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Refined Semilattice Structure of Left C-Wrpp Semigroups

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Abstract In this paper, we explore the refined semilattice of left C-wrpp semigroups, and show that a left C-wrpp semigroup S is a refined semilattice of left-$\mathcal{R}$ cancellative stripes if and only if it is a spined product of a C-wrpp component and a left regular band. It is a generalization of the refined semilattice decomposition of left C-rpp semigroups.

Keywords left C-wrpp semigroup; refined semilattice; left-$\mathcal{R}$ cancellative stripe; spined product.

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1. Introduction

In the last decades, generalizations of the class of Clifford semigroups have been extensively investigated by many authors and some interesting results have been obtained[1−18]. Fountain[2] introduced rpp monoids with central idempotents, briefly called C-rpp semigroups, which are one of significant generalizations of Clifford semigroups. He showed that a semigroup is C-rpp if and only if it is a strong semilattice of left cancellative monoids. The class of C-rpp semigroups includes the class of Clifford semigroups but is out of the range of regular semigroups. In Ref. [18], Zhu, Guo and Shum generalized the class of Clifford semigroups to the class of left C-semigroups which is also in the range of regular semigroups. They showed that a semigroup is a left C-semigroup if and only if it is a semilattice of left groups. Guo, Zhu and Shum in Ref. [7] had defined and investigated the structure of left C-rpp semigroups, where they showed that a rpp semigroup is a left C-rpp semigroup if and only if it is a semilattice of left stripes. By a left stripe, it means that it is a direct product of a left cancellative monoid and a left zero band. On the other hand, in 1997, Tang[13] generalized Fountain's work on C-rpp semigroups to the class of semigroups called C-wrpp semigroups, and he showed that a semigroup is a C-wrpp semigroup if and only if it is a strong semilattice of left-$\mathcal{R}$ cancellative monoids. Du and Shum[1] introduced the concept of left C-wrpp semigroups. The class of left C-wrpp semigroups includes the class of C-wrpp semigroups and the class of left C-rpp semigroups. The authors established the semi–spined product structure for left C-wrpp semigroups.

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Refined semilattice of semigroups was firstly studied by Zhang, Shum and Zhang in Ref. [16]. It is a natural generalization of the notation of strong semilattice of semigroups. Thus, a number of results in the literature concerning strong semilattice decomposition can be further developed[8,14,15,17]. Recently, Zhang[17] has investigated the refined semilattice structure of left C-rpp semigroup, and he showed that a left C-rpp semigroup S is a refined semilattice of left stripes if and only if it is a spined product of C-rpp component and a left regular band.

In this paper, we study the structure of refined semilattice for left C-wrpp semigroups. We shall prove that a left C-wrpp semigroup S is a refined semilattice of left-$\mathcal{R}$ cancellative stripes if and only if it is a spined product of a C-wrpp component and a left regular band. It shows that our main result is a generalization of the refined semilattice decomposition of left C-rpp semigroups. Some methods in Ref. [17] are adopted.

For notation and terminologies not mentioned in this paper, readers are referred to [1], [16], [19] or [20].

2. Preliminaries

It will be convenient to make use of the following notations and lemmas in the remainder of this paper.

Definition 2.1[13] *Let S be a semigroup. We define the $\mathcal{L}^{**}$-relation by $a\mathcal{L}^{**}b$ for $a, b \in S$ if and only if $(ax, ay) \in \mathcal{R} \Leftrightarrow (bx, by) \in \mathcal{R}$ for $x, y \in S^1$, where $\mathcal{R}$ is the usual Green's $\mathcal{R}$-relation on S.*

For $a \in S$, the equivalence relation $\mathcal{L}^{**}$-class containing the element a is denoted by L_a^{**}.

Definition 2.2[1] *A semigroup S is called wrpp semigroup if the following conditions are satisfied:*

(1) Each $\mathcal{L}^{**}$-class of S contains at least one idempotent of S;

(2) For all $e \in E(L_a^{**})$, $a = ae$.

Definition 2.3[13] *A semigroup S is said to be a C-wrpp semigroup if each $\mathcal{L}^{**}$-class of S contains an idempotent and all idempotents of S are central in S.*

Definition 2.4[14] *A wrpp semigroup S is called an adequate wrpp semigroup if for each $a \in S$, there exists a unique idempotent e satisfying $a\mathcal{L}^{**}e$ and $a = ea$.*

Hereafter, we denote the unique idempotent e in Definition 2.4 by e_a.

Definition 2.5[1] *An adequate wrpp semigroup S is said to be a left C-wrpp semigroup if it satisfies $aS \subseteq L^{**}(a)$ for all $a \in S$, where $L^{**}(a)$ represents the smallest left **-ideal of S generated by $a \in S$. By a left **-ideal L of S, we mean that it is a left ideal of S and satisfies that $L_x^{**} \subseteq L$ for all $x \in L$.*

Definition 2.6[13] *A semigroup S is said to be left-$\mathcal{R}$ cancellative if for all $a, b, c \in S$, $(ca, cb) \in \mathcal{R}$ implies $(a, b) \in \mathcal{R}$.*

Lemma 2.7[1] *Let S be an adequate wrpp semigroup. Then the following conditions are equivalent:*

(1) S is a left C-wrpp semigroup;

(2) $\mathcal{L}^{**}$ is a semilattice congruence on S;

(3) $E(S)$ is a left regular band and $\mathcal{L}^{**}$ is a congruence on S;

(4) S is a semilattice of left-$\mathcal{R}$ cancellative stripes.

It is easy to verify the following corollary.

Corollary 2.8 *If an adequate wrpp semigroup S is a semilattice of left-$\mathcal{R}$ cancellative stripes, then every left-$\mathcal{R}$ cancellative stripe is a $\mathcal{L}^{**}$-class.*

Next, we introduce the concept of refined semilattice of semigroups.

Definition 2.9[16,17] *Let Y be a semilattice and $\{S_\alpha : \alpha \in Y\}$ a family of disjoint of semigroups of type T, indexed by Y. For each pair $\alpha, \beta \in Y$ with $\alpha \geq \beta$, let $D(\alpha, \beta)$ be a set of index and*

$$\{S_{d(\alpha,\beta)} : d(\alpha, \beta) \in D(\alpha, \beta)\}$$

a congruence partition of S_β (i.e., the relation σ on S_β defined by $(b_\beta, b_\beta^{'}) \in \sigma$ if and only if $b_\beta, b_\beta^{'} \in S_{d(\alpha,\beta)}$ for some $d(\alpha, \beta) \in D(\alpha, \beta)$ is a congruence on S_β), and for $\alpha \geq \beta \geq \gamma$, the partition

$$\{S_{d(\alpha,\gamma)} : d(\alpha, \gamma) \in D(\alpha, \gamma)\}$$

is dense in the partition

$$\{S_{d(\beta,\gamma)} : d(\beta, \gamma) \in D(\beta, \gamma)\},$$

i.e., for any $d(\beta, \gamma) \in D(\beta, \gamma)$, there exists $D^{'}(\alpha, \gamma) \subseteq D(\alpha, \gamma)$ such that

$$S_{d(\beta,\gamma)} = \cup_{d(\alpha,\gamma) \in D^{'}(\alpha,\gamma)} S_{d(\alpha,\gamma)}.$$

Moreover, let

$$\{\Phi_{d(\alpha,\beta)} : S_\alpha \to S_{d(\alpha,\beta)} : d(\alpha, \beta) \in D(\alpha, \beta)\}$$

be a family of homomorphisms. Suppose the following conditions are satisfied:

(a) $D(\alpha, \alpha)$ is singleton and $\Phi_{d(\alpha,\alpha)}$ is the identical automorphism of S_α for each $\alpha \in Y$, where $d(\alpha, \alpha)$ is the unique element of $D(\alpha, \alpha)$.

(b) (i) For any $\alpha, \beta, \gamma \in Y$ with $\alpha \geq \beta \geq \gamma$,

$$\{\Phi_{d(\alpha,\beta)}\Phi_{d(\beta,\gamma)} : d(\alpha, \beta) \in D(\alpha, \beta), d(\beta, \gamma) \in D(\beta, \gamma)\}$$
$$\subseteq \{\Phi_{d(\alpha,\gamma)} : d(\alpha, \gamma) \in D(\alpha, \gamma)\}.$$

(ii) For any $d(\alpha, \alpha\beta) \in D(\alpha, \alpha\beta)$ and $d(\alpha\beta, \alpha\beta\gamma) \in D(\alpha\beta, \alpha\beta\gamma)$,

$$S_{d(\alpha,\alpha\beta)}\Phi_{d(\alpha\beta,\alpha\beta\gamma)} \subseteq S_{d(\alpha,\alpha\beta\gamma)},$$

where $d(\alpha, \alpha\beta\gamma)$ satisfies

$$\Phi_{d(\alpha,\alpha\beta\gamma)} = \Phi_{d(\alpha,\alpha\beta)}\Phi_{d(\alpha\beta,\alpha\beta\gamma)}.$$

(c) For $\alpha, \beta, \gamma \in Y$ with $\gamma \le \alpha\beta$ and for any fixed $a_\alpha \in S_\alpha$, $d(\alpha\beta, \gamma) \in D(\alpha\beta, \gamma)$, there exists $\bar{d}(\beta, \gamma) \in D(\beta, \gamma)$ such that

$$\{a_\alpha \Phi_{d(\alpha,\gamma)} : d(\alpha, \gamma) \in D(\alpha, \gamma)\} \cap S_{d(\alpha\beta,\gamma)} \subseteq S_{\bar{d}(\beta,\gamma)}.$$

(d) For $\alpha, \beta \in Y$ with $\alpha \ge \beta$ and $a_\alpha \in S_\alpha$, $b_\beta \in S_\beta$, $d(\alpha, \beta) \in D(\alpha, \beta)$, $d^{'}(\alpha, \beta) \in D(\alpha, \beta)$,

$$b_\beta(a_\alpha \Phi_{d'(\alpha,\beta)}) \in S_{d(\alpha,\beta)} \Rightarrow b_\beta(a_\alpha \Phi_{d'(\alpha,\beta)}) = b_\beta(a_\alpha \Phi_{d(\alpha,\beta)}),$$

and

$$(a_\alpha \Phi_{d'(\alpha,\beta)})b_\beta \in S_{d(\alpha,\beta)} \Rightarrow (a_\alpha \Phi_{d'(\alpha,\beta)})b_\beta = (a_\alpha \Phi_{d(\alpha,\beta)})b_\beta.$$

We now form the set $S = \cup\{S_\alpha : \alpha \in Y\}$ and define a multiplication $\circ$ on S by the following statements.

For any $a_\alpha \in S_\alpha$, $b_\beta \in S_\beta$, define

$$a_\alpha \circ b_\beta = (a_\alpha \Phi_{\bar{d}(\alpha,\alpha\beta)})(b_\beta \Phi_{\bar{d}(\beta,\alpha\beta)}),$$

where $\bar{d}(\alpha, \alpha\beta) \in D(\alpha, \alpha\beta)$, $\bar{d}(\beta, \alpha\beta) \in D(\beta, \alpha\beta)$ which satisfy the following conditions:

$$\{a_\alpha \Phi_{d(\alpha,\alpha\beta)} : d(\alpha, \alpha\beta) \in D(\alpha, \alpha\beta)\} \subseteq S_{\bar{d}(\beta,\alpha\beta)}$$

and

$$\{b_\beta \Phi_{d(\beta,\alpha\beta)} : d(\beta, \alpha\beta) \in D(\beta, \alpha\beta)\} \subseteq S_{\bar{d}(\alpha,\alpha\beta)}.$$

Then $(S = \cup_{\alpha\in Y} S_\alpha, \circ)$ is a semigroup as it has been shown in Ref. [16]. Hence, the semigroup $(S, \circ)$ is called the refined semilattice of type T semigroups and is denoted by

$$\{Y; S_{d(\alpha,\beta)}, \Phi_{d(\alpha,\beta)}, D(\alpha, \beta); S_\alpha\}.$$

In the following, we give a lemma of the semi-spined product structure for left C-wrpp semigroups.

Recall that if $T = \cup_{\alpha\in Y} T_\alpha$ and $I = \cup_{\alpha\in Y} I_\alpha$ are semilattice compositions of the semigroups T_α and I_α, respectively, then we can form the set union $S = \cup_{\alpha\in Y} S_\alpha$, where $S_\alpha = T_\alpha \times I_\alpha$ is the Cartesian product of T_α and I_α. Let $\mathcal{T}_l(I)$ be the left transformation semigroup acting on I and define a mapping $\eta : S \to \mathcal{T}_\updownarrow(\mathcal{I})$ by $(a, i) \to \eta(a, i)$ such that $\eta(a, i)j = (a, i)^\sharp j$ for every $j \in I$. Suppose that the mapping η satisfies the following conditions:

(S_1) If $(a, i) \in S_\alpha$, $j \in I_\beta$, then $(a, i)^\sharp j \in I_{\alpha\beta}$;

(S_2) If $(a, i) \in S_\alpha$, $j \in I_\beta$ with $\alpha \le \beta$, then $(a, i)^\sharp j = ij$, where ij is the semigroup product in the semigroup $I = \cup_{\alpha\in Y} I_\alpha$;

(S_3) If $(a, i) \in S_\alpha$, $(b, j) \in S_\alpha$, then $\eta(a, i)\eta(b, j) = \eta(ab, (a, i)^\sharp j)$, where ab is the semigroup product in the semigroup $T = \cup_{\alpha\in Y} T_\alpha$.

Then we define a multiplication $\circ$ on $S = \cup_{\alpha\in Y} S_\alpha$ by $(a, i) \circ (b, j) = (ab, (a, i)^\sharp j)$. It can be easily verified that $\circ$ is a binary associative operation on S, so that S becomes a semigroup under the multiplication $\circ$. We denote the semigroup $(S, \circ)$ by $S = T \times_\eta I$ and call $S = T \times_\eta I$ the semi-spined product of the semigroups T and I with respect to η[1,5].

Definition 2.10[20] *Let M and T be semigroups and also H their common morphic image.*

Let $S = \{(a,b) \in M \times T | a\varphi = b\psi\}$, where $\varphi : M \to H$ and $\psi : T \to H$ are the semigroup homomorphisms which map from M and T onto H respectively. Then we call S the spined product of the semigroups M and T with respect to H, φ and ψ, denoted by $S = M \otimes_{H,\varphi,\psi} T$.

Definition 2.11[1] *Let $T = \cup_{\alpha \in Y} T_\alpha$ be a C-wrpp semigroup (that is, T is a strong semilattice of left-$\mathcal{R}$ cancellative monoids $[Y; T_\alpha; \varphi_{\alpha,\beta}]$ by the theorem of Tang in Ref. [13]) and let I be a left regular band which is expressed as a semilattice of left zero bands I_α (that is, $I = \cup_{\alpha \in Y} I_\alpha$). Then we call the semi-spined product $T \times_\eta I = \cup_{\alpha \in Y} S_\alpha$, where $S_\alpha = T_\alpha \times_\eta I_\alpha$, the curler formed by T and I under the structure mapping η defined by conditions (S_1)–(S_3) if the following condition (Q) is satisfied:*

$$(Q) : \ker\eta(a,i) = \ker\eta(b,j) \quad \textit{for all} \quad (a,i),(b,j) \in S_\alpha.$$

Lemma 2.12[1] *Let I be a left regular band and M a C-wrpp semigroup. Then the curler constructed by $S = M \times_\eta I$ is a left C-wrpp semigroup. Conversely, every left C-wrpp semigroup S can be expressed by a curler $S = M \times_\eta I$, where I is a left regular band and M a C-wrpp semigroup.*

Lemma 2.13[1] *Let I be a left regular band and M a C-wrpp semigroup. If the curler constructed by $S = M \times_\eta I$ is a left C-wrpp semigroup, then the following statements hold:*

(1) S is a spined product of M and I if and only if $\rho = \{((a,i),(b,j)) \in S \times S : i = j\}$ is a congruence on S;

(2) S is a left C-rpp semigroup if and only if S is an rpp semigroup.

Lemma 2.14[16] *A band is regular band if and only if it is a refined semilattice of rectangular bands.*

By Lemma 2.14, we can get the following corollary.

Corollary 2.15 *A band is left regular band if and only if it is a refined semilattice of left zero bands.*

3. Refined semilattice of left C-wrpp semigroups

Before proving our main theorem, we also need the following important properties.

Lemma 3.1[13] *Each left-$\mathcal{R}$ cancellative monoid contains a unique idempotent.*

By Lemma 3.1, we can immediately get the following result.

Corollary 3.2 *Let $S_\alpha = M_\alpha \times I_\alpha$ be a left-$\mathcal{R}$ cancellative stripe. Then we have that $E(S_\alpha) = \{e_\alpha\} \times I_\alpha$, where e_α is the unique idempotent of left-$\mathcal{R}$ cancellative monoid M_α. And in the following, we always use $e_\alpha \times I_\alpha$ to denote $\{e_\alpha\} \times I_\alpha$.*

Proposition 3.3 *Let $S = \{Y; S_{d(\alpha,\beta)}, \Phi_{d(\alpha,\beta)}, D(\alpha,\beta); S_\alpha = M_\alpha \times I_\alpha\}$, where M_α is a left-$\mathcal{R}$ cancellative monoid and I_α a left zero band for any $\alpha \in Y$. For any $d(\alpha,\beta), d^{'}(\alpha,\beta) \in$*

$D(\alpha,\beta)(\alpha \geq \beta)$, if $(a_\alpha, i_\alpha)\Phi_{d(\alpha,\beta)} = (a_\beta, i_\beta)$ and $(a_\alpha, i_\alpha)\Phi_{d'(\alpha,\beta)} = (b_\beta, j_\beta)$, then

(i) $(e_\alpha, i_\alpha)\Phi_{d(\alpha,\beta)} = (e_\beta, i_\beta)$;

(ii) $(e_\beta, i_\beta) \in S_{d(\alpha,\beta)}$;

(iii) $a_\beta = b_\beta$;

(iv) For any $\alpha \geq \beta$ and $d_1(\alpha,\beta) \in D(\alpha,\beta)$, if $(b_\beta, j_\beta) \in S_{d_1(\alpha,\beta)}$, then $(e_\beta, j_\beta) \in S_{d_1(\alpha,\beta)}$.

Proof Firstly, we prove (i) and (ii). To see (i) holds, we observe that

$$(a_\beta, i_\beta) = (a_\alpha, i_\alpha)\Phi_{d(\alpha,\beta)} = (e_\alpha, i_\alpha)\Phi_{d(\alpha,\beta)}(a_\alpha, i_\alpha)\Phi_{d(\alpha,\beta)} = (e_\alpha, i_\alpha)\Phi_{d(\alpha,\beta)}(a_\beta, i_\beta).$$

If put $(e_\alpha, i_\alpha)\Phi_{d(\alpha,\beta)} = (b_\beta, j_\beta)$, then by the above argument we immediately have $j_\beta = i_\beta$. Also, since

$$(b_\beta, j_\beta) = (e_\alpha, i_\alpha)\Phi_{d(\alpha,\beta)} = [(e_\alpha, i_\alpha)(e_\alpha, i_\alpha)]\Phi_{d(\alpha,\beta)} = (e_\alpha, i_\alpha)\Phi_{d(\alpha,\beta)}(e_\alpha, i_\alpha)\Phi_{d(\alpha,\beta)} = (b_\beta, j_\beta)^2,$$

i.e., (b_β, j_β) is an idempotent, by Corollary 3.2, we have $b_\beta = e_\beta$. Hence, we have proved that $(e_\alpha, i_\alpha)\Phi_{d(\alpha,\beta)} = (e_\beta, i_\beta)$, and then (i) holds. At this time, by the definition of refined semilattice, we immediately have $(e_\beta, i_\beta) \in S_{d(\alpha,\beta)}$, and it means that (ii) holds.

Secondly, we prove (iii). Since

$$(e_\beta, i_\beta)(a_\alpha, i_\alpha) = (e_\beta, i_\beta)[(a_\alpha, i_\alpha)\Phi_{d(\alpha,\beta)}] = (a_\beta, i_\beta)$$

and

$$\begin{aligned}(b_\beta, j_\beta) &= (e_\beta, j_\beta)(b_\beta, j_\beta) = (e_\beta, j_\beta)[(a_\alpha, i_\alpha)\Phi_{d'(\alpha,\beta)}] = (e_\beta, j_\beta)(a_\alpha, i_\alpha)\\ &= (e_\beta, j_\beta)(e_\beta, i_\beta)(a_\alpha, i_\alpha) = (e_\beta, j_\beta)(a_\beta, i_\beta) = (a_\beta, j_\beta),\end{aligned}$$

we have $a_\beta = b_\beta$. Hence, (iii) holds.

Finally, we prove (iv). If $(b_\beta, j_\beta) \in S_{d_1(\alpha,\beta)}$, then by Definition 2.9, we have $d''(\alpha,\beta) \in D(\alpha,\beta)$ such that $(e_\beta, j_\beta) \in S_{d''(\alpha,\beta)}$. However, since $(b_\beta, j_\beta)(e_\beta, j_\beta) = (e_\beta, j_\beta)(b_\beta, j_\beta) = (b_\beta, j_\beta)$, we will obtain that

$$S_{d_1(\alpha,\beta)}S_{d''(\alpha,\beta)} \cap S_{d_1(\alpha,\beta)} \neq \varphi$$

and

$$S_{d''(\alpha,\beta)}S_{d_1(\alpha,\beta)} \cap S_{d_1(\alpha,\beta)} \neq \varphi.$$

Also, since $\{S_{d(\alpha,\beta)} : d(\alpha,\beta) \in D(\alpha,\beta)\}$ is a congruence partition of S_β, we have

$$S_{d_1(\alpha,\beta)}S_{d''(\alpha,\beta)} \subseteq S_{d_1(\alpha,\beta)} \tag{1}$$

and

$$S_{d''(\alpha,\beta)}S_{d_1(\alpha,\beta)} \subseteq S_{d_1(\alpha,\beta)}. \tag{2}$$

Clearly, by (ii), for any $d(\alpha,\beta) \in D(\alpha,\beta)$, $E(S_{d(\alpha,\beta)}) \neq \varphi$ and now we denote the element in $E(S_{d_1(\alpha,\beta)})$ as (e_β, i'_β). Let $(e_\beta, i'_\beta) \in E(S_{d_1(\alpha,\beta)})$. Then by (1) and (2), we have $(e_\beta, j_\beta) = (e_\beta, j_\beta)(e_\beta, i'_\beta) \in S_{d_1(\alpha,\beta)}$. Hence, (iv) holds. □

Proposition 3.4 *Let $S = \{Y; S_{d(\alpha,\beta)}, \Phi_{d(\alpha,\beta)}, D(\alpha,\beta); S_\alpha = M_\alpha \times I_\alpha\}$, where M_α is a left-$\mathcal{R}$ cancellative monoid and I_α a left zero band for any $\alpha \in Y$. If $(a,i)\mathcal{R}(b,j)$ for any $(a,i) \in M_\alpha \times I_\alpha$*

and $(b,j)\in M_\beta\times I_\beta$, *then we have* $\alpha=\beta$, $i=j$ *and* $a\mathcal{R}b$.

Proof Let $x=(a,i)\in M_\alpha\times I_\alpha=S_\alpha$, $y=(b,j)\in M_\beta\times I_\beta=S_\beta$. Since $(a,i)\mathcal{R}(b,j)$, there exists $u\in S_\gamma$, $v\in S_\delta$ such that $xu=y, yv=x$. By the multiplication of refined semilattice of semigroups, we know that $xu\in S_{\alpha\gamma}$, but $xu=y\in S_\beta$, thus, $\beta=\alpha\gamma$, and $\beta\le\alpha$. Similarly, we can obtain $\alpha=\beta\delta$, and then $\alpha\le\beta$. Hence, $\alpha=\beta$.

On the other hand, by $xu=y, yv=x$ and $\alpha=\beta$, we have $\alpha\gamma=\alpha, \alpha\delta=\alpha$, and then $\alpha\le\gamma, \alpha\le\delta$. By the above equalities, we also have $x\Phi_{\bar{d}(\alpha,\alpha\gamma)}u\Phi_{\bar{d}(\gamma,\alpha\gamma)}=y$, i.e., $x\Phi_{\bar{d}(\alpha,\alpha)}u\Phi_{\bar{d}(\gamma,\alpha)}=x(u\Phi_{\bar{d}(\gamma,\alpha)})=y$. Now, if we let $u\Phi_{\bar{d}(\gamma,\alpha)}=(u_\alpha,k_\alpha)$, then we immediately have that $(a,i)(u_\alpha,k_\alpha)=(b,j)$, i.e., $au_\alpha=b, i=ik_\alpha=j$. Similarly, we can show that there exists $v_\alpha\in M_\alpha$ such that $bv_\alpha=a$. Thus, $a\mathcal{R}b$. □

Now, we start to give our main theorem.

Theorem 3.5 *A left C-wrpp semigroup S is a refined semilattice of left-$\mathcal{R}$ cancellative stripes if and only if it is a spined product of a C-wrpp component and a left regular band.*

Proof To prove our theorem, by Lemmas 2.12 and 2.13 (1), we only need to show the equivalent statement: a semigroup S can be expressed as a refined semilattice of left-$\mathcal{R}$ cancellative stripes $M_\alpha\times I_\alpha$ if and only if it is a left C-wrpp semigroup such that the semi-spined product decomposition $S=M_S\times_\eta I_S$ of S is the spined product decomposition. Next, we set about to show this equivalent statement.

$\Rightarrow$) Let $S=\{Y;S_{d(\alpha,\beta)},\Phi_{d(\alpha,\beta)},D(\alpha,\beta);S_\alpha=M_\alpha\times I_\alpha\}$, where M_α is a left-$\mathcal{R}$ cancellative monoid and I_α is a left zero band for any $\alpha\in Y$. Clearly, S is a semilattice of $\{M_\alpha\times I_\alpha:\alpha\in Y\}$. In order to show the necessity, we will set about it by the following steps:

(1) S is a wrpp semigroup.

Let $(a_\alpha,i_\alpha)\in M_\alpha\times I_\alpha$. Clearly, $(e_\alpha,i_\alpha)\in M_\alpha\times I_\alpha$ such that

$$(e_\alpha,i_\alpha)(a_\alpha,i_\alpha)=(a_\alpha,i_\alpha)(e_\alpha,i_\alpha)=(a_\alpha,i_\alpha).$$

Now if $(a_\alpha,i_\alpha)x_\beta\mathcal{R}\ (a_\alpha,i_\alpha)x_\gamma$ for $x_\beta\in M_\beta\times I_\beta$ and $x_\gamma\in M_\gamma\times I_\gamma$, then we have

$$(a_\alpha,i_\alpha)(e_\alpha,i_\alpha)x_\beta\mathcal{R}(a_\alpha,i_\alpha)(e_\alpha,i_\alpha)x_\gamma,$$

and so,

$$(a_\alpha,i_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}[(e_\alpha,i_\alpha)x_\beta]\Phi_{\bar{d}(\alpha\beta,\alpha\beta)}\mathcal{R}(a_\alpha,i_\alpha)\Phi_{\bar{d}_1(\alpha,\alpha\gamma)}[(e_\alpha,i_\alpha)x_\gamma]\Phi_{\bar{d}_1(\alpha\gamma,\alpha\gamma)} \tag{3}$$

where $\bar{d}(\alpha,\alpha\beta)$ and $\bar{d}_1(\alpha,\alpha\gamma)$ satisfy:

$$(e_\alpha,i_\alpha)x_\beta\in S_{\bar{d}(\alpha,\alpha\beta)},(e_\alpha,i_\alpha)x_\gamma\in S_{\bar{d}_1(\alpha,\alpha\gamma)}.$$

By Proposition 3.4, we have $\alpha\beta=\alpha\gamma$. Also, if we let $(a_\alpha,i_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}=(a_{\alpha\beta},i_{\alpha\beta})$ and $(a_\alpha,i_\alpha)\Phi_{\bar{d}_1(\alpha,\alpha\gamma)}=(a_\alpha,i_\alpha)\Phi_{\bar{d}_1(\alpha,\alpha\beta)}=(b_{\alpha\beta},j_{\alpha\beta})$, then by Proposition 3.3 (iii), we have $a_{\alpha\beta}=b_{\alpha\beta}$. Further, since $I_{\alpha\beta}$ is a left zero band and $\alpha\beta=\alpha\gamma$, we have that

$$(a_\alpha,i_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}[(e_\alpha,i_\alpha)x_\beta]\Phi_{\bar{d}(\alpha\beta,\alpha\beta)}=(-,i_{\alpha\beta})\mathcal{R}(a_\alpha,i_\alpha)\Phi_{\bar{d}_1(\alpha,\alpha\gamma)}[(e_\alpha,i_\alpha)x_\gamma]\Phi_{\bar{d}_1(\alpha\gamma,\alpha\gamma)}=(-,j_{\alpha\beta}).$$

By Proposition 3.4, we have $i_{\alpha\beta} = j_{\alpha\beta}$. Hence, we obtain $(a_\alpha, i_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)} = (a_\alpha, i_\alpha)\Phi_{\bar{d}_1(\alpha,\alpha\gamma)}$, and consequently, $\bar{d}(\alpha, \alpha\beta) = \bar{d}_1(\alpha, \alpha\beta)$. Recall that $M_{\alpha\beta}$ is a left-$\mathcal{R}$ cancellative monoid, by (3) and Proposition 3.4 we have

$$(e_\alpha, i_\alpha)x_\beta\mathcal{R}(\rceil_\alpha, \rangle_\alpha)\S_\gamma.$$

On the other hand, if $(e_\alpha, i_\alpha)x_\beta\mathcal{R}(e_\alpha, i_\alpha)x_\gamma$, then since $\mathcal{R}$ is a left congruence, we can easily obtain that

$$(a_\alpha, i_\alpha)(e_\alpha, i_\alpha)x_\beta\mathcal{R}(a_\alpha, i_\alpha)(e_\alpha, i_\alpha)x_\gamma,$$

i.e.,

$$(a_\alpha, i_\alpha)x_\beta\mathcal{R}(a_\alpha, i_\alpha)x_\gamma.$$

Hence, we have $(a_\alpha, i_\alpha)\mathcal{L}^{**}(e_\alpha, i_\alpha)$.

Let $x = (a, i) \in M_\alpha \times I_\alpha$. For all $e = (e_\alpha, j) \in E(L^{**}_x)$, where $E(L^{**}_x)$ is the set of idempotents in L^{**}_x, we have

$$xe = (a, i)(e_\alpha, j) = (ae_\alpha, ij) = (a, i) = x.$$

We have proved that S is a wrpp semigroup.

(2) S is an adequate wrpp semigroup.

For all $a \in S$, there exists $\alpha \in Y$ such that $a \in M_\alpha \times I_\alpha$. Put $a = (m_\alpha, i_\alpha)$. By the argument of (1) above, there exists $e = (e_\alpha, i_\alpha) \in M_\alpha \times I_\alpha$ such that $a = (m_\alpha, i_\alpha)\mathcal{L}^{**}(e_\alpha, i_\alpha) = e$ and $ea = (e_\alpha, i_\alpha)(m_\alpha, i_\alpha) = (m_\alpha, i_\alpha) = a$. If there is another idempotent $a^* = (f_\alpha, j_\alpha)$ satisfying $a\mathcal{L}^{**}a^*$ and $a = a^*a$, then $(f_\alpha, j_\alpha)(m_\alpha, i_\alpha) = (m_\alpha, i_\alpha)$ and $(f_\alpha, j_\alpha)^2 = (f_\alpha, j_\alpha)$. Hence, $f_\alpha m_\alpha = m_\alpha, i_\alpha = j_\alpha i_\alpha = j_\alpha$ and $f_\alpha^2 = f_\alpha$. By Lemma 3.1, we have $e_\alpha = f_\alpha$. Hence, $a^* = (e_\alpha, i_\alpha) = e$.

Thus, by (1) and (2), S is an adequate wrpp semigroup. Moreover, by Lemma 2.7, we have proved that S is a left C-wrpp semigroup.

(3) Finally, we show that the relation ρ in Lemma 2.13 is a congruence.

First, we show that $e_x e_y = e_{xy}$ for all $x, y \in S$. In fact, if we let $x = (a_\alpha, i_\alpha)$ and $y = (a_\beta, i_\beta)$, then we have $e_x = (e_\alpha, i_\alpha)$, $e_y = (e_\beta, i_\beta)$ and $e_{xy} = (e_{\alpha\beta}, i_{\alpha\beta})$, and also

$$xy = (a_\alpha, i_\alpha)(a_\beta, i_\beta) = (a_\alpha, i_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}(a_\beta, i_\beta)\Phi_{\bar{d}(\beta,\alpha\beta)},$$

where $\bar{d}(\alpha, \alpha\beta)$ and $\bar{d}(\beta, \alpha\beta)$ satisfy that

$$\{(a_\beta, i_\beta)\Phi_{d(\beta,\alpha\beta)} : d(\beta, \alpha\beta) \in D(\beta, \alpha\beta)\} \subseteq S_{\bar{d}(\alpha,\alpha\beta)}$$

and

$$\{(a_\alpha, i_\alpha)\Phi_{d(\alpha,\alpha\beta)} : d(\alpha, \alpha\beta) \in D(\alpha, \alpha\beta)\} \subseteq S_{\bar{d}(\beta,\alpha\beta)}.$$

By Proposition 3.3 (i) and (iv), we have $(e_\beta, i_\beta)\Phi_{d(\beta,\alpha\beta)}, (e_\alpha, i_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)} \in E(S_{\bar{d}(\alpha,\alpha\beta)})$. Since $E(S_{\alpha\beta})$ is a rectangular band, we have

$$\begin{aligned} xy &= (a_\alpha, i_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}(a_\beta, i_\beta)\Phi_{\bar{d}(\beta,\alpha\beta)} \\ &= (e_\alpha, i_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}(a_\alpha, i_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}(a_\beta, i_\beta)\Phi_{\bar{d}(\beta,\alpha\beta)} \\ &= (e_\alpha, i_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}(e_\alpha, i_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}(a_\alpha, i_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}(a_\beta, i_\beta)\Phi_{\bar{d}(\beta,\alpha\beta)} \end{aligned}$$

$$= (e_\alpha, i_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}(e_\beta, i_\beta)\Phi_{\bar{d}(\beta,\alpha\beta)}(e_\alpha, i_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}(a_\alpha, i_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}(a_\beta, i_\beta)\Phi_{\bar{d}(\beta,\alpha\beta)}$$
$$= (e_\alpha, i_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}(e_\beta, i_\beta)\Phi_{\bar{d}(\beta,\alpha\beta)}(a_\alpha, i_\alpha)\Phi_{\bar{d}(\alpha,\alpha\beta)}(a_\beta, i_\beta)\Phi_{\bar{d}(\beta,\alpha\beta)}$$
$$= (e_\alpha, i_\alpha)(e_\beta, i_\beta)(a_\alpha, i_\alpha)(a_\beta, i_\beta) = e_x e_y xy. \tag{4}$$

Similarly, we can prove that

$$xy = xye_x e_y. \tag{5}$$

On the other hand, since S is a left C-wrpp semigroup, by Corollary 2.8 for any $\alpha \in Y$, S_α is a $\mathcal{L}^{**}$-class of S. Also, since $e_x e_y \in S_{\alpha\beta}$, $xy \in S_{\alpha\beta}$, we have $e_x e_y \mathcal{L}^{**} xy$. By the definition of left C-wrpp semigroup and (4) and (5), we have $e_x e_y = e_{xy}$.

Now, we define a relation on S by

$$(a, i)\rho(b, j) \Leftrightarrow (\exists \alpha \in Y) a, b \in M_\alpha \ \text{ and } \ i = j \in I_\alpha.$$

Then for any $x, y \in S$, $x\rho y$ if and only if $e_x = e_y$. By the above argument, we know $e_x e_y = e_{xy}$, and then we can immediately obtain that ρ is a congruence.

Hence, summing up the above arguments and according to Lemma 2.13(1), we have shown that the semi-spined product decomposition $S = M_S \times_\eta I_S$ is a spined product decomposition.

$\Leftarrow$) The proof is analogous to the proof of sufficiency of Theorem 1.6 in Ref. [17].

Let S be a left C-wrpp semigroup such that the semi-spined product decomposition $S = M_S \times_\eta I_S$ of S is the spined product decomposition. Then by Lemma 2.13(1), $\rho = \{((a, i), (b, j)) \in S \times S : i = j\}$ is a congruence. Also by Lemma 2.7, we have $S = \cup_{\alpha \in Y}(M_\alpha \times I_\alpha)$, where M_α is a left-$\mathcal{R}$ cancellative monoid, I_α is a left zero band and Y is a semilattice. Let e_α be the identity of M_α. Then $E(S) = \cup_{\alpha \in Y}(e_\alpha \times I_\alpha)$. By Lemma 2.7, we know it is a left regular band, and also by Corollary 2.15, it is a refined semilattice of left zero bands $e_\alpha \times I_\alpha$ for all $\alpha \in Y$.

Define the multiplication $\circ$ on $I = \cup_{\alpha \in Y} I_\alpha$ by

$$i_\alpha \circ i_\beta = k \ \text{ if and only if } \ (e_\alpha, i_\alpha)(e_\beta, i_\beta) = (e_{\alpha\beta}, k).$$

Then $(I, \circ)$ forms a band which is clearly isomorphic to $E(S)$.

In fact, if we define a mapping

$$\varphi : E(S) \to I, \ \ (e_\alpha, i_\alpha) \to i_\alpha,$$

it is easy to verify that φ is an isomorphism from $E(S)$ to I. Firstly, it is clear to see that φ is surjective. Secondly, if $(e_\alpha, i_\alpha)\varphi = i_\alpha, (e_\beta, i_\beta)\varphi = i_\beta$ and $i_\alpha = i_\beta \in I$, then we have $\alpha = \beta$, and so $e_\alpha = e_\beta$ by Lemma 3.1, thus $(e_\alpha, i_\alpha) = (e_\beta, i_\beta)$, i.e., φ is injective. Finally, for any $(e_\alpha, i_\alpha), (e_\beta, i_\beta) \in E(S)$, if we denote $(e_\alpha, i_\alpha)(e_\beta, i_\beta) = (e_{\alpha\beta}, k)$,then $[(e_\alpha, i_\alpha)(e_\beta, i_\beta)]\varphi = (e_{\alpha\beta}, k)\varphi = k = i_\alpha \circ i_\beta = (e_\alpha, i_\alpha)\varphi \circ (e_\beta, i_\beta)\varphi$. Hence, φ is an isomorphism.

Therefore, I is a left regular band which is a refined semilattice of I_α.

Further, by the proof of Theorem 4.3 in [1], $E(S) \cong I_S$, hence, we have $I \cong I_S$. And then I_S is a left regular band which can also be regarded as a refined semilattice of I_α under isomorphism.

Let $I_S = \{Y; I_{d(\alpha,\beta)}, \Phi_{d(\alpha,\beta)}, D(\alpha, \beta); I_\alpha\}$. Also, by the proof of Theorem 4.3 in Ref. [1], we know $M_S = \cup_{\alpha \in Y} M_\alpha$. Now, for any $\alpha, \beta \in Y$ with $\alpha \geq \beta$, we can define a mapping $\Phi_{\alpha,\beta}$ from

M_α to M_β as follows:

$$\text{for any } \ a \in M_\alpha, a\Phi_{\alpha,\beta} = a^{'} \ \text{ if and only if } \ (e_\beta, j)(a, i) = (a^{'}, j).$$

Analogously to the proof of step (b) of Theorem 3.4 in Ref. [7], we can see that $M_S = \{Y; M_\alpha, \Phi_{\alpha,\beta}\}$ is a strong semilattice of the left $-\mathcal{R}$ cancellative monoids M_α.

Let $S_{d(\alpha,\beta)} = M_\beta \times I_{d(\alpha,\beta)}$. We define a mapping $\Psi_{\alpha,\beta}$ from $M_\alpha \times I_\alpha$ to $M_\beta \times I_\beta$ for any $\alpha, \beta \in Y$ with $\alpha \geq \beta$ as follows:

$$\text{for any } \ (a_\alpha, i_\alpha) \in M_\alpha \times I_\alpha, (a_\alpha, i_\alpha)\Psi_{\alpha,\beta} = (a_\alpha\Phi_{\alpha,\beta}, i_\alpha\Phi_{d(\alpha,\beta)}).$$

We now show that $S = \{Y; S_{d(\alpha,\beta)}, \Psi_{d(\alpha,\beta)}, D(\alpha, \beta); S_\alpha = M_\alpha \times I_\alpha\}$, and the verification will be done by the following steps.

Firstly, it is easy to check that $\Psi_{d(\alpha,\beta)}$ is a homomorphism. Also, we can check that $\{S_{d(\alpha,\beta)} : d(\alpha, \beta) \in D(\alpha, \beta)\}$ is a congruence partition of $S_\beta = M_\beta \times I_\beta$, and for $\alpha \geq \beta \geq \gamma$, the partition $\{S_{d(\alpha,\gamma)} : d(\alpha, \gamma) \in D(\alpha, \gamma)\}$ is clearly dense in the partition $\{S_{d(\beta,\gamma)} : d(\beta, \gamma) \in D(\beta, \gamma)\}$.

Secondly,we show that Conditions (a), (b), (c) and (d) hold.

(i) Condition (a) holds.

For any $\alpha \in Y$, $D(\alpha, \alpha)$ is clearly singleton, also since $\Phi_{\alpha,\alpha}$ and $\Phi_{d(\alpha,\alpha)}$ are the identical automorphisms,where $d(\alpha, \alpha)$ is the unique element of $D(\alpha, \alpha)$, we immediately obtain that $\Psi_{d(\alpha,\alpha)}$ is the identical automorphism,and so (a) holds.

(ii) Condition (b) holds.

First, for any $\alpha, \beta, \gamma \in Y$ with $\alpha \geq \beta \geq \gamma$, and any $(a_\alpha, i_\alpha) \in M_\alpha \times I_\alpha$,$d(\alpha, \beta) \in D(\alpha, \beta), d(\beta, \gamma) \in D(\beta, \gamma)$, since $M_S = \{Y; M_\alpha, \Phi_{\alpha,\beta}\}$ is a strong semilattice of M_α and

$$I_S = \{Y; I_{d(\alpha,\beta)}, \Phi_{d(\alpha,\beta)}, D(\alpha, \beta); I_\alpha\}$$

is a refined semilattice of I_α, we can obtain that

$$(a_\alpha, i_\alpha)\Psi_{\alpha,\beta}\Psi_{\beta,\gamma} = (a_\alpha\Phi_{\alpha,\beta}\Phi_{\beta,\gamma}, i_\alpha\Phi_{d(\alpha,\beta)}\Phi_{d(\beta,\gamma)}) = (a_\alpha\Phi_{\alpha,\gamma}, i_\alpha^{'}\Phi_{d(\alpha,\gamma)}) = (a_\alpha, i_\alpha^{'})\Psi_{\alpha,\gamma}.$$

Hence,we have

$$\begin{aligned}&\{\Psi_{d(\alpha,\beta)}\Psi_{d(\beta,\gamma)} : d(\alpha, \beta) \in D(\alpha, \beta), d(\beta, \gamma) \in D(\beta, \gamma)\}\\ &\quad \subseteq \{\Psi_{d(\alpha,\gamma)} : d(\alpha, \gamma) \in D(\alpha, \gamma)\}.\end{aligned}$$

Also, for any $d(\alpha, \alpha\beta) \in D(\alpha, \alpha\beta)$ and $d(\alpha\beta, \alpha\beta\gamma) \in D(\alpha\beta, \alpha\beta\gamma)$, and any $(a_{\alpha\beta}, i_{\alpha\beta}) \in S_{d(\alpha,\alpha\beta)} = M_{\alpha\beta} \times I_{d(\alpha,\alpha\beta)}$, we have

$$(a_{\alpha\beta}, i_{\alpha\beta})\Psi_{d(\alpha\beta,\alpha\beta\gamma)} = (a_{\alpha\beta}\Phi_{\alpha\beta,\alpha\beta\gamma}, i_{\alpha\beta}\Phi_{d(\alpha\beta,\alpha\beta\gamma)}).$$

Notice that $a_{\alpha\beta}\Phi_{\alpha\beta,\alpha\beta\gamma} \in M_{\alpha\beta\gamma}$, $i_{\alpha\beta}\Phi_{d(\alpha\beta,\alpha\beta\gamma)} \in I_{d(\alpha,\alpha\beta\gamma)}$ since $I_{d(\alpha,\alpha\beta)}\Phi_{d(\alpha\beta,\alpha\beta\gamma)} \subseteq I_{d(\alpha,\alpha\beta\gamma)}$. we have $(a_{\alpha\beta}, i_{\alpha\beta})\Psi_{d(\alpha\beta,\alpha\beta\gamma)} \in S_{d(\alpha,\alpha\beta\gamma)}$, i.e.,

$$S_{d(\alpha,\alpha\beta)}\Psi_{d(\alpha\beta,\alpha\beta\gamma)} \subseteq S_{d(\alpha,\alpha\beta\gamma)}.$$

On the other hand, we can check that $d(\alpha, \alpha\beta\gamma)$ satisfies $\Psi_{d(\alpha,\alpha\beta\gamma)} = \Psi_{d(\alpha,\alpha\beta)}\Psi_{d(\alpha\beta,\alpha\beta\gamma)}$ since $\Phi_{\alpha,\alpha\beta\gamma} = \Phi_{\alpha,\alpha\beta}\Phi_{\alpha\beta,\alpha\beta\gamma}$ and $\Phi_{d(\alpha,\alpha\beta\gamma)} = \Phi_{d(\alpha,\alpha\beta)}\Phi_{d(\alpha\beta,\alpha\beta\gamma)}$. Hence, (b) holds.

(iii) Condition (c) holds.

For $\alpha, \beta, \gamma \in Y$ with $\gamma \leq \alpha\beta$, and for any fixed $(a_\alpha, i_\alpha) \in S_\alpha$ and $d(\alpha\beta, \gamma) \in D(\alpha\beta, \gamma)$, there exists $\bar{d}(\beta, \gamma) \in D(\beta, \gamma)$ such that

$$\{i_\alpha \Phi_{d(\alpha,\gamma)} : d(\alpha, \gamma) \in D(\alpha, \gamma)\} \cap I_{d(\alpha\beta,\gamma)} \subseteq I_{\bar{d}(\beta,\gamma)}.$$

And then, we have

$$\begin{aligned}
&\{(a_\alpha, i_\alpha)\Psi_{d(\alpha,\gamma)} : d(\alpha, \gamma) \in D(\alpha, \gamma)\} \cap S_{d(\alpha\beta,\gamma)} \\
&\quad = \{(a_\alpha \Phi_{\alpha,\gamma}, i_\alpha \Phi_{d(\alpha,\gamma)}) : d(\alpha, \gamma) \in D(\alpha, \gamma)\} \cap (M_\gamma \times I_{d(\alpha\beta,\gamma)}) \\
&\quad = \{a_\alpha \Phi_{\alpha,\gamma}\} \times (\{i_\alpha \Phi_{d(\alpha,\gamma)} : d(\alpha, \gamma) \in D(\alpha, \gamma)\} \cap I_{d(\alpha\beta,\gamma)}) \\
&\quad \subseteq M_\gamma \times I_{\bar{d}(\beta,\gamma)} = S_{\bar{d}(\beta,\gamma)}.
\end{aligned}$$

Hence, Condition (c) holds.

(iv) Condition (d) holds.

For $\alpha, \beta \in Y$ with $\alpha \geq \beta$ and $(a_\alpha, i_\alpha) \in S_\alpha$, $(b_\beta, i_\beta) \in S_\beta$, $d(\alpha, \beta) \in D(\alpha, \beta)$, $d^{'}(\alpha, \beta) \in D(\alpha, \beta)$, we have

$$i_\beta(i_\alpha \Phi_{d'(\alpha,\beta)}) \in I_{d(\alpha,\beta)} \Rightarrow i_\beta(i_\alpha \Phi_{d'(\alpha,\beta)}) = i_\beta(i_\alpha \Phi_{d(\alpha,\beta)}),$$

and

$$(i_\alpha \Phi_{d'(\alpha,\beta)})i_\beta \in I_{d(\alpha,\beta)} \Rightarrow (i_\alpha \Phi_{d'(\alpha,\beta)})i_\beta = (i_\alpha \Phi_{d(\alpha,\beta)})i_\beta.$$

Let $(b_\beta, i_\beta)[(a_\alpha, i_\alpha)\Psi_{d'(\alpha,\beta)}] \in S_{d(\alpha,\beta)}$, that is,

$$(b_\beta, i_\beta)(a_\alpha \Phi_{\alpha,\beta}, i_\alpha \Phi_{d'(\alpha,\beta)}) = (b_\beta(a_\alpha \Phi_{\alpha,\beta}), i_\beta(i_\alpha \Phi_{d'(\alpha,\beta)})) \in S_{d(\alpha,\beta)}.$$

Then we have

$$\begin{aligned}
&(b_\beta, i_\beta)(a_\alpha \Phi_{\alpha,\beta}, i_\alpha \Phi_{d'(\alpha,\beta)}) = (b_\beta(a_\alpha \Phi_{\alpha,\beta}), i_\beta(i_\alpha \Phi_{d'(\alpha,\beta)})) \\
&\quad = (b_\beta(a_\alpha \Phi_{\alpha,\beta}), i_\beta(i_\alpha \Phi_{d_{(\alpha,\beta)}})) = (b_\beta, i_\beta)(a_\alpha \Phi_{\alpha,\beta}, i_\alpha \Phi_{d(\alpha,\beta)}) \\
&\quad = (b_\beta, i_\beta)(a_\alpha, i_\alpha)\Psi_{d(\alpha,\beta)}.
\end{aligned}$$

Moreover, we can dually deduce that, $((a_\alpha, i_\alpha)\Psi_{d'(\alpha,\beta)})(b_\beta, i_\beta) \in S_{d(\alpha,\beta)}$ implies

$$((a_\alpha, i_\alpha)\Psi_{d'(\alpha,\beta)})(b_\beta, i_\beta) = ((a_\alpha, i_\alpha)\Psi_{d(\alpha,\beta)})(b_\beta, i_\beta).$$

Hence, condition (d) holds.

To finish our proof, we remain to show the following step:

(v) For $\alpha, \beta \in Y$ and $x = (a_\alpha, i_\alpha) \in S_\alpha$,$y = (b_\beta, i_\beta) \in S_\beta$, we have $e_x = (e_\alpha, i_\alpha)$, $e_y = (e_\beta, i_\beta)$ and $e_{xy} = (e_{\alpha\beta}, i_{\alpha\beta})$. Notice that ρ in Lemma 2.13 is a congruence, we can get $e_{xy} = e_x e_y$ and then

$$\begin{aligned}
xy &= (a_\alpha, i_\alpha)(b_\beta, i_\beta) = e_{xy} xy = e_x e_y xy \\
&= (e_\alpha, i_\alpha)(e_\beta, i_\beta)(a_\alpha, i_\alpha)(b_\beta, i_\beta) \\
&= (e_\alpha, i_\alpha)(e_\beta, i_\beta)(a_\alpha \Phi_{\alpha,\alpha\beta}, -)(b_\beta \Phi_{\beta,\alpha\beta}, -) \\
&= (e_{\alpha\beta}, i_\alpha i_\beta)(a_\alpha \Phi_{\alpha,\alpha\beta}, -)(b_\beta \Phi_{\beta,\alpha\beta}, -)
\end{aligned}$$

$$= (e_{\alpha\beta}, i_\alpha\Phi_{\bar{d}(\alpha,\alpha\beta)} i_\beta\Phi_{\bar{d}(\beta,\alpha\beta)})(a_\alpha\Phi_{\alpha,\alpha\beta}, -)(b_\beta\Phi_{\beta,\alpha\beta}, -)$$
$$= (e_{\alpha\beta}, i_\alpha\Phi_{\bar{d}(\alpha,\alpha\beta)})(e_{\alpha\beta}, i_\beta\Phi_{\bar{d}(\beta,\alpha\beta)})(a_\alpha\Phi_{\alpha,\alpha\beta}, -)(b_\beta\Phi_{\beta,\alpha\beta}, -)$$
$$= (a_\alpha\Phi_{\alpha,\alpha\beta}, i_\alpha\Phi_{\bar{d}(\alpha,\alpha\beta)})(b_\beta\Phi_{\beta,\alpha\beta}, i_\beta\Phi_{\bar{d}(\beta,\alpha\beta)})$$
$$= (a_\alpha, i_\alpha)\Psi_{\bar{d}(\alpha,\alpha\beta)}(b_\beta, i_\beta)\Psi_{\bar{d}(\beta,\alpha\beta)},$$

where $\bar{d}(\alpha,\alpha\beta)$ and $\bar{d}(\beta,\alpha\beta)$ satisfy that

$$\{i_\alpha\Phi_{d(\alpha,\alpha\beta)} : d(\alpha,\alpha\beta) \in D(\alpha,\alpha\beta)\} \subseteq I_{\bar{d}(\beta,\alpha\beta)}$$

and

$$\{i_\beta\Phi_{d(\beta,\alpha\beta)} : d(\beta,\alpha\beta) \in D(\beta,\alpha\beta)\} \subseteq I_{\bar{d}(\alpha,\alpha\beta)}.$$

Consequently, we have

$$\{(a_\alpha, i_\alpha)\Psi_{d(\alpha,\alpha\beta)} : d(\alpha,\alpha\beta) \in D(\alpha,\alpha\beta)\}$$
$$= \{(a_\alpha\Phi_{\alpha,\alpha\beta}, i_\alpha\Phi_{d(\alpha,\alpha\beta)}) : d(\alpha,\alpha\beta) \in D(\alpha,\alpha\beta)\} \subseteq S_{\bar{d}(\beta,\alpha\beta)}$$

and

$$\{(b_\beta, i_\beta)\Psi_{d(\beta,\alpha\beta)} : d(\beta,\alpha\beta) \in D(\beta,\alpha\beta)\}$$
$$= \{(b_\beta\Phi_{\beta,\alpha\beta}, i_\beta\Phi_{d(\beta,\alpha\beta)}) : d(\beta,\alpha\beta) \in D(\beta,\alpha\beta)\} \subseteq S_{\bar{d}(\alpha,\alpha\beta)}.$$

Hence, summing up the above steps, we have shown that S is a refined semilattice of left-$\mathcal{R}$ cancellative stripes $M_\alpha \times I_\alpha$. □

Now, if we let the semigroup S in Theorem 3.5 be a rpp semigroup, then by Lemma 2.13(2), we immediately have the following corollary which is an equivalent description of Theorem 1.6 in Ref. [17]:

Corollary 3.6 *A left C-rpp semigroup S is a refined semilattice of left cancellative stripes if and only if it is a spined product of a C-rpp component and a left regular band.*

Remark From the above arguments, we immediately obtain that our results actually generalize the ones of Zhang in Ref. [17].

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