Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 7, Issue 2, Article 53, 2006

ON A GENERALIZED $n-$INNER PRODUCT AND THE CORRESPONDING CAUCHY-SCHWARZ INEQUALITY

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Received 22 November, 2004; accepted 27 February, 2006
Communicated by C.P. Niculescu

ABSTRACT. In this paper is defined an n-inner product of type $\langle \mathbf{a}_1,\dots,\mathbf{a}_n|\mathbf{b}_1\cdots\mathbf{b}_n\rangle$ where $\mathbf{a}_1,\dots,\mathbf{a}_n, \mathbf{b}_1,\dots,\mathbf{b}_n$ are vectors from a vector space V. This definition generalizes the definition of Misiak of n-inner product [5], such that in special case if we consider only such pairs of sets $\{\mathbf{a}_1,\dots,\mathbf{a}_1\}$ and $\{\mathbf{b}_1\cdots\mathbf{b}_n\}$ which differ for at most one vector, we obtain the definition of Misiak. The Cauchy-Schwarz inequality for this general type of n-inner product is proved and some applications are given.

Key words and phrases: Cauchy-Schwarz inequality, n-inner product, n-norm.

2000 *Mathematics Subject Classification.* 46C05, 26D20.

1. INTRODUCTION

A. Misiak [5] has introduced an n-inner product by the following definition.

Definition 1.1. Assume that n is a positive integer and V is a real vector space such that $\dim V \geq n$ and $(\bullet,\bullet|\underbrace{\bullet,\dots,\bullet}_{n-1})$ is a real function defined on $\underbrace{V\times V\times\cdots\times V}_{n+1}$ such that:

i) $(\mathbf{x}_1,\mathbf{x}_1|\mathbf{x}_2,\dots,\mathbf{x}_n)\geq 0$, for any $\mathbf{x}_1,\mathbf{x}_2,\dots,\mathbf{x}_n\in V$ and $(\mathbf{x}_1,\mathbf{x}_1|\mathbf{x}_2,\dots,\mathbf{x}_n)=0$ if and only if $\mathbf{x}_1,\mathbf{x}_2,\dots,\mathbf{x}_n$ are linearly dependent vectors;

ii) $(\mathbf{a},\mathbf{b}|\mathbf{x}_1,\dots,\mathbf{x}_{n-1}) = (\varphi(\mathbf{a}),\varphi(\mathbf{b})|\pi(\mathbf{x}_1),\dots,\pi(\mathbf{x}_{n-1}))$, for any $\mathbf{a},\mathbf{b},\mathbf{x}_1,\dots,\mathbf{x}_{n-1}\in V$ and for any bijections

$$\pi:\{\mathbf{x}_1,\dots,\mathbf{x}_{n-1}\}\to\{\mathbf{x}_1,\dots,\mathbf{x}_{n-1}\} \quad \text{and} \quad \varphi:\{\mathbf{a},\mathbf{b}\}\to\{\mathbf{a},\mathbf{b}\};$$

ISSN (electronic): 1443-5756

219-04

iii) If $n > 1$, then $(\mathbf{x}_1, \mathbf{x}_1 | \mathbf{x}_2, \dots, \mathbf{x}_n) = (\mathbf{x}_2, \mathbf{x}_2 | \mathbf{x}_1, \mathbf{x}_3, \dots, \mathbf{x}_n)$, for any $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n \in V$;

iv) $(\alpha\mathbf{a}, \mathbf{b} | \mathbf{x}_1, \dots, \mathbf{x}_{n-1}) = \alpha(\mathbf{a}, \mathbf{b} | \mathbf{x}_1, \dots, \mathbf{x}_{n-1})$, for any $\mathbf{a}, \mathbf{b}, \mathbf{x}_1, \dots, \mathbf{x}_{n-1} \in V$ and any scalar $\alpha \in \mathbb{R}$;

v) $(\mathbf{a}+\mathbf{a}_1, \mathbf{b} | \mathbf{x}_1, \dots, \mathbf{x}_{n-1}) = (\mathbf{a}, \mathbf{b} | \mathbf{x}_1, \dots, \mathbf{x}_{n-1}) + (\mathbf{a}_1, \mathbf{b} | \mathbf{x}_1, \dots, \mathbf{x}_{n-1})$, for any $\mathbf{a}, \mathbf{b}, \mathbf{a}_1, \mathbf{x}_1, \dots, \mathbf{x}_{n-1} \in V$.

Then $(\bullet, \bullet | \underbrace{\bullet, \dots, \bullet}_{n-1})$ is called the n-inner product and $(V, (\bullet, \bullet | \underbrace{\bullet, \dots, \bullet}_{n-1}))$ is called the n-prehilbert space.

If $n = 1$, then Definition 1.1 reduces to the ordinary inner product.

This n-inner product induces an n-norm ([5]) by

$$\|\mathbf{x}_1, \dots, \mathbf{x}_n\| = \sqrt{(\mathbf{x}_1, \mathbf{x}_1 | \mathbf{x}_2, \dots, \mathbf{x}_n)}.$$

In the next section we introduce a more general and more convenient definition of n-inner product and prove the corresponding Cauchy-Schwarz inequality. In the last section some related results are given.

Although in this paper we only consider real vector spaces, the results of this paper can easily be generalized for the complex vector spaces.

2. n-INNER PRODUCT AND THE CAUCHY-SCHWARZ INEQUALITY

First we give the following definition of n-inner products.

Definition 2.1. Assume that n is a positive integer, V is a real vector space such that $\dim V \geq n$ and $\langle \bullet, \dots, \bullet | \bullet, \dots, \bullet \rangle$ is a real function on V^{2n} such that

i)

$$\langle \mathbf{a}_1, \dots, \mathbf{a}_n | \mathbf{a}_1, \dots, \mathbf{a}_n \rangle > 0 \tag{2.1}$$

if $\mathbf{a}_1, \dots, \mathbf{a}_n$ are linearly independent vectors,

ii)

$$\langle \mathbf{a}_1, \dots, \mathbf{a}_n | \mathbf{b}_1, \dots, \mathbf{b}_n \rangle = \langle \mathbf{b}_1, \dots, \mathbf{b}_n | \mathbf{a}_1, \dots, \mathbf{a}_n \rangle \tag{2.2}$$

for any $\mathbf{a}_1, \dots, \mathbf{a}_n, \mathbf{b}_1, \dots, \mathbf{b}_n \in V$,

iii)

$$\langle \lambda\mathbf{a}_1, \dots, \mathbf{a}_n | \mathbf{b}_1, \dots, \mathbf{b}_n \rangle = \lambda \langle \mathbf{a}_1, \dots, \mathbf{a}_n | \mathbf{b}_1, \dots, \mathbf{b}_n \rangle \tag{2.3}$$

for any scalar $\lambda \in \mathbb{R}$ and any $\mathbf{a}_1, \dots, \mathbf{a}_n, \mathbf{b}_1, \dots, \mathbf{b}_n \in V$,

iv)

$$\langle \mathbf{a}_1, \dots, \mathbf{a}_n | \mathbf{b}_1, \dots, \mathbf{b}_n \rangle = -\langle \mathbf{a}_{\sigma(1)}, \dots, \mathbf{a}_{\sigma(n)} | \mathbf{b}_1, \dots, \mathbf{b}_n \rangle \tag{2.4}$$

for any odd permutation σ in the set $\{1, \dots, n\}$ and any $\mathbf{a}_1, \dots, \mathbf{a}_n, \mathbf{b}_1, \dots, \mathbf{b}_n \in V$,

v)

$$\begin{aligned} &\langle \mathbf{a}_1 + \mathbf{c}, \mathbf{a}_2, \dots, \mathbf{a}_n | \mathbf{b}_1, \dots, \mathbf{b}_n \rangle \\ &\qquad = \langle \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n | \mathbf{b}_1, \dots, \mathbf{b}_n \rangle + \langle \mathbf{c}, \mathbf{a}_2, \dots, \mathbf{a}_n | \mathbf{b}_1, \dots, \mathbf{b}_n \rangle \end{aligned} \tag{2.5}$$

for any $\mathbf{a}_1, \dots, \mathbf{a}_n, \mathbf{b}_1, \dots, \mathbf{b}_n, \mathbf{c} \in V$,

vi) if

$$\langle \mathbf{a}_1, \mathbf{b}_1, \dots, \mathbf{b}_{i-1}, \mathbf{b}_{i+1}, \dots, \mathbf{b}_n | \mathbf{b}_1, \dots, \mathbf{b}_n \rangle = 0 \tag{2.6}$$

for each $i \in \{1, 2, \dots, n\}$, then

$$\langle \mathbf{a}_1, \dots, \mathbf{a}_n | \mathbf{b}_1, \dots, \mathbf{b}_n \rangle = 0 \tag{2.7}$$

for arbitrary vectors $\mathbf{a}_2, \dots, \mathbf{a}_n$.

Then the function $\langle \bullet, \dots, \bullet | \bullet, \dots, \bullet \rangle$ is called an n-inner product and the pair $(V, \langle \bullet, \dots, \bullet | \bullet, \dots, \bullet \rangle)$ is called an n-prehilbert space.

We give some consequences from the conditions i) – vi) of Definition 2.1.

From (2.4) it follows that if two of the vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$ are equal, then $\langle \mathbf{a}_1, \dots, \mathbf{a}_n | \mathbf{b}_1, \dots, \mathbf{b}_n \rangle = 0$.

From (2.3) it follows that

$$\langle \mathbf{a}_1, \dots, \mathbf{a}_n | \mathbf{b}_1, \dots, \mathbf{b}_n \rangle = 0$$

if there exists i such that $\mathbf{a}_i = 0$.

From (2.4) and (2.2) it follows more generally that

iv')

$$\langle \mathbf{a}_1, \dots, \mathbf{a}_n | \mathbf{b}_1, \dots, \mathbf{b}_n \rangle = (-1)^{\operatorname{sgn}(\pi)+\operatorname{sgn}(\tau)} \langle \mathbf{a}_{\pi(1)}, \dots, \mathbf{a}_{\pi(n)} | \mathbf{b}_{\tau(1)}, \dots, \mathbf{b}_{\tau(n)} \rangle$$

for any permutations π and τ on $\{1, \dots, n\}$ and $\mathbf{a}_1, \dots, \mathbf{a}_n, \mathbf{b}_1, \dots, \mathbf{b}_n \in V$.

From (2.3), (2.4) and (2.5) it follows that

$$\langle \mathbf{a}_1, \dots, \mathbf{a}_n | \mathbf{b}_1, \dots, \mathbf{b}_n \rangle = 0$$

if $\mathbf{a}_1, \dots, \mathbf{a}_n$ are linearly dependent vectors. Thus i) can be replaced by

i') $\langle \mathbf{a}_1, \dots, \mathbf{a}_n | \mathbf{a}_1, \dots, \mathbf{a}_n \rangle \geq 0$ for any $\mathbf{a}_1, \dots, \mathbf{a}_n \in V$ and $\langle \mathbf{a}_1, \dots, \mathbf{a}_n | \mathbf{a}_1, \dots, \mathbf{a}_n \rangle = 0$ if and only if $\mathbf{a}_1, \dots, \mathbf{a}_n$ are linearly dependent vectors.

Note that the n-inner product on V induces an n-normed space by

$$\|\mathbf{x}_1, \dots, \mathbf{x}_n\| = \sqrt{\langle \mathbf{x}_1, \dots, \mathbf{x}_n | \mathbf{x}_1, \dots, \mathbf{x}_n \rangle},$$

and it is the same norm induced by Definition 1.1.

In the special case if we consider only such pairs of sets $\mathbf{a}_1, \dots, \mathbf{a}_1$ and $\mathbf{b}_1, \dots, \mathbf{b}_n$ which differ for at most one vector, for example $\mathbf{a}_1 = \mathbf{a}$, $\mathbf{b}_1 = \mathbf{b}$ and $\mathbf{a}_2 = \mathbf{b}_2 = \mathbf{x}_1, \dots, \mathbf{a}_n = \mathbf{b}_n = \mathbf{x}_{n-1}$, then by putting

$$(\mathbf{a}, \mathbf{b} | \mathbf{x}_1, \dots, \mathbf{x}_{n-1}) = \langle \mathbf{a}, \mathbf{x}_1, \dots, \mathbf{x}_{n-1} | \mathbf{b}, \mathbf{x}_1, \dots, \mathbf{x}_{n-1} \rangle$$

we obtain an n-inner product according to Definition 1.1 of Misiak. Indeed, the conditions i), iv) and v) are triavially satisfied. The condition ii) is satisfied for an arbitrary permutation π, because according to iv$'$)

$$\langle \mathbf{a}_1, \dots, \mathbf{a}_n | \mathbf{b}_1, \dots, \mathbf{b}_n \rangle = \langle \mathbf{a}_1, \mathbf{a}_{\pi(2)}, \dots, \mathbf{a}_{\pi(n)} | \mathbf{b}_1, \mathbf{b}_{\pi(2)}, \dots, \mathbf{b}_{\pi(n)} \rangle$$

for any permutation $\pi : \{2, 3, \dots, n\} \to \{2, 3, \dots, n\}$. Similarly the condition iii) is satisfied. Moreover, in this special case of Definition 2.1 we do not have any restriction of Definition 1.1. For example, then the condition vi) does not say anything. Namely, if $\mathbf{a}_1 \notin \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$, then the vectors $\mathbf{a}_2, \dots, \mathbf{a}_n$ must be from the set $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$, and $(2.6'')$ is satisfied because the assumption $(2.6')$ is satisfied. If $\mathbf{a}_1 \in \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$, for example $\mathbf{a}_1 = \mathbf{b}_j$, then $(2.6')$ implies that $\langle \mathbf{b}_j, \mathbf{b}_1, \dots, \mathbf{b}_{j-1}, \mathbf{b}_{j+1}, \dots, \mathbf{b}_n | \mathbf{b}_1, \dots, \mathbf{b}_n \rangle = 0$, and it is possible only if $\mathbf{b}_1, \dots, \mathbf{b}_n$ are linearly dependent vectors. However, then $(2.6'')$ is satisfied. Thus Definition 2.1 generalizes Definition 1.1.

Now we give the following example of the n-inner product.

Example 2.1. We refer to the classical known example, as an n-inner product according to Definition 2.1. Let V be a space with inner product $\langle\cdot|\cdot\rangle$. Then

$$\langle \mathbf{a}_1,\dots,\mathbf{a}_n|\mathbf{b}_1,\dots,\mathbf{b}_n\rangle = \begin{vmatrix} \langle\mathbf{a}_1|\mathbf{b}_1\rangle & \langle\mathbf{a}_1|\mathbf{b}_2\rangle & \cdots & \langle\mathbf{a}_1|\mathbf{b}_n\rangle \\ \langle\mathbf{a}_2|\mathbf{b}_1\rangle & \langle\mathbf{a}_2|\mathbf{b}_2\rangle & \cdots & \langle\mathbf{a}_2|\mathbf{b}_n\rangle \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \langle\mathbf{a}_n|\mathbf{b}_1\rangle & \langle\mathbf{a}_n|\mathbf{b}_2\rangle & \cdots & \langle\mathbf{a}_n|\mathbf{b}_n\rangle \end{vmatrix}$$

satisfies the conditions i) - vi) and hence it defines an n-inner product on V. The conditions i) - v) are trivial, and we will prove vi). If $\mathbf{b}_1,\dots,\mathbf{b}_n$ are linearly independent vectors and

$$\begin{aligned}\langle \mathbf{a}_1,\mathbf{b}_1,\dots,\mathbf{b}_{i-1},\mathbf{b}_{i+1},\dots,\mathbf{b}_n|\mathbf{b}_1,\dots,\mathbf{b}_n\rangle \\ \equiv (-1)^{i-1}\langle \mathbf{b}_1,\dots,\mathbf{b}_{i-1},\mathbf{a}_1,\mathbf{b}_{i+1},\dots,\mathbf{b}_n|\mathbf{b}_1,\dots,\mathbf{b}_n\rangle \\ \equiv (-1)^{i-1}\begin{vmatrix} \langle\mathbf{b}_1|\mathbf{b}_1\rangle & \langle\mathbf{b}_1|\mathbf{b}_2\rangle & \cdots & \langle\mathbf{b}_1|\mathbf{b}_n\rangle \\ \langle\mathbf{b}_2|\mathbf{b}_1\rangle & \langle\mathbf{b}_2|\mathbf{b}_2\rangle & \cdots & \langle\mathbf{b}_2|\mathbf{b}_n\rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle\mathbf{a}_1|\mathbf{b}_1\rangle & \langle\mathbf{a}_1|\mathbf{b}_2\rangle & \cdots & \langle\mathbf{a}_1|\mathbf{b}_n\rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle\mathbf{b}_n|\mathbf{b}_1\rangle & \langle\mathbf{b}_n|\mathbf{b}_2\rangle & \cdots & \langle\mathbf{b}_n|\mathbf{b}_n\rangle \end{vmatrix} = 0,\end{aligned}$$

then the vector

$$(\langle\mathbf{a}_1|\mathbf{b}_1\rangle,\langle\mathbf{a}_1|\mathbf{b}_2\rangle,\dots,\langle\mathbf{a}_1|\mathbf{b}_n\rangle)\in\mathbb{R}^n$$

is a linear combination of

$$\begin{aligned}&(\langle\mathbf{b}_1|\mathbf{b}_1\rangle,\dots,\langle\mathbf{b}_1|\mathbf{b}_n\rangle),\dots,(\langle\mathbf{b}_{i-1}|\mathbf{b}_1\rangle,\dots,\langle\mathbf{b}_{i-1}|\mathbf{b}_n\rangle),\\ &(\langle\mathbf{b}_{i+1}|\mathbf{b}_1\rangle,\dots,\langle\mathbf{b}_{i+1}|\mathbf{b}_n\rangle),\dots,(\langle\mathbf{b}_n|\mathbf{b}_1\rangle,\dots,\langle\mathbf{b}_n|\mathbf{b}_n\rangle).\end{aligned}$$

Since this is true for each $i\in\{1,2,\dots,n\}$, it must be that $\langle\mathbf{a}_1|\mathbf{b}_1\rangle=\cdots=\langle\mathbf{a}_1|\mathbf{b}_n\rangle=0$. Hence

$$\langle \mathbf{a}_1,\dots,\mathbf{a}_n|\mathbf{b}_1,\dots,\mathbf{b}_n\rangle = 0$$

for arbitrary $\mathbf{a}_2,\dots,\mathbf{a}_n$.

Note that the inner product defined by

$$\langle \mathbf{a}_1\wedge\cdots\wedge\mathbf{a}_n|\mathbf{b}_1\wedge\cdots\wedge\mathbf{b}_n\rangle = \begin{vmatrix} \langle\mathbf{a}_1|\mathbf{b}_1\rangle & \langle\mathbf{a}_1|\mathbf{b}_2\rangle & \cdots & \langle\mathbf{a}_1|\mathbf{b}_n\rangle \\ \langle\mathbf{a}_2|\mathbf{b}_1\rangle & \langle\mathbf{a}_2|\mathbf{b}_2\rangle & \cdots & \langle\mathbf{a}_2|\mathbf{b}_n\rangle \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \langle\mathbf{a}_n|\mathbf{b}_1\rangle & \langle\mathbf{a}_n|\mathbf{b}_2\rangle & \cdots & \langle\mathbf{a}_n|\mathbf{b}_n\rangle \end{vmatrix}$$

can uniquely be extended to ordinary inner products over the space $\Lambda_n(V)$ of n-forms over V [4]. Indeed, if $\{\mathbf{e}_i\}_{i\in I}$, I an index set, is an orthonormal basis of $(V,\langle *|*\rangle)$, then

$$\langle \mathbf{e}_{i_1}\wedge\cdots\wedge\mathbf{e}_{i_n}|\mathbf{e}_{j_1}\wedge\cdots\wedge\mathbf{e}_{j_n}\rangle = \delta^{i_1\cdots i_n}_{j_1\cdots j_n}$$

where the expression $\delta^{i_1\cdots i_n}_{j_1\cdots j_n}$ is equal to 1 or -1 if $\{i_1,\dots,i_n\}=\{j_1,\dots,j_n\}$ with different $i_1,\dots,i_n$ and additionally the permutation $\binom{i_1\ i_2\ \cdots\ i_n}{j_1\ j_2\ \cdots\ j_n}$ is even or odd respectively, and where the above expression is 0 otherwise. It implies an inner product over $\Lambda_n(V)$.

Before we prove the next theorem, we give the following remarks assuming that $\dim V > n$.

Let $\mathbf{b}_1,\dots,\mathbf{b}_n$ be linearly independent vectors. If a vector $\mathbf{a}$ is such that

$$\langle \mathbf{a},\mathbf{b}_1,\dots,\mathbf{b}_{i-1},\mathbf{b}_{i+1},\dots,\mathbf{b}_n|\mathbf{b}_1,\dots,\mathbf{b}_n\rangle = 0,\quad (1\le i\le n)$$

then we say that the vector $\mathbf{a}$ is orthogonal to the subspace generated by $\mathbf{b}_1, \ldots, \mathbf{b}_n$. Note that the set of orthogonal vectors to this n-dimensional subspace is a vector subspace of V, and the orthogonality of $\mathbf{a}$ to the considered vector subspace is invariant of the base vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n$. If $\mathbf{x}$ is an arbitrary vector, then there exist unique $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $\mathbf{x} - \lambda_1 \mathbf{b}_1 - \cdots - \lambda_n \mathbf{b}_n$ is orthogonal to the vector subspace generated by $\mathbf{b}_1, \ldots, \mathbf{b}_n$. Namely, the orthogonality conditions

$$\langle \mathbf{b}_1, \ldots, \mathbf{b}_{i-1}, \mathbf{x} - \lambda_1 \mathbf{b}_1 - \cdots - \lambda_n \mathbf{b}_n, \mathbf{b}_{i+1}, \ldots, \mathbf{b}_n | \mathbf{b}_1, \ldots, \mathbf{b}_n \rangle = 0, \quad (1 \le i \le n)$$

have unique solutions

$$\lambda_i = \frac{\langle \mathbf{b}_1, \ldots, \mathbf{b}_{i-1}, \mathbf{x}, \mathbf{b}_{i+1}, \ldots, \mathbf{b}_n | \mathbf{b}_1, \ldots, \mathbf{b}_n \rangle}{\langle \mathbf{b}_1, \ldots, \mathbf{b}_n | \mathbf{b}_1, \ldots, \mathbf{b}_n \rangle}, \quad (1 \le i \le n).$$

Hence each vector $\mathbf{x}$ can uniquely be decomposed as $\mathbf{x} = \lambda_1 \mathbf{b}_1 + \cdots + \lambda_n \mathbf{b}_n + \mathbf{c}$, where the vector $\mathbf{c}$ is orthogonal to the vector subspace generated by $\mathbf{b}_1, \ldots, \mathbf{b}_n$. According to this definition, the condition vi) of Definition 2.1 says that if the vector $\mathbf{a}_1$ is orthogonal to the vector subspace generated by $\mathbf{b}_1, \ldots, \mathbf{b}_n$, then $(2.6'')$ holds for arbitrary vectors $\mathbf{a}_2, \ldots, \mathbf{a}_n$.

Now we prove the Cauchy-Schwarz inequality as a consequence of Definition 2.1.

Theorem 2.1. *If $\langle \bullet, \ldots, \bullet | \bullet, \ldots, \bullet \rangle$ is an n-inner product on V, then the following inequality*

$$\langle \mathbf{a}_1, \ldots, \mathbf{a}_n | \mathbf{b}_1, \ldots, \mathbf{b}_n \rangle^2 \le \langle \mathbf{a}_1, \ldots, \mathbf{a}_n | \mathbf{a}_1, \ldots, \mathbf{a}_n \rangle \langle \mathbf{b}_1, \ldots, \mathbf{b}_n | \mathbf{b}_1, \ldots, \mathbf{b}_n \rangle, \tag{2.8}$$

is true for any vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$, $\mathbf{b}_1, \ldots, \mathbf{b}_n \in V$. Moreover, equality holds if and only if at least one of the following conditions is satisfied

i) *the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are linearly dependent,*
ii) *the vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ are linearly dependent,*
iii) *the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ and $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ generate the same vector subspace of dimension n.*

Proof. If $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are linearly dependent vectors or $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are linearly dependent vectors, then both sides of (2.8) are zero and hence equality holds. Thus, suppose that $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and also $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are linearly independent vectors. Note that the inequality (2.8) does not depend on the choice of the basis $\mathbf{a}_1, \ldots, \mathbf{a}_n$ of the subspace generated by these n vectors. Indeed, each vector row operation preserves the inequality (2.8), because both sides are invariant or both sides are multiplied by a positive real scalar after any elementary vector row operation. We assume that $\dim V > n$, because if $\dim V = n$, then the theorem is obviously satisfied.

Let Σ be a space generated by the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and Σ^* be the orthogonal subspace to Σ. Let us decompose the vectors $\mathbf{b}_i$ as $\mathbf{b}_i = \mathbf{c}_i + \mathbf{d}_i$ where $\mathbf{c}_i \in \Sigma$ and $\mathbf{d}_i \in \Sigma^*$. Thus

$$\mathbf{b}_i = \sum_{j=1}^{n} P_{ij} \mathbf{a}_j + \mathbf{d}_i, \qquad (1 \le i \le n)$$

$$\begin{aligned}
\langle \mathbf{a}_1, \ldots, \mathbf{a}_n | \mathbf{b}_1, \ldots, \mathbf{b}_n \rangle &= \left\langle \mathbf{a}_1, \ldots, \mathbf{a}_n \,\middle|\, \sum_{j_1=1}^{n} P_{1j_1} \mathbf{a}_{j_1} + \mathbf{d}_1, \ldots, \sum_{j_n=1}^{n} P_{nj_n} \mathbf{a}_{j_n} + \mathbf{d}_n \right\rangle \\
&= \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} P_{1j_1} P_{2j_2} \cdots P_{nj_n} \langle \mathbf{a}_1, \ldots, \mathbf{a}_n | \mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_n} \rangle \\
&= \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} P_{1j_1} P_{2j_2} \cdots P_{nj_n} (-1)^{sgn\sigma} \langle \mathbf{a}_1, \ldots, \mathbf{a}_n | \mathbf{a}_1, \ldots, \mathbf{a}_n \rangle \\
&= detP \cdot \langle \mathbf{a}_1, \ldots, \mathbf{a}_n | \mathbf{a}_1, \ldots, \mathbf{a}_n \rangle
\end{aligned}$$

where we used the conditions ii) - vi) from Definition 2.1 and we denoted by P the matrix with entries P_{ij}, and $\sigma = \binom{1\,2\,\cdots\,n}{j_1 j_2 \cdots j_n}$.

If $\det P = 0$, then the left side of (2.8) is 0, the right side is positive and hence the inequality (2.8) is true. So, let us suppose that P is a non-singular matrix and $Q = P^{-1}$. Now the inequality (2.8) is equivalent to

$$(\det P)^2 \langle \mathbf{a}_1, \dots, \mathbf{a}_n | \mathbf{a}_1, \dots, \mathbf{a}_n \rangle^2 \le \langle \mathbf{a}_1, \dots, \mathbf{a}_n | \mathbf{a}_1, \dots, \mathbf{a}_n \rangle \langle \mathbf{b}_1, \dots, \mathbf{b}_n | \mathbf{b}_1, \dots, \mathbf{b}_n \rangle,$$

$$\langle \mathbf{a}_1, \dots, \mathbf{a}_n | \mathbf{a}_1, \dots, \mathbf{a}_n \rangle \le \langle \mathbf{b}'_1, \dots, \mathbf{b}'_n | \mathbf{b}'_1, \dots, \mathbf{b}'_n \rangle, \tag{2.9}$$

where $\mathbf{b}'_i = \sum_{j=1}^n Q_{ij} \mathbf{b}_j, \quad (1 \le i \le n)$. Note that $\mathbf{b}'_i$ decomposes as

$$\mathbf{b}'_i = \sum_{j=1}^n Q_{ij} \left(\sum_{l=1}^n P_{jl} \mathbf{a}_l + \mathbf{d}_j \right) = \mathbf{a}_i + \mathbf{d}'_i$$

where $\mathbf{d}'_i = \sum_{j=1}^n Q_{ij} \mathbf{d}_j \in \Sigma^*$. Now we will prove (2.9), i.e.

$$\langle \mathbf{a}_1, \dots, \mathbf{a}_n | \mathbf{a}_1, \dots, \mathbf{a}_n \rangle \le \langle \mathbf{a}_1 + \mathbf{d}'_1, \dots, \mathbf{a}_n + \mathbf{d}'_n | \mathbf{a}_1 + \mathbf{d}'_1, \dots, \mathbf{a}_n + \mathbf{d}'_n \rangle \tag{2.10}$$

and equality holds if and only if $\mathbf{b}'_1 = \mathbf{a}_1, \dots, \mathbf{b}'_n = \mathbf{a}_n$, i.e., $\mathbf{d}'_1 = \cdots = \mathbf{d}'_n = 0$. More precisely, we will prove that (2.10) is true for at least one basis $\mathbf{a}_1, \dots, \mathbf{a}_n$ of Σ.

Using (2.5) and (2.2) we obtain

$$\begin{aligned}
&\langle \mathbf{a}_1 + \mathbf{d}'_1, \dots, \mathbf{a}_n + \mathbf{d}'_n | \mathbf{a}_1 + \mathbf{d}'_1, \dots, \mathbf{a}_n + \mathbf{d}'_n \rangle \\
&= \langle \mathbf{a}_1, \mathbf{a}_2 + \mathbf{d}'_2, \dots, \mathbf{a}_n + \mathbf{d}'_n | \mathbf{a}_1, \mathbf{a}_2 + \mathbf{d}'_2, \dots, \mathbf{a}_n + \mathbf{d}'_n \rangle \\
&\qquad + \langle \mathbf{d}'_1, \mathbf{a}_2 + \mathbf{d}'_2, \dots, \mathbf{a}_n + \mathbf{d}'_n | \mathbf{d}'_1, \mathbf{a}_2 + \mathbf{d}'_2, \dots, \mathbf{a}_n + \mathbf{d}'_n \rangle \\
&\qquad + 2 \langle \mathbf{a}_1, \mathbf{a}_2 + \mathbf{d}'_2, \dots, \mathbf{a}_n + \mathbf{d}'_n | \mathbf{d}'_1, \mathbf{a}_2 + \mathbf{d}'_2, \dots, \mathbf{a}_n + \mathbf{d}'_n \rangle \\
&= \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 + \mathbf{d}'_3, \dots, \mathbf{a}_n + \mathbf{d}'_n | \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 + \mathbf{d}'_3, \dots, \mathbf{a}_n + \mathbf{d}'_n \rangle \\
&\qquad + \langle \mathbf{a}_1, \mathbf{d}'_2, \mathbf{a}_3 + \mathbf{d}'_3, \dots, \mathbf{a}_n + \mathbf{d}'_n | \mathbf{a}_1, \mathbf{d}'_2, \mathbf{a}_3 + \mathbf{d}'_3, \dots, \mathbf{a}_n + \mathbf{d}'_n \rangle \\
&\qquad + \langle \mathbf{d}'_1, \mathbf{a}_2 + \mathbf{d}'_2, \dots, \mathbf{a}_n + \mathbf{d}'_n | \mathbf{d}'_1, \mathbf{a}_2 + \mathbf{d}'_2, \dots, \mathbf{a}_n + \mathbf{d}'_n \rangle \\
&\qquad + 2 \langle \mathbf{a}_1, \mathbf{a}_2 + \mathbf{d}'_2, \dots, \mathbf{a}_n + \mathbf{d}'_n | \mathbf{d}'_1, \mathbf{a}_2 + \mathbf{d}'_2, \dots, \mathbf{a}_n + \mathbf{d}'_n \rangle \\
&\qquad + 2 \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 + \mathbf{d}'_3, \dots, \mathbf{a}_n + \mathbf{d}'_n | \mathbf{a}_1, \mathbf{d}'_2, \mathbf{a}_3 + \mathbf{d}'_3, \dots, \mathbf{a}_n + \mathbf{d}'_n \rangle \\
&= \cdots \\
&= \langle \mathbf{a}_1, \dots, \mathbf{a}_n | \mathbf{a}_1, \dots, \mathbf{a}_n \rangle + \langle \mathbf{a}_1, \dots, \mathbf{a}_{n-1}, \mathbf{d}'_n | \mathbf{a}_1, \dots, \mathbf{a}_{n-1}, \mathbf{d}'_n \rangle \\
&\qquad + \cdots + \langle \mathbf{a}_1, \mathbf{d}'_2, \dots, \mathbf{a}_n + \mathbf{d}'_n | \mathbf{a}_1, \mathbf{d}'_2, \dots, \mathbf{a}_n + \mathbf{d}'_n \rangle \\
&\qquad + \langle \mathbf{d}'_1, \mathbf{a}_2 + \mathbf{d}'_2, \dots, \mathbf{a}_n + \mathbf{d}'_n | \mathbf{d}'_1, \mathbf{a}_2 + \mathbf{d}'_2, \dots, \mathbf{a}_n + \mathbf{d}'_n \rangle + S,
\end{aligned}$$

where

$$\begin{aligned}
S = {} & 2 \langle \mathbf{a}_1, \mathbf{a}_2 + \mathbf{d}'_2, \dots, \mathbf{a}_n + \mathbf{d}'_n | \mathbf{d}'_1, \mathbf{a}_2 + \mathbf{d}'_2, \dots, \mathbf{a}_n + \mathbf{d}'_n \rangle \\
& + 2 \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 + \mathbf{d}'_3, \dots, \mathbf{a}_n + \mathbf{d}'_n | \mathbf{a}_1, \mathbf{d}'_2, \mathbf{a}_3 + \mathbf{d}'_3, \dots, \mathbf{a}_n + \mathbf{d}'_n \rangle \\
& + \cdots + 2 \langle \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{n-1}, \mathbf{a}_n | \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{n-1}, \mathbf{d}'_n \rangle .
\end{aligned}$$

We can change the basis $\mathbf{a}_1, \dots, \mathbf{a}_n$ of Σ such that the sum S vanishes. Indeed, if we replace $\mathbf{a}_1$ by $\lambda \mathbf{a}_1$ we can choose almost always a scalar λ such that $S = 0$. The other cases can be considered by another analogous linear transformations. Thus without loss of generality we can put $S = 0$.

According to i′) the inequality (2.10) is true and equality holds if and only if the following sets of vectors $\{\mathbf{a}_1, \dots, \mathbf{a}_{n-1}, \mathbf{d}'_n\}, \dots, \{\mathbf{a}_1, \mathbf{d}'_2, \mathbf{a}_3 + \mathbf{d}'_3, \dots, \mathbf{a}_n + \mathbf{d}'_n\}, \{\mathbf{d}'_1, \mathbf{a}_2 + \mathbf{d}'_2, \dots, \mathbf{a}_n + \mathbf{d}'_n\}$

are linearly dependent. This is satisfied if and only if $\mathbf{d}_1' = \mathbf{d}_2' = \cdots = \mathbf{d}_n' = 0$, i.e. if and only if $\mathbf{b}_1' = \mathbf{a}_1, \ldots, \mathbf{b}_n' = \mathbf{a}_n$. □

3. SOME APPLICATIONS

Let Σ_1 and Σ_2 be two subspaces of V of dimension n. We define the angle φ between Σ_1 and Σ_2 by

$$\cos\varphi = \frac{\langle \mathbf{a}_1, \ldots, \mathbf{a}_n | \mathbf{b}_1, \ldots, \mathbf{b}_n \rangle}{\|\mathbf{a}_1, \ldots, \mathbf{a}_n\| \cdot \|\mathbf{b}_1, \ldots, \mathbf{b}_n\|}, \tag{3.1}$$

where $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are linearly independent vectors of Σ_1, $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are linearly independent vectors of Σ_2 and

$$\|\mathbf{a}_1, \ldots, \mathbf{a}_n\| = \sqrt{\langle \mathbf{a}_1, \ldots, \mathbf{a}_n | \mathbf{a}_1, \ldots, \mathbf{a}_n \rangle}, \qquad \|\mathbf{b}_1, \ldots, \mathbf{b}_n\| = \sqrt{\langle \mathbf{b}_1, \ldots, \mathbf{b}_n | \mathbf{b}_1, \ldots, \mathbf{b}_n \rangle}.$$

The angle φ does not depend on the choice of the bases $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and $\mathbf{b}_1, \ldots, \mathbf{b}_n$.

Note that any n-inner product induces an ordinary inner product over the vector space $\Lambda_n(V)$ of n-forms on V as follows. Let $\{\mathbf{e}_\alpha\}$, be a basis of V. Then we define

$$\left\langle \sum_{i_1,\ldots,i_n} a_{i_1\cdots i_n} \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_n} \middle| \sum_{j_1,\ldots,j_n} b_{j_1\cdots j_n} \mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_n} \right\rangle \\ = \sum_{i_1,\ldots,i_n,j_1,\ldots,j_n} a_{i_1\cdots i_n} b_{j_1\cdots j_n} \langle \mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_n} | \mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_n} \rangle.$$

The first requirement for the inner product is a consequence of Theorem 2.1. For example, if

$$w = p\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_n} - q\mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_n},$$

then

$$\langle w|w \rangle = p^2 \langle \mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_n} | \mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_n} \rangle + q^2 \langle \mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_n} | \mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_n} \rangle \\ - 2pq \langle \mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_n} | \mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_n} \rangle \geq 0$$

and moreover, the last expression is 0 if and only if

$$\langle \mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_n} | \mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_n} \rangle = \sqrt{\langle \mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_n} | \mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_n} \rangle} \sqrt{\langle \mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_n} | \mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_n} \rangle}$$

which means that $\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_n}$ and $\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_n}$ generate the same subspace, and $p\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_n} = q\mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_n}$, i.e. if and only if $w = 0$. The other requirements for inner products are obviously satisfied. Hence we obtain an induced ordinary inner product on the vector space $\Lambda_n(V)$ of n-forms on V.

Remark 3.1. Note that the inner product on $\Lambda_n(V)$ introduced in Example 2.1 is only a special case of an inner product on $\Lambda_n(V)$ and also n-inner product. It is induced via the existence of an ordinary inner product on V.

The angle between subspaces defined by (3.1) coincides with the angle between two n-forms in the vector space $\Lambda_n(V)$. Since the angle between two "lines" in any vector space with ordinary inner product can be considered as a distance, we obtain that

$$\varphi = \arccos \frac{\langle \mathbf{a}_1, \ldots, \mathbf{a}_n | \mathbf{b}_1, \ldots, \mathbf{b}_n \rangle}{\|\mathbf{a}_1, \ldots, \mathbf{a}_n\| \cdot \|\mathbf{b}_1, \ldots, \mathbf{b}_n\|} \tag{3.2}$$

determines a metric among the n-dimensional subspaces of V. Indeed, it induces a metric on the Grassmann manifold $G_n(V)$, which is compatible with the ordinary topology of the

Grassman manifold $G_n(V)$. This metric over Grassmann manifolds appears natural and appears convenient also for the infinite dimensional vector spaces V.

Further, we shall consider a special case of an n-inner product for which there exists a basis $\{\mathbf{e}_\alpha\}$ of V such that the vector $\mathbf{e}_i$ is orthogonal to the subspace generated by the vectors $\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_n}$ for different values of $i, i_1, \dots, i_n$. For such an n-inner product we have

$$\langle \mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_n} | \mathbf{e}_{j_1}, \dots, \mathbf{e}_{j_n} \rangle = C_{i_1 \cdots i_n} \delta^{i_1 \cdots i_n}_{j_1 \cdots j_n} \tag{3.3}$$

where $\delta^{i_1 \cdots i_n}_{j_1 \cdots j_n}$ is equal to 1 or -1 if $\{i_1, \dots, i_n\} = \{j_1, \dots, j_n\}$ with different $i_1, \dots, i_n$, the permutation $\binom{i_1\ i_2\ \cdots\ i_n}{j_1\ j_2\ \cdots\ j_n}$ is even or odd respectively, the expression is 0 otherwise, and where $C_{i_1 \cdots i_n} > 0$. Moreover, one can verify that the previous formula induces an n-inner product, i.e. the six conditions i) - vi) are satisfied if and only if all the coefficients $C_{i_1 \cdots i_n}$ are equal to a positive constant $C > 0$. Moreover, we can assume that $C = 1$, because otherwise we can consider the basis $\{\mathbf{e}_\alpha / C^{1/2n}\}$ instead of the basis $\{\mathbf{e}_\alpha\}$ of V. Hence this special case of n-inner product reduces to the n-inner product given by the Example 2.1. Indeed, the ordinary inner product is uniquely defined such that $\{\mathbf{e}_\alpha\}$ has an orthonormal system of vectors.

If the dimension of V is finite, for example $\dim V = m > n$, then the previous n-inner product induces a dual $(m-n)$-inner product on V which is induced by

$$\langle \mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_{m-n}} | \mathbf{e}_{j_1}, \dots, \mathbf{e}_{j_{m-n}} \rangle^* = \delta^{i_1 \cdots i_{m-n}}_{j_1 \cdots j_{m-n}}. \tag{3.4}$$

The dual $(m-n)$-inner product is defined using the "orthonormal basis" $\{\mathbf{e}_\alpha\}$ of V. If we have chosen another "orthonormal basis", the result will be the same. Further we prove the following theorem.

Theorem 3.2. *Let V be a finite dimensional vector space and let the n-inner product on V be defined as in Example 2.1. Then*

$$\varphi(\Sigma_1, \Sigma_2) = \varphi(\Sigma_1^*, \Sigma_2^*),$$

where Σ_1 and Σ_2 are arbitrary n-dimensional subspaces of V and Σ_1^* and Σ_2^* are their orthogonal subspaces in V.

Proof. Let $\Sigma_1 = \langle \omega_1 \rangle$, $\Sigma_2 = \langle \omega_2 \rangle$, $\Sigma_1^* = \langle \omega_1^* \rangle$, $\Sigma_2^* = \langle \omega_2^* \rangle$, where $\|\omega_1\| = \|\omega_2\| = \|\omega_1^*\| = \|\omega_2^*\| = 1$. We will prove that

$$\omega_1 \cdot \omega_2 = \pm \omega_1^* \cdot \omega_2^*.$$

Indeed, $\omega_1 \cdot \omega_2 = \omega_1^* \cdot \omega_2^*$ if $\omega_1 \wedge \omega_2$ and $\omega_1^* \wedge \omega_2^*$ have the same orientation in V and $\omega_1 \cdot \omega_2 = -\omega_1^* \cdot \omega_2^*$ if $\omega_1 \wedge \omega_2$ and $\omega_1^* \wedge \omega_2^*$ have the opposite orientations in V.

Assume that the dimension of V is m. Without loss of generality we can assume that

$$\omega_1 = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_n \quad \text{and} \quad \omega_1^* = \mathbf{e}_{n+1} \wedge \mathbf{e}_{n+2} \wedge \cdots \wedge \mathbf{e}_m.$$

Without loss of generality we can assume that

$$\omega_2 = \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_n \quad \text{and} \quad \omega_2^* = \mathbf{a}_{n+1} \wedge \mathbf{a}_{n+2} \wedge \cdots \wedge \mathbf{a}_m,$$

where $\mathbf{a}_1, \ldots, \mathbf{a}_m$ is an orthonormal system. Suppose that $\mathbf{a}_i = (a_{i1}, \ldots, a_{im})$ $(1 \leq i \leq m)$, and let us introduce an orthogonal $m \times m$ matrix

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} & a_{1,n+1} & \cdots & a_{1m} \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ a_{n1} & \cdots & a_{nn} & a_{n,n+1} & \cdots & a_{nm} \\ a_{n+1,1} & \cdots & a_{n+1,n} & a_{n+1,n+1} & \cdots & a_{n+1,m} \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ a_{m1} & \cdots & a_{mn} & a_{m,n+1} & \cdots & a_{mm} \end{bmatrix}.$$

We denote by $A_{i_1 \cdots i_n}$ $(1 \leq i_1 < i_2 < \cdots < i_n \leq m)$, the $n \times n$ submatrix of A whose rows are the first n rows of A and whose columns are the i_1-th,...,i_n-th column of A. We denote by $A^*_{i_1 \cdots i_n}$ the $(m-n) \times (m-n)$ submatrix of A which is obtained by deleting the rows and the columns corresponding to the submatrix $A_{i_1 \cdots i_n}$. It is easy to verify that

$$\omega_1 \cdot \omega_2 = \det A_{12\ldots n} \quad \text{and} \quad \omega_1^* \cdot \omega_2^* = \det A^*_{12\ldots n}$$

and thus we have to prove that

$$\det A_{12\ldots n} = \pm \det A^*_{12\ldots n}, \tag{3.5}$$

i.e.

$$\det A_{12\ldots n} = \det A^*_{12\ldots n} \quad \text{if} \quad \det A = 1$$

and

$$\det A_{12\ldots n} = -\det A^*_{12\ldots n} \quad \text{if} \quad \det A = -1.$$

Assume that $\det A = 1$. Let us consider the expression

$$F = \sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq m} \Big[(\det A_{i_1 i_2 \cdots i_n} - (-1)^{1+2+\cdots+n} (-1)^{i_1+i_2+\cdots+i_n} \det A^*_{i_1 i_2 \cdots i_n}) \Big]^2.$$

Using $\|\omega_2\| = 1$ and $\|\omega_2^*\| = 1$ we get

$$\sum_{1 \leq i_1 < \cdots < i_n \leq m} (\det A_{i_1 i_2 \cdots i_n})^2 = \sum_{1 \leq i_1 < \cdots < i_n \leq m} (\det A^*_{i_1 i_2 \cdots i_n})^2 = 1$$

and using the Laplace formula for decomposition of determinants, we obtain

$$\begin{aligned} F = & \sum_{1 \leq i_1 < \cdots < i_n \leq m} (\det A_{i_1 i_2 \cdots i_n})^2 + \sum_{1 \leq i_1 < \cdots < i_n \leq m} (\det A^*_{i_1 i_2 \cdots i_n})^2 \\ & \quad - 2 \sum_{1 \leq i_1 < \cdots < i_n \leq m} (-1)^{n(n+1)/2} (-1)^{i_1+i_2+\cdots+i_n} \det A_{i_1 i_2 \cdots i_n} \det A^*_{i_1 i_2 \cdots i_n} \\ & = 1 + 1 - 2 \cdot \det A = 2 - 2 = 0. \end{aligned}$$

Hence $F = 0$ implies that

$$\det A_{i_1 i_2 \cdots i_n} = (-1)^{n(n+1)/2} (-1)^{i_1+i_2+\cdots+i_n} \det A^*_{i_1 i_2 \cdots i_n}.$$

In particular, for $i_1 = 1, \ldots, i_n = n$ we obtain

$$\det A_{i_1 i_2 \cdots i_n} = \det A^*_{i_1 i_2 \cdots i_n}.$$

Assume that $\det A = -1$. Then we consider the expression

$$F' = \sum_{1\le i_1<i_2<\cdots<i_n\le m} \Big[(\det A_{i_1 i_2\cdots i_n} + (-1)^{1+2+\cdots+n}(-1)^{i_1+i_2+\cdots+i_n}\det A^*_{i_1 i_2\cdots i_n})\Big]^2$$

and analogously we obtain that

$$\det A_{i_1 i_2\cdots i_n} = -(-1)^{n(n+1)/2}(-1)^{i_1+i_2+\cdots+i_n}\det A^*_{i_1 i_2\cdots i_n}.$$

In particular, for $i_1 = 1, \dots, i_n = n$ we obtain

$$\det A_{i_1 i_2\cdots i_n} = -\det A^*_{i_1 i_2\cdots i_n}.$$

□

Finally we make the following remark. The presented approach to n-inner products appears to be essential for applications in functional analysis. Since the corresponding n-norm is the same as the corresponding n-norm from the definition of Misiak, we have the same results in the normed spaces. It is an open question whether from Definition 2.1 a generalized n-inner product and n-semi-inner product with characteristic p can be introduced. It may also be of interest to research the strong convexity in the possibly introduced space with n-semi-inner product with characteristic p.

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