^2 \ge \|qy\|^2. \tag{2.13}$$

A closure argument (cf. [2, Lemma 1]) shows that the same inequalities are true on the minimal domain $\mathcal{D}_0$. □

Remark 2.2.

(i) It is well-known that the existence of a positive solution u, the existence of a continuously differentiable solution z of the inequality $z' + z^2/p + q \le 0$, or the identity $M[y] = L^*(pL(y))$ for y having a continuous second derivative are each equivalent to the disconjugacy of M on I; see e.g. [15, Corollary 6.1, Theorem 7.2] or Coppel [5, p.6].

(ii) We may require that both the conditions $q \ge 0$ and (2.1) – (2.3) hold "eventually", i.e. on $I_{a'}$ for sufficiently large $a' > a$. In this case the restriction of M to $I_{a'}$ will be separated on its maximal domain. Since q is bounded on $(a, a']$ it is immediate that separation holds also for I_a (cf. [2, Remark 1 and Proposition 2]) although the corresponding inequality may be of the form (1.1) rather than (2.13).

(iii) If we retrace the proof of Theorem 2.1 with $p = 1$ (2.1) – (2.3) becomes

$$(1-\delta)(u')^2 \le u''u \le 2(u')^2 \Leftrightarrow (1-\delta)(u')^2 \le qu^2 \le 2(u')^2, \quad \delta \in [0, 1/3)\,, \tag{2.14}$$

with a corresponding change in (2.6) – (2.8).

(iv) If q is positive and u satisfies (2.1) or (2.2) then u' is strictly positive or negative, for if $u'(x_0) = 0$ either $u(x_0) = 0$ or one of $(pu')'$ or u vanishes at x_0. In either case $q > 0 \Rightarrow u(x_0) = 0$, implying that $u \equiv 0$.

In the remainder of the paper "separated" means separated on $\mathcal{D}$ unless the restriction to $\mathcal{D}_0$ is stated. Also, in proving separation inequalities on $\mathcal{D}_0$ such as (1.2) we will generally start with $y \in C_0^\infty(I_a)$ and omit the routine closure argument which extends the inequality to $\mathcal{D}_0$.

We now show that information about the asymptotic behavior of positive solutions of $M[y] = 0$ can yield criteria for separation based on the stable conditions of p and q.

Theorem 2.3. *Suppose that p, q are positive and twice differentiable with p' nonnegative or nonpositive. Set*

$$t(x) := \int_a^x \sqrt{\frac{q}{p}},$$

$$\mu(x) := (pq)^{-1/4},$$

and assume that $\lim_{x\to\infty} t(x) = \infty$, $\mu(p\mu')' \in L^1(I_a)$, and $\limsup_{x\to\infty} |p'|/\sqrt{pq} = \delta < \frac{1}{3}$. Then M is separated.

Proof. By Coppel [6, Theorem 13], M has fundamental solutions u such that for $x \to \infty$

$$u \sim= \mu \exp(\pm t(x)), \quad u' \sim= \pm(p\mu)^{-1} \exp(\pm t(x)).$$

It follows that $(pu')' \sim qy_1$ and so

$$u(pu')' \sim qu^2 \sim \sqrt{\frac{p}{q}} \exp(\pm 2t(x)) \sim p(u')^2.$$

Clearly (2.1) is satisfied on $I_{a'}$ for sufficiently large $a' > a$. To derive (2.2) observe that the asymptotic equivalence of $p(u')^2$ and $(pu')'$ implies that

$$(u')^2 \sim u''u + \frac{p'}{p} u'u.$$

But

$$\begin{aligned}(p'/p)(u'u/(u')^2) = \frac{p'u}{pu'} &\sim p' \frac{u}{p\sqrt{q/pu^2}} \\ &\sim p'\mu^2 \le \frac{|p'|}{\sqrt{pq}} \le \delta + \epsilon < \frac{1}{3}\end{aligned}$$

as $x \to \infty$. Thus for $\epsilon > 0$ and on some $I_{a'}$ with a' sufficiently large we have that

$$(u')^2 \le (u''u + (\delta + \epsilon)(1 + \epsilon)(u')^2) \Rightarrow (1 - (\delta + \epsilon)(1 + \epsilon))(u')^2 \le u''u$$

which obviously implies (2.2) if ϵ is small enough. Finally, if $p' \ge 0$ we choose $y_1 = \mu(x) \exp(t(x))$ and if $p' \le 0$ we choose $y_1 = \mu \exp(-t(x))$. In either case (2.3) holds. By Remark 2.2 (ii), the fact that M is LP at ∞, and Theorem 2.1, separation follows. □

In 1970 [8] Everitt and Giertz showed:

Corollary 2.4. *If* $p = 1$, $q \ge d > 0$, *and*

$$\int_{I_a} q^{-1/4} \left|(q^{-1/4})''\right| < \infty,$$

then M *is separated.*

Proof. Evidently this condition is a special case of Theorem 2.1 with $p = 1$, cf. [6, Theorem 14]. □

Remark 2.5. The hypothesis of Corollary 2.4 can be shown to be equivalent to (see [6, p. 122]

$$\int_{I_a} \left|q^{-3/2} q''\right| < \infty,$$

unless $q(x) \sim cx^{-4}$ and $q'(x) \sim -4cx^{-5}$ for c a positive constant. But in this case M is trivially separated on I_a if $a > 0$.

A similar result using the asymptotic properties of solutions but requiring less smoothness on q is given by:

Theorem 2.6. *Suppose that* $p = 1$, $q \ge d > 0$ *is differentiable, and*

$$\int_{I_a} \frac{|q'|}{q^{3r/2 - 1/2}} < \infty$$

for some r, $1 \le r \le 2$. *Then* M *is separated.*

Proof. By a result of Hartman and Winter [15, p. 320] M has solutions u such that

$$u \sim q^{-1/4}\exp(\pm t(x)),$$
$$u' \sim \pm q^{1/2} u,$$

where $t(x) = \int_a^x \sqrt{q}$. Since $u'' \sim q^{3/4}\exp(\pm\int_a^x q^{1/2})$, it is clear that (2.14) is satisfied on some $I_{a'}$, $a' > a$. □

In most cases however it is difficult to verify (2.1) – (2.3) or (2.14) directly, which motivates us to seek an equivalent formulation of Theorem 2.1 for which knowledge of properties of positive solutions of $M[y] = 0$ is not required.

Theorem 2.7. *Let $p > 0$ and z be $C^1(I)$ functions. Then if (2.6) – (2.8) hold and $q = pz^2 - (pz)'$. $M[y]$ is separated and the inequality (2.13) holds.*

Proof. The fact that Theorem 2.1 implies Theorem 2.7 is clear. On the other hand, if we set $u = e^{-\int z}$, then u is a positive solution of $M[y] = 0$. $z = -u'/u$, and the conditions (2.1) – (2.3) hold as they are equivalent to (2.6) – (2.8). Thus all the assumptions of Theorem 2.1 are satisfied. □

Theorem 2.8. *Suppose that M is separated, $q \geq d > 0$, and that h is a weight. Assume further that either*

$$\lim_{x\to\infty} \frac{q^2}{h} = \infty \tag{2.15}$$

or $\lim_{x\to\infty} h = \infty$, *and*

$$K\left\|\sqrt{p}y'\right\| \geq \left\|h^{\theta/2}y\right\| \tag{2.16}$$

for some $\theta > 1$ and all $y \in C_0^\infty$. Let $G_M(y) := \{(y, M[y]), y \in \mathcal{D}\}$, equipped with the graph norm. Then the mapping $\lambda : G_M \to L^2(h; I)$ given by $\lambda(G_M(y) = y$ is compact, and M satisfies an inequality of the form

$$\epsilon\|M[y]\| + K(\epsilon)\|y\| \geq \left\|\sqrt{h}y\right\| \tag{2.17}$$

on $\mathcal{D}$ for $\epsilon > 0$.

Proof. If (2.15) holds and M is separated, then by (1.2) of Proposition 1.1 there is an inequality of the form

$$\begin{aligned}\frac{L}{C}\|y\| + \frac{K}{C}\|M[y]\| &\geq \|qy\| \\ &= \left(\int_{I_a}\left(\frac{q^2}{h}\right)hy^2\right)^{\frac{1}{2}} \\ &\geq n\left(\int_{x_n}^\infty hy^2\right)^{\frac{1}{2}}.\end{aligned} \tag{2.18}$$

for any positive integer and where the sequence $\{x_n\} \to \infty$. Let $\lambda_n : G_M \to L^2(I_a^n)$ be given by the characteristic function on I_a^n composed with λ, where $I_a^n = [a, x_n]$. Since the solutions of $M[y] = 0$ and q are continuous on I_a^n, a Green's function argument shows that the maps $\lambda_n : G_M \to L^2(h; I_a^n)$ are compact. By (2.18) the λ_n converge in operator norm to a compact limit λ. Also since $q \geq d > 0$, q is closed, considered as a multiplication operator $\tilde{q} : L^2(I_a) \to L^2(I_a)$, and since M is separated $\mathcal{D} \subset \mathcal{D}(\tilde{q})$. In this situation Corollary V.3.8 of Goldberg [14, p. 123] applies and gives (2.17).

Under the second set of conditions we have from the Cauchy-Schwartz inequality, integration by parts, and since $\lim_{x\to\infty} h = \infty$ that on some $I_{a'}$, $a' > a$, and for $y \in C_0^\infty(I_{a'})$ that

$$\begin{aligned}\|(py')'\|\,\|h^{\theta/2}y\| &\geq \|(py')'\|\,\|y\| \\ &\geq [(py')', y] \\ &= \|\sqrt{p}y'\|^2 \\ &\geq K^{-2}\|h^{\theta/2}y\|^2.\end{aligned}$$

Hence

$$\begin{aligned}\|(py')'\| &\geq K^{-2}\|h^{\theta/2}y\| \\ &\geq K^{-2}\|h^{(\theta-1)/2}\sqrt{h}y\| \\ &\geq K^{-2}n\|\sqrt{h}y\|_{(x_n,\infty)}.\end{aligned}$$

Since M is separated we obtain from (1.2) the inequality

$$\frac{L}{C}\|y\| + \frac{K}{C}\|M[y]\| \geq \|(py')'\| \geq K^{-1}n\left\|\sqrt{h}y\right\|_{(x_n,\infty)}$$

on the $C_0^\infty(I_a)$ functions and therefore also on $\mathcal{D}_0$; the proof that λ restricted to $\mathcal{D}_0$ is compact continues as in the first part. But since $\mathcal{D}$ is a finite dimensional extension of $\mathcal{D}_0$, λ is also compact. □

Remark 2.9. Following Everitt and Giertz [8] we say that q is in the class $P(\gamma)$ or $q \in P(\gamma)$ if whenever $y \in \mathcal{D}$ then $|q|^\gamma \in L^2(I_a)$. Thus the separation of M on $\mathcal{D}$ is equivalent to $q \in P(1)$. It is also easy to verify by thinking of $q = q_1 + q_2$ where $q_1(x) \leq 1$ and $q_2(x) > 1$ that $q \in P(\gamma) \Rightarrow q \in P(\beta)$ for any $\beta \in (0, \gamma]$. Suppose now $q \in P(1)$ and $\lim_{x\to\infty} q = \infty$. Then from the first part of Theorem 2.8 not only will $q \in P(\theta)$, $\theta < 1$, but the "compactness" inequality (2.17) will hold if $h = q^\theta$. If M is separated, $q \to \infty$, and (2.16) holds for $h = q^2$ and $\theta > 1$ then $q \in P(\theta)$, and we have the interesting consequence that the mapping $\lambda : G_M \to L^2(q; I_a)$ is compact. In general, if $q \in P(\gamma)$ and $q \to \infty$ then $\lambda : G_M \to L^2(q^\beta; I_a)$ is compact.

A disadvantage of Theorem 2.7 is that although q has the form $pz^2 - (pz)'$, since M is disconjugate, it may be difficult to determine z and to verify (2.6) – (2.8). We attempt to remedy this problem in the next three corollaries and obtain additional usable tests.

Corollary 2.10. *If $M_1[y] = -(py')' + q_1 y$ where $q_1(z_1) = (pz_1^2 - (pz_1)')$ satisfies the hypotheses of Theorem 2.7 and $M_{1,c}[y] = -(py')' + q_{1,c}y$ where $q_{1,c}(z_1) = (pc^2z_1^2 - (pcz_1)')y$, where $c > 1$ then $M_{1,c}[y]$ is separated. More generally, if g is a differentiable function such that $g, g' \geq 0$ and*

$$\frac{g'(x)x^2}{g(x)^2} \leq 1 \tag{2.19}$$

then if $q_2 = z_2^2 - z_2'$ where $z_2 = g(z_1)$ M_2 is separated. Conversely, if $M_2[y]$ satisfies the hypotheses of Theorem 2.7 and

$$\frac{g'(x)x^2}{g(x)^2} \geq 1, \tag{2.20}$$

then M_1 is separated.

Proof. Let $z_2 = cz_1$. Then since $c > 1$, z_2 satisfies (2.6) – (2.8) and $q_2 = pz_2^2 - (pz_2)'$. Also $p'z_2 \leq 0$. Separation follows by Theorem 2.7.

For the second part, since z_2 satisfies (2.6) – (2.8) and by (2.19) we have that

$$\begin{aligned} z_2' = g'(z_1)z_1' &\geq -g'(z_1)z_1^2 \geq -g(z_1)^2 = -z_2^2 \\ &\leq \delta g'(z_1)z_1^2 \leq \delta g(z_1)^2 = \delta z_2^2. \end{aligned}$$

Thus z_2 satisfies (2.6) – (2.8) and we can again apply Theorem 2.7. On the other hand, using (2.20)

$$z_2' \geq -z_2^2 \Leftrightarrow z_1' \geq -\frac{g(z_1)^2}{g'(z_1)} \geq -z_1^2,$$

$$z_2' \leq \delta z_2^2 \Leftrightarrow z_1' \leq \frac{\delta g(z_1)^2}{g'(z_1)} \leq \delta z_1^2.$$

□

Example 2.1. Let $p = 1$, $z_1(x) = \sqrt{x}$, and $q_1(x) = x - \left(\frac{1}{2}\right) x^{-\frac{1}{2}}$. If $a > \left(\frac{3}{2}\right)^{\frac{2}{3}}$, then (2.6) – (2.8) is satisfied for some $\delta < \frac{1}{3}$. If $g(x) = \exp(x^2)$, (2.19) is satisfied for say $a > 2$. Taking $z_2(x) = g(z_1) = \exp(x)$ we get that $q_2(x) = \exp(2x) - \exp(x)$ and there is an inequality of the form

$$K\|M_2[y]\| \geq \|q_2 y\|$$

on $\mathcal{D}_0$ defined on I_a. That M_2 is separated on $\mathcal{D}_0$ also follows from Theorem A, but the inequality seems new.

The next two lemmas are useful.

Lemma 2.11. *Suppose that $M_1[y] = -(py')' + q_1 y$ is separated on $\mathcal{D}_0$. If*

$$\limsup_{x\to\infty} \frac{q_2}{q_1} < 1 + \gamma,$$

$$\liminf_{x\to\infty} \frac{q_2}{q_1} > 1 - \gamma,$$

where γ is sufficiently small, then $M_2[y]$ is also separated on $\mathcal{D}_0$.

Proof. Choose a' large enough so that on $I_{a'}$

$$\left|\frac{q_2}{q_1} - 1\right| < \gamma.$$

Since $M_2[y] = M_1(y) + (q_2 - q_1)y$ by the triangle inequality and inequality (1.2) we have that

$$L\|y\| + K\|M_2[y]\| + K\left\|q_1\left(\frac{q_2}{q_1} - 1\right)y\right\| \geq K\|M_1[y]\| + L\|y\|$$

for $y \in C_0^\infty$. Hence on $I_{a'}$

$$\begin{aligned} L\|y\| + K\|M_2[y]\| + K\gamma\|q_1 y\| &\geq C\|q_1 y\| \\ &\geq C(1+\gamma)^{-1}\|q_2 y\|. \end{aligned}$$

Thus

$$L\|y\| + K\|M_2[y]\| \geq d\|q_2 y\|,$$

where $d = (1+\gamma)^{-1}(C - K\gamma)$, which is positive for small enough γ. □

Lemma 2.12. *Suppose that $M_1[y] = -(py')' + q_1 y$ satisfies the separation inequality (2.17) with $h = q_1^2$ for any $\epsilon > 0$ on $\mathcal{D}_0$. If also there are constants $K_1, K_2 > 0$ such that $K_1 \leq |q_1/q_2| \leq K_2$ then $M_2[y] = -(py')' + q_2 y$ satisfies the same separation inequality on $\mathcal{D}_0$ with $h = q_2^2$ for sufficiently small $\epsilon > 0$.*

Proof. Since

$$M_2[y] = M_1[y] + q_2\left(1 - \frac{q_1}{q_2}\right)y$$

for $y \in C_0^\infty(I_a)$, we arrive at the inequality

$$\begin{aligned}\epsilon\|M_2[y]\| + \epsilon\left\|q_2\left(1 - \frac{q_1}{q_2}\right)y\right\| + K(\epsilon)\|y\| &\geq \epsilon\|M_1[y]\| + K(\epsilon)\|y\| \\ &\geq \|q_1 y\| \geq K_1\|q_2 y\|\end{aligned}$$

for any $\epsilon > 0$. Hence also

$$\epsilon\|M_2[y]\| + K(\epsilon)\|y\| \geq d_1\|q_2 y\|,$$

where $d_1 = (K_1 - (1 + K_2)\epsilon > 0$ for small enough ϵ. □

Remark 2.13. Taking $q_2 = -q_1$ and $K_1 = K_2 = 1$ in Lemma 2.12, we see that if M_1 satisfies (2.17) then so does M_2 which means that we can have separation for a potential q which is negative and unbounded below provided the expression constructed with potential $|q|$ satisfies (2.17).

Example 2.2. Suppose $p(x) = 1$ and $q_1(x) = \exp(x)$. Then by Theorem A or B M_1 is separated. Let $t_n(x) = \exp(\exp(\cdots\exp(x))\cdots)$ be a n-fold iteration of $\exp(x)$ and set $q_2(x) = \exp(x)(1 + \epsilon\sin(t_n(x))$, $\epsilon > 0$. Then Theorems A and B do not apply because (1.3) and (1.4) are unbounded. However, by Lemma 2.11 M_2 is separated if ϵ is sufficiently small. Clearly $t_n(x)$ can be replaced by any other rapidly increasing function.

Example 2.3. Let $p_1(x) = \exp(x)$ and $q_1(x) = x^{1/3}$ on I_a. By Theorem C M_1 is separated. It is easy to verify that p_1 and q_1 satisfy the Muckenhoupt condition

$$\sup_{x\in I_a}\int_x^\infty p_1^{-1}\int_a^x q_1^\theta < \infty, \quad \theta > 1,$$

and therefore (cf. Opić and Kufner [18, Theorem 6.2]) the Hardy inequality $K\|\exp(x/2)y'\| \geq \|q_1^{\theta/2}y\|$ holds on C_0^∞. Therefore from the second part of Theorem 2.8 we obtain an inequality of the form (2.17). If now $q_2(x) = -q_1(x)(2 + \sin(\exp(x^n))$ we will have from Lemma 2.12 the same kind of inequality but with

$$M_2[y] = -(\exp(x)y')' - x^{1/3}(2 + \sin(\exp(x^n))y.$$

Theorem 2.14. *If $p > 0$, z is a C^1 function, $p'z \leq 0$ and*

$$-K_1 z^2 \leq z' \leq K_2 z^2$$

for positive constants K_1, K_2 then the operators

$$M_{1,c}[y] = -(py')' + q_{1,c}y,$$
$$M_{2,c}[y] = -(py')' + q_{2,c}y,$$

where $q_{1,c} = c^2pz^2 - c(pz)'$ and $q_{2,c} = c^2pz^2$ are separated for sufficiently large $c \geq 1$.

Proof. To prove that $M_{1,c}$ is separated we retrace the proof of Theorem 2.1. Let $L_c(y) = y' + czy$ and $L_c^*(y) = -y' + czy$, where $y \in C_0^\infty(I)$. Then

$$\|L_c^*(y)\|^2 = \int_1^\infty (y')^2 + (cz' + c^2z^2)y^2.$$

If $c \geq K_1$, then $cz' + c^2z^2 \geq 0$ and as before,

$$\|cz\|^2 \leq 4\|L_c^*[y]\|^2.$$

Likewise $L_c^*(pL_c(y)) = M_{1,c}[y]$ and

$$\begin{aligned} q_{1,c} &= -pcz' - p'cz + pc^2z^2 \\ &\geq -pcz' + pc^2z^2 \\ &\geq pcz^2(c - K_2) \\ &\geq 0 \end{aligned}$$

if $c > K_2$. From the definition of $q_{1,c}$ we also have that $(pz)' \leq K_2pz^2$. And so

$$\begin{aligned} \|M_{1,c}[y]\|^2 &\geq \frac{1}{4}\int_{I_a}[(czp)^2(y')^2 + (c^4z^4p^2 - ((cz)^3p^2)'y^2] \\ &\geq \int_{I_a}[c^4z^4p^2 - 3c^3z^2z'p^2]y^2 \\ &\geq \left[1 - \frac{3K_2}{c}\right]\int_{I_a} c^4p^2z^4y^2 \end{aligned}$$

for $c > 3K_2$. Now also

$$\|(pcz)'y\| + \|M_{2,c}[y]\| \geq \|M_{1,c}[y]\| \geq K_3\left\|c^2pz^2y\right\|$$

where $K_3 = \sqrt{1 - 3K_2/c}$, so that

$$\begin{aligned} \|M_{2,c}[y]\| &\geq K_3\left\|c^2pz^2y\right\| - \|(pcz)'y\| \\ &\geq \left(\sqrt{1 - \frac{3K_2}{c}} - \sqrt{\frac{K_2}{c}}\right)\left\|c^2pz^2y\right\|. \end{aligned}$$

Since the constant is positive for large enough c the inequality (2.13) for $M_{2,c}[y]$ is established. Since

$$\begin{aligned} \frac{q_{1,c}}{q_{2,c}} &= (1 - (pz)'(c^2pz^2) \\ &\leq \left(1 + \frac{K_2}{c^2}\right) \\ &\geq \left(1 - \frac{K_2}{c^2}\right), \end{aligned}$$

Lemma 2.11 may be applied to conclude that $M_{1,c}$ is separated and satisfies an inequality like (2.13). □

Example 2.4. If p' is of constant sign, let $z = -\text{sgn}(p')\sqrt{q/p}$ then $p'z \leq 0$ as required and $q_{2,c} = c^2q$. A calculation shows that the hypothesis of Theorem 2.14 becomes

$$-2K_1 \geq \left(\frac{p'}{\sqrt{pq}} - \frac{p^{1/2}q'}{q^{3/2}}\right)\text{sgn}(p') \leq 2K_2.$$

Equivalently we can require that

$$\eta = \sup_{x\in I_a}\left|\frac{p'}{\sqrt{pq}} - \frac{p^{1/2}q'}{q^{3/2}}\right| < \infty$$

to conclude that $M_d[y] = -(py')' + dqy$ is separated for sufficiently large d. For example, if $p(x) = q(x) = \exp(x^2)$ both Theorem A and B fail for any M_d yet $\eta = 0$ and so we have an inequality of the form

$$K\| - (\exp(x^2)y')' + d\exp(x^2)y\| \geq \|d\exp(x^2)y\|$$

for large enough d.

Corollary 2.15. *Let p, z, h, and g be functions such that $p > 0$ and p, z are C^1, $p'z \leq 0$, $z' \leq \delta z^2$ for $\delta \in [0, 1/3)$, $h \geq d > 0$, g is bounded, and*

$$\lim_{x\to\infty} \left| \frac{h(pz)'}{pz^2} \right| = 0, \tag{2.21}$$

then $M_1[y] = -(py')' + q_1 y$, where $q_1 = pz^2 - (pz)'$ is separated on $\mathcal{D}$ and $M_2[y] = -(py')' + q_2$, where $q_2 = pz^2 + hg(pz)'$ is separated on at least on $\mathcal{D}_0$. If we assume additionally that

$$\lim_{x\to\infty} pz^2 = \infty, \tag{2.22}$$

then the inequalities

$$\epsilon \|M_i[y]\| + K(\epsilon)\|y\| \geq \|q_j^\theta y\|$$

hold for $1 \leq i,\ j \leq 2$ and $\theta < 1$ on $\mathcal{D}$ if $i = 1$ and on $\mathcal{D}_0$ if $i = 2$.

Proof. Since

$$\left| \frac{h(pz)'}{pz^2} \right| \geq d \left| \frac{(pz)'}{pz^2} \right|,$$

$(pz)'/pz^2 \to 0$ as $x \to \infty$ which implies that for $I_{a'} = [a', \infty)$ and a' sufficiently large, $-pz^2 \leq (pz)'$. Since the assumptions of Theorem 2.7 are satisfied, $M_1[y]$ is separated on $\mathcal{D}$ relative $I_{a'}$ and by Remark 2.2(ii) also on I_a. Since

$$\lim_{x\to\infty} \frac{q_1}{q_2} = \lim_{x\to\infty} \frac{1 - (pz)'/pz^2}{1 - hg(pz)'/pz^2} = 1$$

the separation of M_2 and M_3 on $\mathcal{D}_0$ follows from Lemma 2.11.

To prove the second claim, a calculation will show that

$$\lim_{x\to\infty} \frac{q_i^2}{q_j^{2\theta}} = \lim_{x\to\infty} (pz^2)^{2(1-\theta)} T(z, p, \theta) = \infty, \quad 1 \leq i, j \leq 2,$$

where $T(z, p, \theta)$ is a term going to 1 as $x \to \infty$. For example,

$$\lim_{x\to\infty} \frac{q_3^2}{q_2^2} = \lim_{x\to\infty} (pz^2)^{2(1-\theta)} \left[\frac{1 + g(pz)'/pz^2}{(1 + hg(pz)'/pz^2)^\theta} \right]^2 = \infty.$$

The inequalities follow from the second part of Theorem 2.8. □

Example 2.5. Set $p(x) = \exp(x/3)$, $z(x) = -\exp(x/3)$, $h(x) = \exp((1 - 3\epsilon)x/3)$, and $g(x) = -\sin(t_n(x))$, where $t_n(x)$ is as in Example 2.3. Then

$$p'z = -\frac{1}{3} \exp\left(\frac{2x}{3}\right) \leq 0, \quad z' = -\frac{1}{3} \exp\left(\frac{x}{3}\right) \leq \delta \exp\left(\frac{2x}{3}\right) = z^2,$$

and (2.21) holds. Then

$$M[y] := -\left(\exp\left(\frac{x}{3}\right) y'\right)' + \exp(x) \left[1 + \frac{2}{3} \exp(-\epsilon x) \sin(t_n(x))\right] y$$

is separated on $\mathcal{D}_0$. Since M is LP at ∞ the separation actually holds on $\mathcal{D}$.

The final result of this paper is quite different from Theorem 2.1 but it reinforces the connection between disconjugacy and separation. In addition, the proof is quite elementary.

Theorem 2.16. *Let $p > 0$ and $q \geq d > 0$ be continuous. Suppose that $M^\lambda[y] = -(py')' + (q - \lambda q^2)y$ is disconjugate on I_a for some $\lambda > 0$. Then $M[y] = -(py')' + qy$ is separated.*

Proof. It is well known (see e.g. [15, Theorem 6.2]) that the disconjugacy of M^λ is equivalent to the positive definiteness of the functional $Q^\lambda[y] = \int_{I_a} (p|y'|^2 + (q - \lambda q^2)|y|^2)$ for $y \in C_0^\infty(I_a)$. In other words, we must have the inequality

$$Q^0[y] \equiv \int_{I_a} \left(p|y'|^2 + qy^2\right) \geq \lambda \int_{I_a} q^2|y|^2 \tag{2.23}$$

with equality holding if and only if $y = 0$. Now consider the expression $M_{q^2} = q^{-2}[-py')'+qy]$, where y is an appropriate function in $L^2(q^2; I_a)$. If $y \in C_0^\infty(I_a)$ then the Cauchy-Schwartz inequality and (2.23) yields that

$$\|M_{q^2}[y]\|_{q^2}\|y\|_{q^2} \geq Q^0[y] \geq \lambda\|y\|_{q^2}^2 \equiv \lambda\|qy\|^2.$$

It follows that the inequality

$$d^{-2}\|M[y]\| \geq \|M_{q^2}[y]\|_{q^2} \geq \lambda\|qy\|$$

holds on the C_0^∞ functions and also therefore on $\mathcal{D}_0$. Because M is LP at ∞ we again conclude that it is separated on $\mathcal{D}$. □

Remark 2.17. (i) If $p^{-1} \in L(I_a)$ and

$$\sup_{x \in I_a} \left(\int_x^\infty p^{-1}\right)\left(\int_a^x \lambda q^2 - q\right) < \frac{1}{4} \tag{2.24}$$

then (cf. Example 2.4) the Hardy inequality $\left\|p^{1/2}y'\right\|^2 \geq \left\|(\lambda q^2 - q)^{1/2}y\right\|$ holds on $C_0^\infty(I_a)$ with equality if and only if $y = 0$. This inequality implies the positive definiteness of Q^λ.

(ii) If $w \geq d >$ is a weight and we require that $(-py')' + (q - \lambda w)$ be disconjugate, then the proof of Theorem 2.16 will yield the inequality

$$d^{-1}\|M[y]\| \geq \lambda\left\|\sqrt{w}y\right\|. \tag{2.25}$$

Substituting w for q^2 in (2.24) will give a sufficient condition for (2.25).

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