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ON A CERTAIN RETARDED INTEGRAL INEQUALITY AND ITS APPLICATIONS

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ABSTRACT. In the present paper explicit bound on a new retarded integral inequality in two independent variables is established. Applications are given to illustrate the usefulness of the inequality.

Key words and phrases: Retarded integral inequality, explicit bound, two independent variables, Volterra-Fredholm integral equation, uniqueness of solutions, continuous dependence of solution.

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1. INTRODUCTION

In the study of differential, integral and finite difference equations, one has often to deal with certain integral and finite difference inequalities, which provide explicit bounds on the unknown functions. A detailed account on such inequalities and some of their applications can be found in [2, 3, 6, 7, 9]. In [8] the present author has established the following useful integral inequality.

Lemma 1.1. *Let* $u(t) \in C(I,\mathbb{R}_+)$, $a(t,s), b(t,s) \in C(D,\mathbb{R}_+)$ *and* $a(t,s), b(t,s)$ *are nondecreasing in* t *for each* $s \in I$, *where* $I = [\alpha,\beta]$, $\mathbb{R}_+ = [0,\infty)$, $D = \{(t,s) \in I^2 : \alpha \le s \le t \le \beta\}$ *and suppose that*

$$u(t) \le k + \int_\alpha^t a(t,s)\, u(s)\, ds + \int_\alpha^\beta b(t,s)\, u(s)\, ds,$$

for $t \in I$, *where* $k \ge 0$ *is a constant. If*

$$p(t) = \int_\alpha^\beta b(t,s) \exp\left(\int_\alpha^s a(s,\sigma)\, d\sigma\right) ds < 1,$$

for $t \in I$, *then*

$$u(t) \le \frac{k}{1-p(t)} \exp\left(\int_\alpha^t a(t,s)\, ds\right),$$

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for $t \in I$.

A version of the above inequality when $a(t,s) = a(s), b(t,s) = b(s)$ is first given in [2, p. 11]. In a recent paper [10] a useful general retarded version of the above inequality is given. The aim of the present paper is to establish a general two independent variable retarded version of the above inequality which can be used as a tool to study the behavior of solutions of a general retarded Volterra-Fredholm integral equation in two independent variables. Applications are given to convey the importance of our result to the literature.

2. MAIN RESULT

In what follows, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = [0,\infty)$, $I_1 = [x_0, M]$ and $I_2 = [y_0, N]$ are the given subsets of $\mathbb{R}$. Let $\Delta = I_1 \times I_2$ and

$$E = \left\{(x,y,s,t) \in \Delta^2 : x_0 \le s \le x \le M, y_0 \le t \le y \le N\right\}.$$

Our main result is established in the following theorem.

Theorem 2.1. *Let* $u(x,y) \in C(\Delta,\mathbb{R}_+)$, $a(x,y,s,t)$, $b(x,y,s,t) \in C(E,\mathbb{R}_+)$ *and* $a(x,y,s,t)$, $b(x,y,s,t)$ *be nondecreasing in* x *and* y *for each* $s \in I_1$, $t \in I_2$, $\alpha \in C^1(I_1,I_1)$, $\beta \in C^1(I_2,I_2)$ *be nondecreasing with* $\alpha(x) \le x$ *on* I_1, $\beta(y) \le y$ *on* I_2 *and suppose that*

$$u(x,y) \le c + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t)\, u(s,t)\, dtds + \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} b(x,y,s,t)\, u(s,t)\, dtds, \tag{2.1}$$

for $x \in I_1, y \in I_2$, *where* $c \ge 0$ *is a constant. If*

$$p(x,y) = \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} b(x,y,s,t) \exp\left(\int_{\alpha(x_0)}^{\alpha(s)} \int_{\beta(y_0)}^{\beta(t)} a(s,t,\sigma,\tau)\, d\tau d\sigma\right) dtds < 1, \tag{2.2}$$

for $x \in I_1, y \in I_2$, *then*

$$u(x,y) \le \frac{c}{1 - p(x,y)} \exp\left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t)\, dtds\right), \tag{2.3}$$

for $x \in I_1, y \in I_2$.

Proof. Fix any arbitrary $(X,Y) \in \Delta$. Then for $x_0 \le x \le X$, $y_0 \le y \le Y$ we have

$$u(x,y) \le c + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(X,Y,s,t)\, u(s,t)\, dtds + \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} b(X,Y,s,t)\, u(s,t)\, dtds. \tag{2.4}$$

Let

$$k = c + \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} b(X,Y,s,t)\, u(s,t)\, dtds, \tag{2.5}$$

then (2.4) can be restated as

$$u(x,y) \le k + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(X,Y,s,t)\, u(s,t)\, dtds, \tag{2.6}$$

for $x_0 \leq x \leq X$, $y_0 \leq y \leq Y$. Now a suitable application of the inequality (c_1) given in Theorem 3 in [9, p. 51] to (2.6) yields

$$u(x,y) \leq k \exp\left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(X,Y,s,t)\,dtds\right), \tag{2.7}$$

for $x_0 \leq x \leq X, y_0 \leq y \leq Y$. Since $(X,Y) \in \Delta$ is arbitrary, from (2.7) and (2.5) with X and Y replaced by x and y we have

$$u(x,y) \leq k \exp\left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t)\,dtds\right), \tag{2.8}$$

where

$$k = c + \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} b(x,y,s,t)\,u(s,t)\,dtds, \tag{2.9}$$

for all $x \in I_1$, $y \in I_2$. Using (2.8) on the right side of (2.9) and in view of (2.2) we have

$$k \leq \frac{c}{1-p(x,y)}. \tag{2.10}$$

Using (2.10) in (2.8) we get the desired inequality in (2.3). The proof is complete. □

By taking $b(x,y,s,t) = 0$ in Theorem 2.1, we get the following useful inequality.

Corollary 2.2. *Let $u(x,y), a(x,y,s,t), \alpha(x), \beta(y)$ and c be as in Theorem 2.1. If*

$$u(x,y) \leq c + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t)\,u(s,t)\,dtds, \tag{2.11}$$

for $x \in I_1, y \in I_2$, then

$$u(x,y) \leq c \exp\left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t)\,dtds\right), \tag{2.12}$$

for $x \in I_1, y \in I_2$.

The following corollaries of Theorem 2.1 and Corollary 2.2 are also useful in certain applications.

Corollary 2.3. *Let $u(x,y), a(x,y,s,t), b(x,y,s,t)$ and c be as in Theorem 2.1 and suppose that*

$$u(x,y) \leq c + \int_{x_0}^{x} \int_{y_0}^{y} a(x,y,s,t)\,u(s,t)\,dtds + \int_{x_0}^{M} \int_{y_0}^{N} b(x,y,s,t)\,u(s,t)\,dtds, \tag{2.13}$$

for $x \in I_1, y \in I_2$. If

$$q(x,y) = \int_{x_0}^{M} \int_{y_0}^{N} b(x,y,s,t) \exp\left(\int_{x_0}^{s} \int_{y_0}^{t} a(s,t,\sigma,\tau)\,d\tau d\sigma\right) dtds < 1, \tag{2.14}$$

for $x \in I_1, y \in I_2$, then

$$u(x,y) \leq \frac{c}{1-q(x,y)} \exp\left(\int_{x_0}^{x} \int_{y_0}^{y} a(x,y,s,t)\,dtds\right), \tag{2.15}$$

for $x \in I_1, y \in I_2$.

Corollary 2.4. *Let* $u(x,y)$, $a(x,y,s,t)$ *and* c *be as in Corollary 2.2. If*

$$u(x,y) \leq c + \int_{x_0}^{x} \int_{y_0}^{y} a(x,y,s,t)\, u(s,t)\, dtds, \tag{2.16}$$

for $x \in I_1, y \in I_2$*, then*

$$u(x,y) \leq c \exp\left(\int_{x_0}^{x} \int_{y_0}^{y} a(x,y,s,t)\, dtds\right), \tag{2.17}$$

for $x \in I_1, y \in I_2$*.*

The proofs of Corollaries 2.3 and 2.4 follow by taking $\alpha(x) = x, \beta(y) = y$ in Theorem 2.1 and Corollary 2.2.

3. Applications

In this section, we present applications of Theorem 2.1 to study certain properties of solutions of the retarded Volterra-Fredholm integral equation in two independent variables of the form

$$\begin{aligned} z(x,y) = f(x,y) &+ \int_{x_0}^{x} \int_{y_0}^{y} A(x,y,s,t,z(s-h_1(s), t-h_2(t)))\, dtds \\ &+ \int_{x_0}^{M} \int_{y_0}^{N} B(x,y,s,t,z(s-h_1(s), t-h_2(t)))\, dtds, \end{aligned} \tag{3.1}$$

where $z, f \in C(\Delta,\mathbb{R})$, $A, B \in C(E\times\mathbb{R},\mathbb{R})$ and $h_1 \in C(I_1,\mathbb{R}_+)$, $h_2 \in C(I_2,\mathbb{R}_+)$, are nonincreasing, $x - h_1(x) \geq 0$, $y - h_2(y) \geq 0$, $x - h_1(x) \in C^1(I_1, I_1)$, $y - h_2(y) \in C^1(I_2, I_2)$, $h_1'(x) < 1$, $h_2'(x) < 1$, $h_1(x_0) = h_2(y_0) = 0$.

The following theorem gives the bound on the solution of equation (3.1).

Theorem 3.1. *Suppose that the functions* f, A, B *in equation (3.1) satisfy the conditions*

$$|f(x,y)| \leq c, \tag{3.2}$$

$$|A(x,y,s,t,z)| \leq a(x,y,s,t)\, |z|, \tag{3.3}$$

$$|B(x,y,s,t,z)| \leq b(x,y,s,t)\, |z|, \tag{3.4}$$

where c, $a(x,y,s,t)$, $b(x,y,s,t)$ *are as in Theorem 2.1. Let*

$$M_1 = \max_{x\in I_1} \frac{1}{1-h_1'(x)}, \quad M_2 = \max_{y\in I_2} \frac{1}{1-h_2'(y)}, \tag{3.5}$$

and

$$\bar{p}(x,y) = \int_{\phi(x_0)}^{\phi(M)} \int_{\psi(y_0)}^{\psi(N)} \bar{b}(x,y,s,t) \exp\left(\int_{\phi(x_0)}^{\phi(s)} \int_{\psi(y_0)}^{\psi(t)} \bar{a}(s,t,\sigma,\tau)\, d\tau d\sigma\right) dtds < 1, \tag{3.6}$$

where $\phi(x) = x - h_1(x)$, $x \in I_1$, $\psi(y) = y - h_2(y)$, $y \in I_2$ *and*

$$\bar{a}(x,y,\sigma,\tau) = M_1 M_2 a(x,y,\sigma + h_1(s), \tau + h_2(t)),$$
$$\bar{b}(x,y,\sigma,\tau) = M_1 M_2 b(x,y,\sigma + h_1(s), \tau + h_2(t)).$$

If $z(x,y)$ *is a solution of equation (3.1) on* Δ*, then*

$$|z(x,y)| \leq \frac{c}{1-\bar{p}(x,y)} \exp\left(\int_{\phi(x_0)}^{\phi(x)} \int_{\psi(y_0)}^{\psi(y)} \bar{a}(x,y,\sigma,\tau)\, d\tau d\sigma\right), \tag{3.7}$$

for $x \in I_1, y \in I_2$.

Proof. Since $z(x,y)$ is a solution of equation (3.1), from (3.1) – (3.4) we have

$$|z(x,y)| \le c + \int_{x_0}^{x}\int_{y_0}^{y} a(x,y,s,t)\,|z(s-h_1(s),t-h_2(t))|\,dtds + \int_{x_0}^{M}\int_{y_0}^{N} b(x,y,s,t)\,|z(s-h_1(s),t-h_2(t))|\,dtds. \tag{3.8}$$

Now by making the change of variables on the right side of (3.8) and using (3.5) we have

$$|z(x,y)| \le c + \int_{\phi(x_0)}^{\phi(x)}\int_{\psi(y_0)}^{\psi(y)} \bar{a}(x,y,\sigma,\tau)\,|z(\sigma,\tau)|\,d\tau d\sigma + \int_{\phi(x_0)}^{\phi(M)}\int_{\psi(y_0)}^{\psi(N)} \bar{b}(x,y,\sigma,\tau)\,|z(\sigma,\tau)|\,d\tau d\sigma. \tag{3.9}$$

A suitable application of Theorem 2.1 to (3.9) yields (3.7). □

The next result deals with the uniqueness of solutions of (3.1).

Theorem 3.2. *Suppose that the functions A, B in equation (3.1) satisfy the conditions*

$$|A(x,y,s,t,z) - A(x,y,s,t,\bar{z})| \le a(x,y,s,t)\,|z-\bar{z}|, \tag{3.10}$$

$$|B(x,y,s,t,z) - B(x,y,s,t,\bar{z})| \le b(x,y,s,t)\,|z-\bar{z}|, \tag{3.11}$$

where $a(x,y,s,t), b(x,y,s,t)$ are as in Theorem 2.1. Let $M_1, M_2, \phi, \psi, \bar{a}, \bar{b}, \bar{p}$ be as in Theorem 3.1. Then the equation (3.1) has at most one solution on Δ.

Proof. Let $z(x,y)$ and $\bar{z}(x,y)$ be two solutions of equation (3.1) on Δ. From (3.1), (3.10), (3.11) we have

$$|z(x,y) - \bar{z}(x,y)| \le \int_{x_0}^{x}\int_{y_0}^{y} a(x,y,s,t)\,|z(s-h_1(s),t-h_2(t)) - \bar{z}(s-h_1(s),t-h_2(t))|\,dtds + \int_{x_0}^{x}\int_{y_0}^{y} b(x,y,s,t)\,|z(s-h_1(s),t-h_2(t)) - \bar{z}(s-h_1(s),t-h_2(t))|\,dtds. \tag{3.12}$$

By making the change of variables on the right side of (3.12) and using (3.5) we have

$$|z(x,y) - \bar{z}(x,y)| \le \int_{\phi(x_0)}^{\phi(x)}\int_{\psi(y_0)}^{\psi(y)} \bar{a}(x,y,\sigma,\tau)\,|z(\sigma,\tau) - \bar{z}(\sigma,\tau)|\,d\tau d\sigma + \int_{\phi(x_0)}^{\phi(M)}\int_{\psi(y_0)}^{\psi(N)} \bar{b}(x,y,\sigma,\tau)\,|z(\sigma,\tau) - \bar{z}(\sigma,\tau)|\,d\tau d\sigma. \tag{3.13}$$

Now a suitable application of Theorem 2.1 to (3.13) yields

$$|z(x,y) - \bar{z}(x,y)| \le 0.$$

Therefore $z(x,y) = \bar{z}(x,y)$, i.e. there is at most one solution to the equation (3.1). □

The following theorem deals with the continuous dependence of solution of equation (3.1) on the right side.

Consider the equation (3.1) and the following equation

$$w(x,y) = g(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} F(x,y,s,t,w(s-h_1(s),t-h_2(t)))\,dtds + \int_{x_0}^{M}\int_{y_0}^{N} G(x,y,s,t,w(s-h_1(s),t-h_2(t)))\,dtds, \tag{3.14}$$

where $w, g \in C(\Delta,\mathbb{R})$, $F, G \in C(E\times\mathbb{R},\mathbb{R})$ and h_1, h_2 are as in equation (3.1).

Theorem 3.3. *Suppose that the functions A, B in equation (3.1) satisfy the conditions (3.10), (3.11) in Theorem 3.2 and further assume that*

$$|f(x,y) - g(x,y)| \le \varepsilon, \tag{3.15}$$

$$\int_{x_0}^{x}\int_{y_0}^{y} |A(x,y,s,t,w(s-h_1(s),t-h_2(t))) - F(x,y,s,t,w(s-h_1(s),t-h_2(t)))|\,dtds \le \varepsilon, \tag{3.16}$$

$$\int_{x_0}^{M}\int_{y_0}^{N} |B(x,y,s,t,w(s-h_1(s),t-h_2(t))) - G(x,y,s,t,w(s-h_1(s),t-h_2(t)))|\,dtds \le \varepsilon, \tag{3.17}$$

where $\varepsilon > 0$ is an arbitrary small constant, and let $M_1, M_2, \phi, \psi, \bar{a}, \bar{b}, \bar{p}$ be as in Theorem 3.1. Then the solution of equation (3.1) depends continuously on the functions involved on the right side of equation (3.1).

Proof. Let $z(x,y)$ and $w(x,y)$ be the solutions of (3.1) and (3.14) respectively. Then we have

$$\begin{aligned} z(x,y) - w(x,y) &= f(x,y) - g(x,y) \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} \{A(x,y,s,t,z(s-h_1(s),t-h_2(t))) - A(x,y,s,t,w(s-h_1(s),t-h_2(t)))\}dtds \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} \{A(x,y,s,t,w(s-h_1(s),t-h_2(t))) - F(x,y,s,t,w(s-h_1(s),t-h_2(t)))\}dtds \\ &\quad + \int_{x_0}^{M}\int_{y_0}^{N} \{B(x,y,s,t,z(s-h_1(s),t-h_2(t))) - B(x,y,s,t,w(s-h_1(s),t-h_2(t)))\}dtds \\ &\quad + \int_{x_0}^{M}\int_{y_0}^{N} \{B(x,y,s,t,w(s-h_1(s),t-h_2(t))) - G(x,y,s,t,w(s-h_1(s),t-h_2(t)))\}dtds. \end{aligned} \tag{3.18}$$

Using (3.10), (3.11), (3.15) – (3.17) in (3.18) we get

$$\begin{aligned} (3.19)\quad & |z(x,y)-w(x,y)| \\ & \le 3\varepsilon + \int_{x_0}^{x}\int_{y_0}^{y} a(x,y,s,t)\,|z(s-h_1(s),t-h_2(t)) - w(s-h_1(s),t-h_2(t))|\,dtds \\ & \quad + \int_{x_0}^{M}\int_{y_0}^{N} b(x,y,s,t)\,|z(s-h_1(s),t-h_2(t)) - w(s-h_1(s),t-h_2(t))|\,dtds. \end{aligned}$$

By making the change of variables on the right side of (3.19) and using (3.5) we get

$$\begin{aligned} (3.20)\quad |z(x,y)-w(x,y)| \le 3\varepsilon & + \int_{\phi(x_0)}^{\phi(x)}\int_{\psi(y_0)}^{\psi(y)} \bar{a}(x,y,s,t)\,|z(\sigma,\tau)-w(\sigma,\tau)|\,d\tau d\sigma \\ & + \int_{\phi(x_0)}^{\phi(M)}\int_{\psi(y_0)}^{\psi(N)} \bar{b}(x,y,s,t)\,|z(\sigma,\tau)-w(\sigma,\tau)|\,d\tau d\sigma. \end{aligned}$$

Now a suitable application of Theorem 2.1 to (3.20) yields

$$(3.21)\qquad |z(x,y)-w(x,y)| \le 3\varepsilon\left[\frac{1}{1-\bar{p}(x,y)}\exp\left(\int_{\phi(x_0)}^{\phi(x)}\int_{\psi(y_0)}^{\psi(y)} \bar{a}(x,y,\sigma,\tau)\,d\tau d\sigma\right)\right],$$

for $x \in I_1, y \in I_2$. On the compact set, the quantity in square brackets in (3.21) is bounded by some positive constant $\overline{M}$. Therefore $|z(x,y)-w(x,y)| \le 3\overline{M}\varepsilon$ on the set, so the solution to equation (3.1) depends continuously on the functions involved on the right side of equation (3.1). If $\varepsilon \to 0$, then $|z(x,y)-w(x,y)| \to 0$ on the set. □

We next consider the following retarded Volterra-Fredholm integral equations

$$\begin{aligned} (3.22)\quad z(x,y) = f(x,y) & + \int_{x_0}^{x}\int_{y_0}^{y} A(x,y,s,t,z(s-h_1(s),t-h_2(t)),\mu)\,dtds \\ & + \int_{x_0}^{M}\int_{y_0}^{N} B(x,y,s,t,z(s-h_1(s),t-h_2(t)),\mu)\,dtds, \end{aligned}$$

$$\begin{aligned} (3.23)\quad z(x,y) = f(x,y) & + \int_{x_0}^{x}\int_{y_0}^{y} A(x,y,s,t,z(s-h_1(s),t-h_2(t)),\mu_0)\,dtds \\ & + \int_{x_0}^{M}\int_{y_0}^{N} B(x,y,s,t,z(s-h_1(s),t-h_2(t)),\mu_0)\,dtds, \end{aligned}$$

where $z, f \in C(\Delta,\mathbb{R})$, $A, B \in C(E\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and μ, μ_0 are real parameters.

The following theorem shows the dependency of solutions of equations (3.22) and (3.23) on parameters.

Theorem 3.4. *Suppose that*

$$(3.24)\qquad |A(x,y,s,t,z,\mu) - A(x,y,s,t,\bar{z},\mu)| \le a(x,y,s,t)\,|z-\bar{z}|,$$

$$(3.25)\qquad |A(x,y,s,t,\bar{z},\mu) - A(x,y,s,t,\bar{z},\mu_0)| \le r(x,y,s,t)\,|\mu-\mu_0|,$$

$$(3.26)\qquad |B(x,y,s,t,z,\mu) - B(x,y,s,t,\bar{z},\mu)| \le b(x,y,s,t)\,|z-\bar{z}|,$$

(3.27) $$|B(x,y,s,t,\bar{z},\mu)-B(x,y,s,t,\bar{z},\mu_0)|\leq e(x,y,s,t)\,|\mu-\mu_0|,$$

where $a(x,y,s,t)$, $b(x,y,s,t)$ *are as in Theorem 2.1 and* $r,e\in C(E,\mathbb{R}_+)$ *are such that*

(3.28) $$\int_{x_0}^{x}\int_{y_0}^{y} r(x,y,s,t)\,dtds\leq k_1,$$

(3.29) $$\int_{x_0}^{M}\int_{y_0}^{N} e(x,y,s,t)\,dtds\leq k_2,$$

where k_1,k_2 *are positive constants. Let* $M_1,M_2,\phi,\psi,\bar{a},\bar{b},\bar{p}$ *be as in Theorem 3.1. Let* $z_1(x,y)$ *and* $z_2(x,y)$ *be the solutions of (3.22) and (3.23) respectively. Then*

(3.30) $$|z_1(x,y)-z_2(x,y)|\leq\frac{(k_1+k_2)\,|\mu-\mu_0|}{1-\bar{p}(x,y)}\exp\left(\int_{\phi(x_0)}^{\phi(x)}\int_{\psi(y_0)}^{\psi(y)}\bar{a}(x,y,s,t)\,dtds\right),$$

for $x\in I_1, y\in I_2$.

Proof. Let $z(x,y)=z_1(x,y)-z_2(x,y)$, $(x,y)\in\Delta$. Then

(3.31) $$\begin{aligned} z(x,y)=&\int_{x_0}^{x}\int_{y_0}^{y}\{A(x,y,s,t,z_1(s-h_1(s),t-h_2(t)),\mu)\\ &-A(x,y,s,t,z_2(s-h_1(s),t-h_2(t)),\mu)\}\,dtds\\ &+\int_{x_0}^{x}\int_{y_0}^{y}\{A(x,y,s,t,z_2(s-h_1(s),t-h_2(t)),\mu)\\ &-A(x,y,s,t,z_2(s-h_1(s),t-h_2(t)),\mu_0)\}\,dtds\\ &+\int_{x_0}^{M}\int_{y_0}^{N}\{B(x,y,s,t,z_1(s-h_1(s),t-h_2(t)),\mu)\\ &-B(x,y,s,t,z_2(s-h_1(s),t-h_2(t)),\mu)\}\,dtds\\ &+\int_{x_0}^{M}\int_{y_0}^{N}\{B(x,y,s,t,z_2(s-h_1(s),t-h_2(t)),\mu)\\ &-B(x,y,s,t,z_2(s-h_1(s),t-h_2(t)),\mu_0)\}\,dtds. \end{aligned}$$

Using (3.24) – (3.29) in (3.31) we get

(3.32) $$\begin{aligned} |z(x,y)|\leq&\,|\mu-\mu_0|\,k_1+|\mu-\mu_0|\,k_2\\ &+\int_{x_0}^{x}\int_{y_0}^{y}a(x,y,s,t)\,|z(s-h_1(s),t-h_2(t))|\,dtds\\ &+\int_{x_0}^{M}\int_{y_0}^{N}b(x,y,s,t)\,|z(s-h_1(s),t-h_2(t))|\,dtds. \end{aligned}$$

By using the change of variables on the right side of (3.32) and (3.5) we get

(3.33) $$\begin{aligned} |z(x,y)|\leq&\,(k_1+k_2)\,|\mu-\mu_0|+\int_{\phi(x_0)}^{\phi(x)}\int_{\psi(y_0)}^{\psi(y)}\bar{a}(x,y,\sigma,\tau)\,|z(\sigma,\tau)|\,d\tau d\sigma\\ &+\int_{\phi(x_0)}^{\phi(M)}\int_{\psi(y_0)}^{\psi(N)}\bar{b}(x,y,\sigma,\tau)\,|z(\sigma,\tau)|\,d\tau d\sigma. \end{aligned}$$

Now a suitable application of Theorem 2.1 to (3.33) yields (3.30), which shows the dependency of solutions of (3.22) and (3.23) on parameters. □

In conclusion, we note that the results given in this paper can be extended very easily to functions involving many independent variables. Since the formulations of such results are quite straightforward in view of the results given above (see also [6]) and hence we omit the details. For the study of behavior of solutions of Volterra-Fredholm integral equations involving functions of one independent variable, see [1, 4, 5].

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