journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 8 (2007), Issue 3, Article 74, 5 pp.

ON AN OPEN PROBLEM CONCERNING AN INTEGRAL INEQUALITY

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Received 09 December, 2006; accepted 11 August, 2007
Communicated by P.S. Bullen

ABSTRACT. In this note, we generalize an open problem posed by Q. A. Ngô in the paper, Notes on an integral inequality, *J. Inequal. Pure & Appl. Math.*, **7**(4) (2006), Art. 120 and give a positive answer to it using an analytic approach.

Key words and phrases: Integral inequality, Cauchy inequality.

2000 *Mathematics Subject Classification.* 26D15.

1. INTRODUCTION

In the paper [2], Q.A. Ngô studied a very interesting integral inequality and proved the following result.

Theorem 1.1. *Let $f(x) \geq 0$ be a continuous function on $[0,1]$ satisfying*

$$\int_x^1 f(t)dt \geq \int_x^1 t\,dt, \quad \forall\, x \in [0,1]. \tag{1.1}$$

Then the inequalities

$$\int_0^1 f^{\alpha+1}(x)dx \geq \int_0^1 x^\alpha f(x)dx \tag{1.2}$$

The first author was supported by the Science Research Foundation of NUIST and the Natural Science Foundation of Jiangsu Province Education Department under Grant No.07KJD510133, and the third author was supported by Youth Natural Science Foundation of Zhengzhou Institute of Aeronautical Industry Management under Grant No. Q05K066.

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and

$$\int_0^1 f^{\alpha+1}(x)dx \geq \int_0^1 x f^{\alpha}(x)dx \tag{1.3}$$

hold for every positive real number $\alpha > 0$.

Next, they proposed the following open problem.

Problem 1.1. Let $f(x)$ be a continuous function on $[0,1]$ satisfying

$$\int_x^1 f(t)dt \geq \int_x^1 t\,dt, \quad \forall\, x \in [0,1]. \tag{1.4}$$

Under what conditions does the inequality

$$\int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^{\alpha} f^{\beta}(x)dx, \tag{1.5}$$

hold for α and β?

We note that, as an open problem, the condition (1.4) maybe result in an unreasonable restriction on $f(x)$. We remove it herein and propose another more general open problem.

Problem 1.2. Under what conditions does the inequality

$$\int_0^b f^{\alpha+\beta}(x)dx \geq \int_0^b x^{\alpha} f^{\beta}(x)dx, \tag{1.6}$$

hold for $b,\ \alpha$ and β?

In this note, we give an answer to Problem 1.2 using an analytic approach. Our main results are Theorem 2.1 and Theorem 2.4 which will be proved in Section 2.

2. MAIN RESULTS AND PROOFS

Firstly, we have

Theorem 2.1. *Let* $f(x) \geq 0$ *be a continuous function on* $[0,1]$ *satisfying*

$$\int_x^1 f^{\beta}(t)dt \geq \int_x^1 t^{\beta}\,dt, \qquad \forall\, x \in [0,1]. \tag{2.1}$$

Then the inequality

$$\int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^{\alpha} f^{\beta}(x)dx, \tag{2.2}$$

holds for every positive real number $\alpha > 0$ *and* $\beta > 0$.

To prove Theorem 2.1, we need the following lemmas.

Lemma 2.2 (General Cauchy inequality, [2]). *Let* α *and* β *be positive real numbers satisfying* $\alpha + \beta = 1$. *Then for all positive real numbers* x *and* y, *we have*

$$\alpha x + \beta y \geq x^{\alpha} y^{\beta}. \tag{2.3}$$

Lemma 2.3. *Under the conditions of Theorem 2.1, we have*

$$\int_0^1 x^{\alpha} f^{\beta}(x)dx \geq \frac{1}{\alpha+\beta+1}. \tag{2.4}$$

Proof. Integrating by parts, we have

$$\begin{aligned}(2.5)\qquad \int_0^1 x^{\alpha-1}\left(\int_x^1 f^\beta(t)dt\right)dx &= \frac{1}{\alpha}\int_0^1\left(\int_x^1 f^\beta(t)dt\right)d(x^\alpha)\\ &= \frac{1}{\alpha}\left[x^\alpha\int_x^1 f^\beta(t)dt\right]_{x=0}^{x=1}+\frac{1}{\alpha}\int_0^1 x^\alpha f^\beta(x)dx\\ &= \frac{1}{\alpha}\int_0^1 x^\alpha f^\beta(x)dx,\end{aligned}$$

which yields

$$(2.6)\qquad \int_0^1 x^\alpha f^\beta(x)dx = \alpha\int_0^1 x^{\alpha-1}\left(\int_x^1 f^\beta(t)dt\right)dx.$$

On the other hand, by (2.1), we get

$$\begin{aligned}(2.7)\qquad \int_0^1 x^{\alpha-1}\left(\int_x^1 f^\beta(t)dt\right)dx &\geq \int_0^1 x^{\alpha-1}\left(\int_x^1 t^\beta\,dt\right)dx\\ &= \frac{1}{\beta+1}\int_0^1 (x^{\alpha-1}-x^{\alpha+\beta})dx\\ &= \frac{1}{\alpha(\alpha+\beta+1)}.\end{aligned}$$

Therefore, (2.4) holds. □

We now give the proof of Theorem 2.1.

Proof of Theorem 2.1. Using Lemma 2.2, we obtain

$$(2.8)\qquad \frac{\beta}{\alpha+\beta}f^{\alpha+\beta}(x)+\frac{\alpha}{\alpha+\beta}x^{\alpha+\beta}\geq x^\alpha f^\beta(x),$$

which gives

$$(2.9)\qquad \beta\int_0^1 f^{\alpha+\beta}(x)dx+\alpha\int_0^1 x^{\alpha+\beta}dx\geq(\alpha+\beta)\int_0^1 x^\alpha f^\beta(x)dx.$$

Moreover, by using Lemma 2.3, we get

$$\begin{aligned}(2.10)\qquad (\alpha+\beta)\int_0^1 x^\alpha f^\beta(x)dx &= \alpha\int_0^1 x^\alpha f^\beta(x)dx+\beta\int_0^1 x^\alpha f^\beta(x)dx\\ &\geq \frac{\alpha}{\alpha+\beta+1}+\beta\int_0^1 x^\alpha f^\beta(x)dx,\end{aligned}$$

that is

$$(2.11)\qquad \beta\int_0^1 f^{\alpha+\beta}(x)dx+\frac{\alpha}{\alpha+\beta+1}\geq\frac{\alpha}{\alpha+\beta+1}+\beta\int_0^1 x^\alpha f^\beta(x)dx,$$

which completes this proof. □

Lastly, we generalize our result.

Theorem 2.4. *Let $f(x)\geq 0$ be a continuous function on $[0,b], b\geq 0$ satisfying*

$$(2.12)\qquad \int_x^b f^\beta(t)dt\geq\int_x^b t^\beta\,dt,\quad \forall\, x\in[0,b].$$

Then the inequality

$$\int_0^b f^{\alpha+\beta}(x)dx \geq \int_0^b x^\alpha f^\beta(x)dx \tag{2.13}$$

hold for every positive real number $\alpha > 0$ *and* $\beta > 0$.

To prove Theorem 2.4, we need the following lemma.

Lemma 2.5. *Under the conditions of Theorem 2.4, we have*

$$\int_0^b x^\alpha f^\beta(x)dx \geq \frac{b^{\alpha+\beta+1}}{\alpha+\beta+1}. \tag{2.14}$$

Proof. Integrating by parts, we have

$$\begin{aligned} \int_0^b x^{\alpha-1}\left(\int_x^b f^\beta(t)dt\right)dx &= \frac{1}{\alpha}\int_0^b\left(\int_x^b f^\beta(t)dt\right)d(x^\alpha) \\ &= \frac{1}{\alpha}\left[x^\alpha\int_x^b f^\beta(t)dt\right]_{x=0}^{x=b} + \frac{1}{\alpha}\int_0^b x^\alpha f^\beta(x)dx \\ &= \frac{1}{\alpha}\int_0^b x^\alpha f^\beta(x)dx, \end{aligned} \tag{2.15}$$

which yields

$$\int_0^b x^\alpha f^\beta(x)dx = \alpha\int_0^b x^{\alpha-1}\left(\int_x^b f^\beta(t)dt\right)dx. \tag{2.16}$$

On the other hand, by (2.12), we get

$$\begin{aligned} \int_0^b x^{\alpha-1}\left(\int_x^b f^\beta(t)dt\right)dx &\geq \int_0^b x^{\alpha-1}\left(\int_x^b t^\beta\,dt\right)dx \\ &= \frac{1}{\beta+1}\int_0^b x^{\alpha-1}(b^{\beta+1}-x^{\beta+1})dx \\ &= \frac{b^{\alpha+\beta+1}}{\alpha(\alpha+\beta+1)}. \end{aligned} \tag{2.17}$$

Therefore, (2.14) holds. □

We now give the proof of Theorem 2.4.

Proof of Theorem 2.4. Using Lemma 2.2, we obtain

$$\beta\int_0^b f^{\alpha+\beta}(x)dx + \alpha\int_0^b x^{\alpha+\beta}dx \geq (\alpha+\beta)\int_0^b x^\alpha f^\beta(x)dx. \tag{2.18}$$

Moreover, by using Lemma 2.5, we get

$$\begin{aligned} (\alpha+\beta)\int_0^b x^\alpha f^\beta(x)dx &= \alpha\int_0^b x^\alpha f^\beta(x)dx + \beta\int_0^b x^\alpha f^\beta(x)dx \\ &\geq \alpha\frac{b^{\alpha+\beta+1}}{\alpha+\beta+1} + \beta\int_0^b x^\alpha f^\beta(x)dx, \end{aligned} \tag{2.19}$$

that is

$$\beta\int_0^b f^{\alpha+\beta}(x)dx + \alpha\frac{b^{\alpha+\beta+1}}{\alpha+\beta+1} \geq \alpha\frac{b^{\alpha+\beta+1}}{\alpha+\beta+1} + \beta\int_0^b x^\alpha f^\beta(x)dx, \tag{2.20}$$

which completes the proof. □

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