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EXISTENCE AND GLOBAL ATTRACTIVITY OF PERIODIC SOLUTIONS IN n-SPECIES FOOD-CHAIN SYSTEM WITH TIME DELAYS

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ABSTRACT. A delayed periodic n-species simple food-chain system with Holling type-II functional response is investigated. By means of Gaines and Mawhin's continuation theorem of coincidence degree theory and by constructing appropriate Lyapunov functionals, sufficient conditions are obtained for the existence and global attractivity of positive periodic solutions of the system.

Key words and phrases: Time delay, Periodic solution, Global attractivity.

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1. INTRODUCION

The traditional Lotka-Volterra type predator-prey model with Michaelis-Menten or Holling type II functional response has received great attention from both theoretical and mathematical biologists, and has been well studied (see, for example, [1] – [12]). Up to now, most of the works on Lotka-Volterra type predator-prey models with Michaelis-Menten or Holling type II functional responses have dealt with autonomous population systems. The analysis of these models has been centered around the coexistence of populations and the local and global stability of equilibria. We note that any biological or environmental parameters are naturally subject to fluctuation in time. As Cushing [13] pointed out, it is necessary and important to consider models with periodic ecological parameters or perturbations which might be quite naturally exposed (for example, those due to seasonal effects of weather, food supply, mating habits, hunting or harvesting seasons, etc.). Thus, the assumption of periodicity of the parameters is a way of incorporating the periodicity of the environment.

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It has been widely argued and accepted that for various reasons, time delay should be taken into consideration in modelling, we refer to the monographs of Cushing [14], Gopalsamy [15], Kuang [16], and MacDonald [17] for general delayed biological systems and to Beretta and Kuang [18], Gopalsamy [19, 20], He [21], Wang and Ma [22], and the references cited therein for studies on delayed biological systems. In general, delay differential equations exhibit much more complicated dynamics than ordinary differential equations since a time delay could cause a stable equilibrium to become unstable and cause the population to fluctuate. Time delay due to gestation is a common example, because generally the consumption of prey by the predator throughout its past history governs the present birth rate of the predator. Therefore, more realistic models of population interactions should take into account the effect of time delays.

The main purpose of this paper is to discuss the combined effects of the periodicity of the ecological and environmental parameters and time delays due to gestation and negative feedback on the dynamics of an n-species food-chain model with Michaelis-Menten or Holling type II functional responses. To do so, we consider the following delay differential equations

$$
(1.1)\qquad \begin{cases} \dot{x}_1(t) = x_1(t)\left[r_1(t) - a_{11}(t)x_1(t-\tau_{11}) - \dfrac{a_{12}(t)x_2(t)}{1+m_1x_1(t)}\right], \\ \dot{x}_j(t) = x_j(t)\left[-r_j(t) + \dfrac{a_{j,j-1}(t)x_{j-1}(t-\tau_{j,j-1})}{1+m_{j-1}x_{j-1}(t-\tau_{j,j-1})}\right. \\ \qquad\qquad\qquad \left. - a_{jj}(t)x_j(t-\tau_{jj}) - \dfrac{a_{j,j+1}(t)x_{j+1}(t)}{1+m_{j+1}x_j(t)}\right], \quad 1<j<n, \\ \dot{x}_n(t) = x_n(t)\left[-r_n(t) + \dfrac{a_{n,n-1}(t)x_{n-1}(t-\tau_{n,n-1})}{1+m_{n-1}x_{n-1}(t-\tau_{n,n-1})} - a_{nn}(t)x_n(t-\tau_{nn})\right], \end{cases}
$$

with initial conditions

$$
(1.2)\qquad x_j(s) = \phi_j(s),\;\; s\in[-\tau,0],\;\; \phi_j(0)>0,\;\; j=1,2,\dots,n.
$$

In system (1.1), $x_i(t)$ denotes the density of the ith population, respectively, $i = 1,\dots,n$. $\tau_{j,j-1}(j = 2,\dots,n)$ are time delays due to gestation, that is, mature adult predators can only contribute to the reproduction of predator biomass. $\tau_{ii} \geq 0$ denotes the delay due to negative feedback of the species x_i. $\tau = \max\{\tau_{ij}, 1 \leq i,j \leq n\}$; $r_i(t), a_{ij}(t)$ $(i,j = 1,2,\dots,n)$ are positively periodic continuous functions with common period $\omega > 0$ and $m_i(i = 1,2,\dots,n-1)$ are positive constants.

It is well known by the fundamental theory of functional differential equations [23] that system (1.1) has a unique solution $x(t) = (x_1,x_2,\dots,x_n)$ satisfying initial conditions (1.2). It is easy to verify that solutions of system (1.1) corresponding to initial conditions (1.2) are defined on $[0,+\infty)$ and remain positive for all $t \geq 0$. In this paper, the solution of system (1.1) satisfying initial conditions (1.2) is said to be positive.

The organization of this paper is as follows. In the next section, by using Gaines and Mawhin's continuation theorem of coincidence degree theory, sufficient conditions are established for the existence of positive periodic solutions of system (1.1) with initial conditions (1.2). In Section 3, by constructing suitable Lyapunov functionals, sufficient conditions are derived for the uniqueness and global attractivity of positive periodic solutions of system (1.1).

2. EXISTENCE OF PERIODIC SOLUTIONS

In this section, by using Gaines and Mawhin's continuation theorem of coincidence degree theory, we show the existence of positive periodic solutions of system (1.1) with initial conditions (1.2). In order to prove our existence result, we need the following notations.

Let X,Y be real Banach spaces, let $L : \operatorname{Dom} L \subset X \to Y$ be a linear mapping, and $N : X \to Y$ be a continuous mapping. The mapping L is called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Y. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : X \to X$, and $Q : Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$, then the restriction L_P of L to $\operatorname{Dom} L \cap \operatorname{Ker} P : (I - P)X \to \operatorname{Im} L$ is invertible. Denote the inverse of L_P by K_P. If Ω is an open bounded subset of X, the mapping N will be called $L-$compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

For convenience of use, we introduce the continuation theorem of coincidence degree theory (see [24, p. 40]) as follows.

Lemma 2.1. *Let $\Omega \subset X$ be an open bounded set. Let L be a Fredholm mapping of index zero and N be L-compact on $\bar{\Omega}$. Assume*

(a) *For each $\lambda \in (0,1), x \in \partial\Omega \cap \operatorname{Dom} L$, $Lx \neq \lambda Nx$;*
(b) *For each $x \in \partial\Omega \cap \operatorname{Ker} L$, $QNx \neq 0$;*
(c) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.

Then $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap \operatorname{Dom} L$.

In what follows we shall also need the following notations:

$$\bar{f} = \frac{1}{\omega}\int_0^\omega f(t)dt, \quad f^L = \min_{t\in[0,\omega]} f(t), \quad f^M = \max_{t\in[0,\omega]} f(t),$$

where f is a continuous ω-periodic function.

Theorem 2.2. *System (1.1) with initial conditions (1.2) has at least one strictly positive ω-periodic solution provided that*

(H1) $$\overline{a_{j+1,j}} > m_j\overline{r_{j+1}},\ \overline{a_{j,j-1}} > m_{j-1}H_j,\ 2 \le j \le n,$$

(H2) $$\overline{r_1} > K_1 + \frac{\overline{a_{11}}H_2}{\overline{a_{21}} - m_1H_2}e^{2\overline{r_1}\omega},$$

where

(2.1) $$\begin{cases} K_1 = \overline{a_{12}}\dfrac{\overline{(a_{21}/m_1)} - \overline{r_2}}{\overline{a_{22}}}e^{2\overline{(a_{21}/m_1)}\omega}, & H_n = \overline{r_n} \\ H_j = K_j + \dfrac{\overline{a_{jj}}H_{j+1}}{\overline{a_{j+i,j}} - m_jH_{j+1}}e^{2\overline{a_{j,j-1}}\omega/m_{j-1}}, & 2 \le j \le n-1, \\ K_j = \overline{r_j} + \overline{a_{j,j+1}}\dfrac{\overline{a_{j+1,j}} - m_j\overline{r_{j+1}}}{m_j\overline{a_{jj}}}e^{2\overline{a_{j+1,j}}\omega/m_j}, & 2 \le j \le n-1. \end{cases}$$

Proof. Let

(2.2) $$y_i(t) = \ln[x_i(t)],\ \ i = 1, \dots, n.$$

On substituting (2.2) into system (1.1), it follows

(2.3) $$\begin{cases} \dot{y}_1(t) = r_1(t) - a_{11}(t)e^{y_1(t-\tau_{11})} - \dfrac{a_{12}(t)e^{y_2(t)}}{1 + m_1e^{y_1(t)}}, \\ \dot{y}_j(t) = -r_j(t) + \dfrac{a_{j,j-1}(t)e^{y_{j-1}(t-\tau_{j,j-1})}}{1 + m_{j-1}e^{y_{j-1}(t-\tau_{j,j-1})}} - a_{jj}(t)e^{y_j(t-\tau_{jj})} - \dfrac{a_{j,j+1}(t)e^{y_{j+1}(t)}}{1 + m_je^{y_j(t)}}, \\ \qquad j = 2, \dots .n-1, \\ \dot{y}_n(t) = -r_n(t) + \dfrac{a_{n,n-1}(t)e^{y_{n-1}(t-\tau_{n,n-1})}}{1 + m_{n-1}e^{y_{n-1}(t-\tau_{n,n-1})}} - a_{nn}(t)e^{y_n(t-\tau_{nn})}. \end{cases}$$

It is easy to see that if system (2.3) has one $\omega-$periodic solution $(y_1^*(t),\dots,y_n^*(t))^T$, then

$$x^*(t)=(x_1^*(t),\dots,x_n^*(t))^T=(\exp[y_1^*(t)],\dots,\exp[y_n^*(t)])^T$$

is a positive ω-periodic solution of system (1.1). Therefore, to complete the proof, it suffices to show that system (2.3) has one ω-periodic solution.

Take

$$X=Y=\{(y_1(t),\dots,y_n(t))^T\in C(\mathbb{R},\mathbb{R}^n):y_i(t+\omega)=y_i(t),i=1,\dots,n\}$$

and

$$\|(y_1(t),\dots,y_n(t))^T\|=\sum_{i=1}^n\max_{t\in[0,\omega]}|y_i(t)|,$$

here $|\cdot|$ denotes $L^\infty-$norm.. It is easy to verify that X and Y are Banach spaces with the norm $\|\cdot\|$. Set

$$L:\operatorname{Dom}L\cap X,\quad L(y_1(t),\dots,y_n(t))^T=\left(\frac{dy_1(t)}{dt},\dots,\frac{dy_n(t)}{dt}\right)^T,$$

where $\operatorname{Dom}L=\{(y_1(t),\dots,y_n(t))^T\in C^1(\mathbb{R},\mathbb{R}^n)\}$ and $N:X\to X$,

$$N\begin{pmatrix}y_1(t)\\ \vdots\\ y_j(t)\\ \vdots\\ y_(t)\end{pmatrix}=\begin{pmatrix}r_1(t)-a_{11}(t)e^{y_1(t-\tau_{11})}-\dfrac{a_{12}(t)e^{y_2(t)}}{1+m_1e^{y_1(t)}}\\ \vdots\\ -r_j(t)+\dfrac{a_{j,j-1}(t)e^{y_1j-1(t-\tau_{j,j-1})}}{1+m_{j-1}e^{y_{j-1}(t-\tau_{j,j-1})}}-a_{jj}(t)e^{y_j(t-\tau_{jj})}-\dfrac{a_{j,j+1}(t)e^{y_{j+1}(t)}}{1+m_je^{y_j(t)}}\\ \vdots\\ -r_n(t)+\dfrac{a_{n,n-1}(t)e^{y_{n-1}(t-\tau_{n,n-1})}}{1+m_{n-1}e^{y_{n-1}(t-\tau_{n,n-1})}}-a_{nn}(t)e^{y_n(t-\tau_{nn})}\end{pmatrix}.$$

Define two projectors P and Q as

$$P\begin{pmatrix}y_1\\ \vdots\\ y_j\\ \vdots\\ y_n\end{pmatrix}=Q\begin{pmatrix}y_1\\ \vdots\\ y_j\\ \vdots\\ y_n\end{pmatrix}=\begin{pmatrix}\frac{1}{\omega}\int_0^\omega y_1(t)dt\\ \vdots\\ \frac{1}{\omega}\int_0^\omega y_j(t)dt\\ \vdots\\ \frac{1}{\omega}\int_0^\omega y_n(t)dt\end{pmatrix},\quad\begin{pmatrix}y_1\\ \vdots\\ y_j\\ \vdots\\ y_n\end{pmatrix}\in X.$$

It is clear that

$$\operatorname{Ker}L=\{x|\ \ x\in X,\ \ x=h,\ \ h\in\mathbb{R}^n\},$$
$$\operatorname{Im}L=\{y|\ \ y\in Y,\ \ \int_0^\omega y(t)dt=0\}\ \text{ is closed in }\ Y,$$

and

$$\text{dim Ker}L=\text{codim Im}L=n.$$

Therefore, L is a Fredholm mapping of index zero. It is easy to show that P and Q are continuous projectors such that

$$\operatorname{Im}P=\operatorname{Ker}L,\quad\operatorname{Ker}Q=\operatorname{Im}L=\operatorname{Im}(I-Q).$$

Furthermore, the inverse K_P of L_P exists and is given by $K_P : \operatorname{Im} L \to \operatorname{Dom} L \cap \operatorname{Ker} P$,

$$K_P(y) = \int_0^t y(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^t y(s)dsdt.$$

Then $QN : X \to Y$ and $K_P(I-Q)N : X \to X$ read

$$QNx = \begin{bmatrix} \frac{1}{\omega}\int_0^\omega \left[r_1(t) - a_{11}(t)e^{y_1(t-\tau_{11})} - \frac{a_{12}(t)e^{y_2(t)}}{1+m_1e^{y_1(t)}} \right] dt \\ \vdots \\ \frac{1}{\omega}\int_0^\omega \left[-r_j(t) + \frac{a_{j,j-1}(t)e^{y_1 j-1(t-\tau_{j,j-1})}}{1+m_{j-1}e^{y_{j-1}(t-\tau_{j,j-1})}} - a_{jj}(t)e^{y_j(t-\tau_{jj})} - \frac{a_{j,j+1}(t)e^{y_{j+1}(t)}}{1+m_je^{y_j(t)}} \right] dt \\ \vdots \\ \frac{1}{\omega}\int_0^\omega \left[-r_n(t) + \frac{a_{n,n-1}(t)e^{y_{n-1}(t-\tau_{n,n-1})}}{1+m_{n-1}e^{y_{n-1}(t-\tau_{n,n-1})}} - a_{nn}(t)e^{y_n(t-\tau_{nn})} \right] dt \end{bmatrix},$$

$$K_P(I-Q)Nx = \int_0^t Nx(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^t Nx(s)dsdt - \left(\frac{t}{\omega} - \frac{1}{2}\right)\int_0^\omega Nx(s)ds.$$

Clearly, QN and $K_P(I-Q)N$ are continuous.

In order to apply Lemma 2.1, we need to search for an appropriate open, bounded subset Ω.

Corresponding to the operator equation $Lx = \lambda Nx$, $\lambda \in (0,1)$, we obtain

(2.4)
$$\begin{cases} \dot{y}_1(t) = \lambda \left[r_1(t) - a_{11}(t)e^{y_1(t-\tau_{11})} - \frac{a_{12}(t)e^{y_2(t)}}{1+m_1e^{y_1(t)}} \right], \\ \dot{y}_j(t) = \lambda \left[-r_j(t) + \frac{a_{j,j-1}(t)e^{y_{j-1}(t-\tau_{j,j-1})}}{1+m_{j-1}e^{y_{j-1}(t-\tau_{j,j-1})}} \right. \\ \qquad\qquad \left. -a_{jj}(t)e^{y_j(t-\tau_{jj})} - \frac{a_{j,j+1}(t)e^{y_{j+1}(t)}}{1+m_je^{y_j(t)}} \right], \ 1 < j < n, \\ \dot{y}_n(t) = \lambda \left[-r_n(t) + \frac{a_{n,n-1}(t)e^{y_{n-1}(t-\tau_{n,n-1})}}{1+m_{n-1}e^{y_{n-1}(t-\tau_{n,n-1})}} - a_{nn}(t)e^{y_n(t-\tau_{nn})} \right]. \end{cases}$$

Suppose that $(y_1(t), \ldots, y_n(t))^T \in X$ is a solution of system (2.4) for some $\lambda \in (0,1)$. Integrating system (2.4) over $[0,\omega]$ gives

(2.5)
$$\int_0^\omega a_{11}(t)e^{y_1(t-\tau_{11})}dt + \int_0^\omega \frac{a_{12}(t)e^{y_2(t)}}{1+m_1e^{y_1(t)}}dt = \int_0^\omega r_1(t)dt,$$

(2.6)
$$\int_0^\omega \frac{a_{j,j-1}(t)e^{y_{j-1}(t-\tau_{j,j-1})}}{1+m_{j-1}e^{y_{j-1}(t-\tau_{j,j-1})}}dt = \int_0^\omega r_j(t)dt + \int_0^\omega \frac{a_{j,j+1}(t)e^{y_{j+1}(t)}}{1+m_je^{y_j(t)}}dt + \int_0^\omega a_{jj}(t)e^{y_j(t-\tau_{jj})}dt, \quad j = 2, 3, \ldots, n-1,$$

(2.7)
$$\int_0^\omega r_n(t)dt + \int_0^\omega a_{nn}(t)e^{y_n(t-\tau_{nn})}dt = \int_0^\omega \frac{a_{n,n-1}(t)e^{y_{n-1}(t-\tau_{n,n-1})}}{1+m_{n-1}e^{y_{n-1}(t-\tau_{n,n-1})}}dt.$$

It follows from (2.5)-(2.7) that

$$\int_0^\omega |\dot{y}_1(t)|dt = \lambda \int_0^\omega \left| r_1(t) - a_{11}(t)e^{y_1(t-\tau_{11})} - \frac{a_{12}(t)e^{y_2(t)}}{1+m_1e^{y_1(t)}} \right| dt \tag{2.8}$$
$$\leq \int_0^\omega \left[r_1(t) + a_{11}(t)e^{y_1(t-\tau_{11})} + \frac{a_{12}(t)e^{y_2(t)}}{1+m_1e^{y_1(t)}} \right] dt$$
$$= 2\overline{r_1}\omega \triangleq d_1,$$

$$\int_0^\omega |\dot{y}_j(t)|dt \tag{2.9}$$
$$= \lambda \int_0^\omega \left| -r_j(t) + \frac{a_{j,j-1}(t)e^{y_{j-1}(t-\tau_{j,j-1})}}{1+m_{j-1}e^{y_{j-1}(t-\tau_{j,j-1})}} - a_{jj}(t)e^{y_j(t-\tau_{jj})} - \frac{a_{j,j+1}(t)e^{y_{j+1}(t)}}{1+m_je^{y_j(t)}} \right| dt$$
$$\leq \int_0^\omega \left[r_j(t) + \frac{a_{j,j-1}(t)e^{y_{j-1}(t-\tau_{j,j-1})}}{1+m_{j-1}e^{y_{j-1}(t-\tau_{j,j-1})}} + a_{jj}(t)e^{y_j(t-\tau_{jj})} + \frac{a_{j,j+1}(t)e^{y_{j+1}(t)}}{1+m_je^{y_j(t)}} \right] dt$$
$$\leq 2\frac{\overline{a_{j,j-1}}}{m_{j-1}}\omega \triangleq d_j, \quad j = 2, \dots, n-1,$$

$$\int_0^\omega |\dot{y}_n(t)|dt = \lambda \int_0^\omega \left| -r_n(t) + \frac{a_{n,n-1}(t)e^{y_{n-1}(t-\tau_{n,n-1})}}{1+m_{n-1}e^{y_{n-1}(t-\tau_{n,n-1})}} - a_{nn}e^{y_n(t-\tau_{nn})} \right| dt \tag{2.10}$$
$$\leq \int_0^\omega \left[r_n(t) + \frac{a_{n,n-1}(t)e^{y_{n-1}(t-\tau_{n,n-1})}}{1+m_{n-1}e^{y_{n-1}(t-\tau_{32})}} + a_{nn}e^{y_n(t-\tau_{nn})} \right] dt$$
$$\leq 2\frac{\overline{a_{n,n-1}}}{m_{n-1}}\omega \triangleq d_n.$$

Since $(y_1(t), \dots, y_n(t))^T \in X$, there exists t_i, T_i such that

$$y_i(t_i) = \min_{t\in[0,\omega]} y_i(t), \ \ y_i(T_i) = \max_{t\in[0,\omega]} y_i(t), \quad i = 1, \dots, n.$$

We derive from (2.5) that

$$\int_0^\omega a_{11}(t)e^{y_1(t-\tau_{11})}dt \leq \int_0^\omega r_1(t)dt,$$

which implies

$$y_1(t_1) \leq \ln\frac{\overline{r_1}}{\overline{a_{11}}} \triangleq \rho_1.$$

This, together with (2.8), leads to

$$y_1(t) \leq y_1(t_1) + \int_0^\omega |\dot{y}_1(t)|dt \tag{2.11}$$

$$\leq \ln\frac{\overline{r_1}}{\overline{a_{11}}} + 2\overline{r_1}\omega.$$

It follows from (2.6) and (2.7) that

$$\int_0^\omega a_{jj}(t)e^{y_j(t-\tau_{jj})}dt \leq \int_0^\omega \frac{a_{j,j-1}(t)e^{y_{j-1}(t-\tau_{j,j-1})}}{1+m_{j-1}e^{y_{j-1}(t-\tau_{j,j-1})}}dt - \overline{r_j}\omega$$
$$\leq \frac{\overline{a_{j,j-1}}}{m_{j-1}}\omega - \overline{r_j}\omega,$$

which implies

$$y_j(t_j) \le \ln \frac{\overline{a_{j,j-1}} - m_{j-1}\overline{r_j}}{m_{j-1}\overline{a_{jj}}} \triangleq \rho_j.$$

This, together with (2.8) and (2.9), leads to

$$\begin{aligned} y_j(t) &\le y_j(t_j) + \int_0^\omega |\dot{y}_j(t)|dt \\ &\le \ln \frac{\overline{a_{j,j-1}} - m_{j-1}\overline{r_j}}{m_{j-1}\overline{a_{jj}}} + 2\frac{\overline{a_{j,j-1}}}{m_{j-1}}\omega, \ j = 2, \dots, n. \end{aligned} \tag{2.12}$$

From (2.5) and (2.12) we obtain

$$\begin{aligned} \overline{a_{11}}e^{y_1(T_1)} &\ge \overline{r_1} - \frac{1}{\omega}\int_0^\omega a_{12}(t)e^{y_2(t)}dt \\ &\ge \overline{r_1} - \overline{a_{12}}\frac{\overline{(a_{21}/m_1)} - \overline{r_2}}{\overline{a_{22}}}e^{2\overline{(a_{21}/m_1)}\omega} \\ &= \overline{r_1} - K_1, \end{aligned} \tag{2.13}$$

that is

$$y_1(T_1) \ge \ln \frac{\overline{r_1} - K_1}{\overline{a_{11}}} \triangleq \delta_1. \tag{2.14}$$

It follows from (2.8) and (2.14) that

$$y_1(t) \ge y_1(T_1) - \int_0^\omega |\dot{y}_1(t)|dt \ge \ln \frac{\overline{r_1} - K_1}{\overline{a_{11}}} - 2\overline{r_1}\omega. \tag{2.15}$$

We obtain from (2.6) and (2.15) that

$$\begin{aligned} \overline{a_{22}}e^{y_2(T_2)} &\ge \frac{1}{\omega}\int_0^\omega \frac{a_{21}(t)e^{y_1(t-\tau_{21})}}{1 + m_1 e^{y_1(t-\tau_{21})}}dt - \overline{r_2} - \frac{1}{\omega}\int_0^\omega a_{23}(t)e^{y_3(t)}dt \\ &\ge \frac{\overline{a_{21}}(\frac{\overline{r_1} - K_1}{\overline{a_{11}}})e^{-2\overline{r_1}\omega}}{1 + m_1(\frac{\overline{r_1} - K_1}{\overline{a_{11}}})e^{-2\overline{r_1}\omega}} - K_2 \\ &= \frac{\overline{a_{21}}(\overline{r_1} - K_1)}{\overline{a_{11}}e^{2\overline{r_1}\omega} + m_1(\overline{r_1} - K_1)} - K_2 \\ &\triangleq \Delta_1 - K_2. \end{aligned} \tag{2.16}$$

If $\Delta_1 - K_2 > 0$ then

$$y_2(T_2) \ge \ln \frac{\Delta_1 - K_2}{\overline{a_{22}}}, \tag{2.17}$$

this, together with (2.9), leads to

$$\begin{aligned} y_2(t) &\ge y_2(T_2) - \int_0^\omega |\dot{y}_2(t)|dt \\ &\ge \ln \frac{\Delta_1 - K_2}{\overline{a_{22}}} - 2\frac{\overline{a_{21}}}{m_1}\omega. \end{aligned} \tag{2.18}$$

It follows from (2.7) and (2.18) that

$$\begin{aligned}\overline{a_{33}}e^{y_3(T_3)} &\geq \frac{1}{\omega}\int_0^{\omega}\frac{a_{32}(t)e^{y_2(t-\tau_{32})}}{1+m_2e^{y_2(t-\tau_{32})}}dt - \overline{r_3} - \int_0^{\omega}a_{34}(t)e^{u_4(t)}dt \\ &\geq \frac{\overline{a_{32}}(\Delta_1 - K_2)e^{-2\overline{(a_{21}/m_1)}\omega}/\overline{a_{22}}}{1+m_2^M(\Delta_1-K_2)e^{-2\overline{(a_{21}/m_1)}\omega}/\overline{a_{22}}} - K_3 \\ &\geq \frac{\overline{a_{32}}(\Delta_1-K_2)}{\overline{a_{22}}e^{2\overline{(a_{21}/m_1)}\omega}+m_2(\Delta_1-K_2)} - K_3 \\ &\triangleq \Delta_2 - K_3.\end{aligned}$$

If $\Delta_2 - K_3 > 0$, together with (2.9), it follows

$$y_3(t) \geq \ln\frac{\Delta_2 - K_3}{\overline{a_{33}}} - 2\frac{\overline{a_{32}}}{m_2}\omega \triangleq \delta_2 - 2\frac{\overline{a_{21}}}{m_1}\omega. \tag{2.19}$$

Similarly, we have

$$\begin{aligned}\overline{a_{jj}}e^{y_j(T_j)} &\geq \frac{\overline{a_{j,j-1}}(\Delta_{j-2}-K_{j-1})}{\overline{a_{j-1,j-1}}e^{2[\overline{a_{j-1,j-2}/m_{j-2}}]\omega}+m_{j-1}(\Delta_{j-2}-K_{j-1})} - K_j \\ &\triangleq \Delta_{j-1} - K_j.\end{aligned}$$

If $\Delta_{j-1} - K_j > 0$, we have

$$\begin{aligned}y_j(t) &\geq y_j(T_j) - \int_0^{\omega}|\dot{y}_j(t)|dt \\ &\geq \ln\frac{\Delta_{j-1}-K_j}{\overline{a_{jj}}} - 2\frac{\overline{a_{j,j-1}}}{m_{j-1}}\omega, \\ &\triangleq \delta_j - 2\frac{\overline{a_{j,j-1}}}{m_{j-1}}\omega,\end{aligned} \tag{2.20}$$

where

$$\Delta_j = \frac{\overline{a_{j+1,j}}(\Delta_{j-1}-K_j)}{\overline{a_{jj}}e^{\frac{2\overline{a_{j,j-1}}}{m_{j-1}}\omega}+m_j(\Delta_{j-1}-K_j)}, \quad 3 \leq j \leq n-1. \tag{2.21}$$

From (2.7) and (2.20), we have

$$\overline{a_{nn}}e^{y_n(T_n)} \geq \Delta_{n-1} - \overline{r_n}.$$

If $\Delta_{n-1} - \overline{r_n} > 0$, together with (2.10), it follows that

$$\begin{aligned}y_n(t) &\geq y_n(T_n) - \int_0^{\omega}|\dot{y}_n(t)|dt \\ &\geq \ln\frac{\Delta_{n-1}-\overline{r_n}}{\overline{a_{nn}}} - 2\frac{\overline{a_{n,n-1}}}{m_{n-1}}\omega \\ &\triangleq \delta_n - 2\frac{\overline{a_{n,n-1}}}{m_{n-1}}\omega.\end{aligned} \tag{2.22}$$

We note that (2.15), (2.18) – (2.20) and (2.22) hold if the following are true:

$$\overline{r_1} > K_1,\ \Delta_j > K_{j+1} (j = 1, 2, \ldots, n-2), \qquad \Delta_{n-1} > \overline{r_n}. \tag{2.23}$$

We now show that assumptions (H1) and (H2) imply (2.23).

Let (H1) and (H2) hold. Then we have

$$\overline{r_1} > K_1 + \frac{\overline{a_{11}} H_2 e^{2\overline{r_1}\omega}}{\overline{a_{21}} - m_1 \overline{r_2}}, \quad \overline{a_{21}} > m_1 H_2,$$

which implies

$$\overline{r_1} > K_1,\ \Delta_1 > H_2.$$

Noting that $\overline{a_{32}} - m_2\overline{r_3} > 0,\ \overline{a_{32}} - m_2 H_3 > 0$, we have

$$\overline{r_1} > K_1,\ \Delta_1 > K_2,\ \Delta_2 > H_3,$$

which, together with $\overline{a_{43}} - m_3\overline{r_4} > 0,\ \overline{a_{43}} - m_3 H_4 > 0$, leads to

$$\overline{r_1} > K_1,\ \Delta_1 > K_2,\ \Delta_2 > K_3,\ \Delta_3 > H_4.$$

Similarly, we have

$$\overline{r_1} > K_1, \quad \Delta_l > K_{l+1} \quad (1 \le l \le j-2), \quad \Delta_{j-1} > H_j,$$

which, together with $\overline{a_{j+1,j}} > m_j\overline{r_{j+1}},\ \overline{a_{j+1,j}} > m_j H_{j+1}$, implies

$$\overline{r_1} > K_1, \quad \Delta_l > K_{l+1} \quad (1 \le l \le j-1), \quad \Delta_j > H_{j+1}.$$

Finally, by a similar argument we obtain

$$\overline{r_1} > K_1, \quad \Delta_l > K_{l+1} \quad (1 \le l \le n-2), \quad \Delta_{n-1} > \overline{r_n}.$$

From what has been discussed above, we finally derive that

$$\max_{t\in[0,\omega]} |y_i(t)| \le \max\{|\rho_i| + d_i, |\delta_i| + d_i\} \triangleq B_i, \quad i = 1, \dots, n. \tag{2.24}$$

Clearly, $B_i\ (i = 1, 2, \dots, n)$ are independent of λ. Denote $B = \sum_{i=1}^n B_i + B_0$, here B_0 is taken sufficiently large such that each solution $(v_1^*, v_2^*, \dots, v_n^*)^T$ of the system of algebraic equations

$$\begin{cases} \overline{r_1} - \overline{a_{11}} e^{v_1} - \dfrac{1}{\omega}\displaystyle\int_0^\omega \frac{a_{12}(t)e^{v_2}}{1+m_1e^{v_1}}dt = 0, \\ -\overline{r_j} + \dfrac{1}{\omega}\displaystyle\int_0^\omega \frac{a_{j,j-1}(t)e^{v_{j-1}}}{1+m_{j-1}e^{v_{j-1}}}dt - \overline{a_{jj}}e^{v_j} - \dfrac{1}{\omega}\displaystyle\int_0^\omega \frac{a_{j,j+1}(t)e^{v_{j+1}}}{1+m_je^{v_j}}dt = 0, \\ \hfill 2 \le j \le n-1 \\ -\overline{r_n} + \dfrac{1}{\omega}\displaystyle\int_0^\omega \frac{a_{n,n-1}(t)e^{v_{n-1}}}{1+m_{n-1}e^{v_{n-1}}}dt - \overline{a_{nn}}e^{v_n} = 0, \end{cases} \tag{2.25}$$

satisfies $\|(v_1^*, v_2^*, \dots, v_n^*)^T\| = \sum_{i=1}^n |v_i^*| < B$ (if it exists). Now, we take $\Omega = \{(y_1, y_2, .., y_n)^T \in X : \|(y_1, y_2, \dots, y_n)^T\| < B\}$. Thus, the condition (a) of Lemma 2.1 is satisfied. When $(y_1, y_2, \dots, y_n)^T \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap \mathbb{R}^n$, $(y_1, y_2, \dots, y_n)^T$ is a constant vector in $\mathbb{R}^n$ with $\sum_{i=1}^n |y_i| = B$. If system (2.25) has solutions, then

$$QN\begin{pmatrix} y_1 \\ \vdots \\ y_j \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \overline{r_1} - \overline{a_{11}}e^{y_1} - \dfrac{1}{\omega}\displaystyle\int_0^\omega \frac{a_{12}(t)e^{y_2}}{1+m_1e^{y_1}}dt \\ \vdots \\ -\overline{r_j} + \dfrac{1}{\omega}\displaystyle\int_0^\omega \frac{a_{j,j-1}(t)e^{y_{j-1}}}{1+m_{j-1}e^{y_{j-1}}}dt - \overline{a_{jj}}e^{y_j} - \dfrac{1}{\omega}\displaystyle\int_0^\omega \frac{a_{j,j+1}(t)e^{y_{j+1}}}{1+m_je^{y_j}}dt \\ \vdots \\ -\overline{r_n} + \dfrac{1}{\omega}\displaystyle\int_0^\omega \frac{a_{n,n-1}(t)e^{y_{n-1}}}{1+m_{n-1}e^{y_{n-1}}}dt - \overline{a_{nn}}e^{y_n} \end{pmatrix} \ne \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

If system (2.25) does not have a solution, then we can directly derive

$$QN\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus, the condition (b) in Lemma 2.1 is satisfied.

In the following, we will prove that the condition (c) in Lemma 2.1 is satisfied. To this end, we define $\phi : \operatorname{Dom} L \times [0,1] \to X$ by

$$\begin{aligned}
&\phi(y_1,\dots,y_n,\mu)\\
&= \begin{pmatrix} \overline{r_1} - \overline{a_{11}}e^{y_1} \\ \vdots \\ -\overline{r_j} + \frac{1}{\omega}\int_0^\omega \frac{a_{j,j-1}(t)e^{y_{j-1}}}{1+m_{j-1}e^{y_{j-1}}}dt - \overline{a_{jj}}e^{y_j} \\ \vdots \\ -\overline{r_n} + \frac{1}{\omega}\int_0^\omega \frac{a_{n,n-1}(t)e^{y_{n-1}}}{1+m_{n-1}e^{y_{n-1}}}dt - \overline{a_{nn}}e^{y_n} \end{pmatrix} + \mu \begin{pmatrix} -\frac{1}{\omega}\int_0^\omega \frac{a_{12}(t)e^{y_2}}{1+m_1e^{y_1}}dt \\ \vdots \\ -\frac{1}{\omega}\int_0^\omega \frac{a_{j,j+1}(t)e^{y_{j+1}}}{1+m_je^{y_j}}dt \\ \vdots \\ 0 \end{pmatrix},
\end{aligned}$$

where μ is a parameter. When $(y_1,y_2,\dots,y_n)^T \in \partial\Omega \cap \mathbb{R}^n, (y_1,y_2,\dots,y_n)^T$ is a constant vector in $\mathbb{R}^n$ with $\sum_{i=1}^n |y_i| = B$. We will show that when

$$(y_1,y_2,\dots,y_n)^T \in \partial\Omega \cap \operatorname{Ker} L,\ \phi(y_1,y_2,\dots,y_n,\mu) \neq 0.$$

Otherwise, there is a constant vector $(y_1,\dots,y_n)^T \in \mathbb{R}^n$ with $\sum_{i=1}^n |y_i| = B$ satisfying $\phi(y_1,y_2,\dots,y_n,\mu) = 0$, that is

$$\begin{gathered}
\overline{r_1} - \overline{a_{11}}e^{y_1} - \mu\frac{1}{\omega}\int_0^\omega \frac{a_{12}(t)e^{y_2}}{1+m_1e^{y_1}}dt = 0,\\
-\overline{r_j} + \frac{1}{\omega}\int_0^\omega \frac{a_{j,j-1}(t)e^{y_{j-1}}}{1+m_{j-1}e^{y_{j-1}}}dt - \overline{a_{jj}}e^{y_j} - \mu\frac{1}{\omega}\int_0^\omega \frac{a_{j,j+1}(t)e^{y_{j+1}}}{1+m_je^{y_j}}dt = 0,\ 2 \le j \le n-1\\
-\overline{r_n} + \frac{1}{\omega}\int_0^\omega \frac{a_{n,n-1}(t)e^{y_{n-1}}}{1+m_{n-1}e^{y_{n-1}}}dt - \overline{a_{nn}}e^{y_n} = 0.
\end{gathered}$$

By some similar arguments in (2.11), (2.12), (2.14), (2.15), (2.17), (2.18), (2.20) and (2.22) we can show that

$$|y_i| \le \max\{|\delta_i|, |\rho_i|\}, i = 1,2,\dots,n.$$

Thus

$$\sum_{i=1}^n |y_i| \le \sum_{i=1}^n \max\{|\rho_i|, |\delta_i|\} < B,$$

which is leads to a contradiction. Using the property of topological degree and taking $J = I : \operatorname{Im} Q \to \operatorname{Ker} L, (y_1,y_2,\dots,y_n)^T \to (y_1,y_2,\dots,y_n)^T$, we have

$$\begin{aligned}
&\deg(JQN(y_1,\dots,y_n)^T, \Omega\cap\operatorname{Ker} L, (0,\dots,0)^T)\\
&= \deg(\phi(y_1,\dots,y_n,1), \Omega\cap\operatorname{Ker} L, (0,\dots,0)^T)\\
&= \deg(\phi(y_1,\dots,y_n,0), \Omega\cap\operatorname{Ker} L, (0,\dots,0)^T)
\end{aligned}$$

$$= \deg\left(\left((\overline{r_1} - \overline{a_{11}}e^{y_1}, \dots, -\overline{r_j} + \frac{1}{\omega}\int_0^\omega \frac{a_{j,j-1}(t)e^{y_{j-1}}}{1+m_{j-1}e^{y_{j-1}}}dt - \overline{a_{jj}}e^{y_j}, \dots,\right.\right.$$
$$\left.\left. -\overline{r_n} + \frac{1}{\omega}\int_0^\omega \frac{a_{n,n-1}(t)e^{y_{n-1}}}{1+m_{n-1}e^{y_{n-1}}}dt - \overline{a_{nn}}e^{y_n}\right)^T,\ \Omega \cap \operatorname{Ker} L, (0,\dots,0)^T\right).$$

Under assumptions (H1) – (H2), one can easily show that the following system of algebraic equations

$$\text{(2.26)} \qquad \begin{cases} \overline{r_1} - \overline{a_{11}}u_1 = 0, \\ -\overline{r_j} + \dfrac{\overline{a_{j,j-1}}u_{j-1}}{1+m_{j-1}u_{j-1}} - \overline{a_{jj}}u_j = 0, \qquad 2 \le j \le n-1 \\ -\overline{r_n} + \dfrac{\overline{a_{n,n-1}}u_{n-1}}{1+m_{n-1}u_{n-1}} - \overline{a_{nn}}u_n = 0, \end{cases}$$

has a unique solution $(u_1^*, \dots, u_n^*)^T$ which satisfies $u_i^* > 0, i = 1, \dots, n$.

A direct calculation shows that

$$\deg(JQN(y_1, y_2, \dots, y_n)^T, \Omega \cap \operatorname{Ker} L, (0,0,\dots,0)^T) = \operatorname{sgn}\left\{\prod_{i=1}^n (-\overline{a_{ii}})\right\} = (-1)^n \neq 0.$$

Finally, it is easy to show that the set $\{K_P(I-Q)Nu | u \in \bar{\Omega}\}$ is equicontinuous and uniformly bounded. By using the Arzela-Ascoli theorem, we see that $K_P(I-Q)N : \overline{\Omega} \to X$ is compact. Moreover, $QN(\bar{\Omega})$ is bounded. Consequently, N is $L-$compact.

By now we have proved that Ω satisfies all the requirements in Lemma 2.1. Hence, system (2.3) has at least one ω-periodic solution. As a consequence, system (1.1) has at least one positive ω-periodic solution. This completes the proof. □

3. GLOBAL ATTRACTIVITY

We now proceed to a discussion on the global attractivity of the positive $\omega-$periodic solution of system (1.1). It is immediate that if any positive periodic solution of system (1.1) is globally attractive, then it is in fact unique. We first derive certain upper and lower bound estimates for solutions of (1.1) – (1.2).

Lemma 3.1. *Let $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ denote any positive solution of system (1.1) with initial conditions (1.2). Then there exists a $T > 0$ such that*

$$\text{(3.1)} \qquad 0 < x_i \le M_i \ \ for \ \ t > T, \ \ i = 1, 2, \dots, n,$$

where

$$\text{(3.2)} \qquad M_1 = \frac{r_1^M}{a_{11}^L} e^{r_1^M \tau_1},$$
$$M_j = \frac{a_{j,j-1}^M/m_{j-1} - r_j^L}{a_{jj}^L} e^{[a_{j,j-1}M/m_{j-1} - r_j^L]\tau_{jj}}, \quad j = 2, 3, \dots, n.$$

The proof of Lemma 3.1 is similar to that of Lemma 2.1 in [22], we therefore omit it here.

We now formulate the global attractivity of the positive $\omega-$periodic solutions of system (1.1) as follows.

Theorem 3.2. *In addition to (H1) – (H2), assume further that*

$$\text{(H3)} \qquad \liminf_{t\to\infty} A_i(t) > 0, i = 1, 2, \dots, n,$$

where

$$A_1(t) = a_{11}(t) - m_1 a_{12}(t) M_2 - a_{21}(t+\tau_{21}) - [r_1(t) + a_{11}(t)M_1 + a_{12}(t)M_2]\int_t^{t+\tau_{11}} a_{11}(s)ds - a_{11}(t+\tau_1)M_1\int_{t+\tau_{11}}^{t+2\tau_{11}} a_{11}(s)ds - a_{21}(t+\tau_{21})M_2\int_{t+\tau_{21}}^{t+\tau_{21}+\tau_{22}} a_{21}(s)ds, \tag{3.3}$$

$$\begin{aligned} A_j(t) = {} & a_{jj}(t) - m_j a_{j,j+1}(t)M_{j+1} - a_{j+1,j}(t+\tau_{j+1,j}) \\ & - \frac{a_{j_1,j}(t)}{m_{j-1}}\int_t^{t+\tau_{j-1,j-1}} a_{j-1,j-1}(s)ds \\ & - \left[r_j(t) + \frac{a_{j,j-1}(t)}{m_{j-1}} + a_{jj}(t)M_j + a_{j,j+1}(t)M_{j+1}\right]\int_t^{t+\tau_{jj}} a_{jj}(s)ds \\ & - a_{jj}(t+\tau_{jj})M_j\int_{t+\tau_{jj}}^{t+2\tau_{jj}} a_{jj}(s)ds \\ & - a_{j+1,j}(t+\tau_{j+1,j})M_{j+1}\int_{t+\tau_{j+1,j}}^{t+\tau_{j+1,j}+\tau_{j+1,j+1}} a_{j+1,j+1}(s)ds, \ 2 \le j \le n-1, \end{aligned} \tag{3.4}$$

$$\begin{aligned} A_n(t) = {} & a_{nn}(t) - a_{n-1,n}(t) \\ & - \left[r_n(t) + \frac{a_{n,n-1}(t)}{m_{n-1}} + a_{nn}(t)M_n\right]\int_t^{t+\tau_{nn}} a_{nn}(s)ds \\ & - a_{nn}(t+\tau_{nn})M_n\int_{t+\tau_{nn}}^{t+2\tau_{nn}} a_{nn}(s)ds \\ & - \frac{a_{n-1,n}(t)}{m_{n-1}}\int_t^{t+\tau_{n-1,n-1}} a_{n-1,n-1}(s)ds. \end{aligned} \tag{3.5}$$

Then system (1.1) has a unique positive $\omega-$*periodic solution* $x^*(t) = (x_1^*(t), \dots, x_n^*(t))^T$ *which is globally attractive.*

Proof. Due to the conclusion of Theorem 2.2, we need only to show the global attractivity of the positive periodic solution of (1.1). Let $x^*(t) = (x_1^*(t), \dots, x_n^*(t))^T$ be a positive $\omega-$ periodic solution of (1.1), and $y(t) = (y_1(t), \dots, y_n(t))^T$ be any positive solution of system (1.1) – (1.2). It follows from Lemma 3.1 that there exist positive constants T and M_i (defined by (3.2)) such that for all $t \ge T$,

$$0 < x_i^*(t) \le M_i, \ \ 0 < y_i(t) \le M_i, \quad i = 1, 2, \dots, n. \tag{3.6}$$

We define

$$V_{11}(t) = |\ln x_1^*(t) - \ln y_1(t)|. \tag{3.7}$$

Calculating the upper right derivative of $V_{11}(t)$ along solution of (1.1), it follows for $t \geq T$ that

$$
\begin{aligned}
(3.8)\quad & D^+V_{11} \\
&= \left(\frac{\dot{x}_1^*(t)}{x_1^*(t)} - \frac{\dot{y}_1(t)}{y_1(t)}\right) \operatorname{sgn}(x_1^*(t) - y_1(t)) \\
&= \operatorname{sgn}(x_1^*(t) - y_1(t)) \Bigg[- a_{11}(t)(x_1^*(t-\tau_{11}) - y_1(t-\tau_{11})) \\
&\qquad\qquad - \frac{a_{12}(t)x_2^*(t)}{1+m_1x_1^*(t)} + \frac{a_{12}(t)y_2(t)}{1+m_1y_1(t)}\Bigg] \\
&= \operatorname{sgn}(x_1^*(t) - y_1(t)) \Bigg[-a_{11}(t)(x_1^*(t) - y_1(t)) - \frac{a_{12}(t)(x_2^*(t) - y_2(t))}{1+m_1x_1^*(t)} \\
&\quad + \frac{m_1a_{12}(t)y_2(t)(x_1^*(t) - y_1(t))}{(1+m_1x_1^*(t))(1+m_1y_1(t))} + a_{11}(t)\int_{t-\tau_{11}}^{t} (\dot{x}_1^*(u) - \dot{y}_1(u))du\Bigg] \\
&\leq -a_{11}(t)|x_1^*(t) - y_1(t)| + a_{12}|x_2^*(t) - y_2(t)| + m_1a_{12}(t)y_2(t)|x_1^*(t) - y_1(t)| \\
&\quad + a_{11}(t)\left|\int_{t-\tau_{11}}^{t} (\dot{x}_1^*(u) - \dot{y}_1(u))du\right| \\
&= -a_{11}(t)|x_1^*(t) - y_1(t)| + a_{12}(t)|x_2^*(t) - y_2(t)| + m_1a_{12}(t)y_2(t)|x_1^*(t) - y_1(t)| \\
&\quad + a_{11}(t)\left|\int_{t-\tau_{11}}^{t} \left\{x_1^*(u)\left[r_1(u) - a_{11}(u)x_1^*(u-\tau_{11}) - \frac{a_{12}(u)x_2^*(u)}{1+m_1x_1^*(u)}\right]\right.\right. \\
&\quad \left.\left. - y_1(u)\left[r_1(u) - a_{11}(u)y_1(u-\tau_{11}) - \frac{a_{12}(u)y_2(u)}{1+m_1y_1(u)}\right]\right\} du\right| \\
&= -a_{11}(t)|x_1^*(t) - y_1(t)| + a_{12}(t)|x_2^*(t) - y_2(t)| + m_1a_{12}(t)y_2(t)|x_1^*(t) - y_1(t)| \\
&\quad + a_{11}(t)\left|\int_{t-\tau_{11}}^{t} \left\{\left[r_1(u) - a_{11}(u)y_1(u-\tau_{11})\right.\right.\right. \\
&\qquad \left. - \frac{a_{12}(u)y_2(u)}{(1+m_1x_1^*(u))(1+m_1y_1(u))}\right](x_1^*(t) - y_1(t)) \\
&\qquad - a_{11}(u)x_1^*(u)(x_1^*(u-\tau_{11}) - y_1(u-\tau_{11})) \\
&\qquad \left.\left. -\frac{a_{12}(u)x_1^*(u)}{1+m_1x_1^*(u)}(x_2^*(u) - y_2^*(u))\right\} du\right| \\
&\leq -a_{11}(t)|x_1^*(t) - y_1(t)| + a_{12}(t)|x_2^*(t) - y_2(t)| + m_1a_{12}(t)y_2(t)|x_1^*(t) - y_1(t)| \\
&\quad + a_{11}(t)\left|\int_{t-\tau_{11}}^{t} \left\{[r_1(u) + a_{11}(u)y_1(u-\tau_{11}) + a_{12}(u)y_2(u)]|x_1^*(u) - y_1(u)|\right.\right. \\
&\quad \left.\left. + a_{11}(u)x_1^*(u)|x_1^*(u-\tau_{11}) - y_1(u-\tau_{11})| + \frac{a_{12}(u)}{m_1}|x_2^*(u) - y_2(u)|\right\} du\right|.
\end{aligned}
$$

We derive from (3.6) and (3.8) that for $t \geq T + \tau$

$$
\begin{aligned}
(3.9)\quad & D^+V_{11}(t) \\
&\leq -a_{11}(t)|x_1^*(t) - y_1(t)| + a_{12}(t)|x_2^*(t) - y_2(t)| + m_1a_{12}(t)M_2|x_1^*(t) - y_1(t)|
\end{aligned}
$$

$$+ a_{11}(t)\left|\int_{t-\tau_{11}}^{t}\left\{[r_1(u)+a_{11}(u)M_1+a_{12}(u)M_2]|x_1^*(u)-y_1(u)|\right.\right.$$
$$\left.\left.+ a_{11}(u)M_1|x_1^*(u-\tau_{11})-y_1(u-\tau_{11})| + \frac{a_{12}(u)}{m_1}|x_2^*(u)-y_2(u)|\right\} du\right|.$$

Define

$$(3.10)\quad V_{12}(t) = \int_t^{t+\tau_{11}}\int_{s-\tau_{11}}^{t} a_{11}(s)\Big\{[r_1(u)+a_{11}(u)M_1+a_{12}(u)M_2]|x_1^*(u)-y_1(u)|$$
$$+a_{11}(u)M_1|x_1^*(u-\tau_{11})-y_1(u-\tau_{11})| + \frac{a_{12}(u)}{m_1}|x_2^*(u)-y_2(u)|\Big\}\}duds.$$

It follows from (3.9) and (3.10) that for $t \geq T+\tau$

$$(3.11)\quad D^+(V_{11}(t)+V_{12}(t))$$
$$\leq -a_{11}(t)|x_1^*(t)-y_1(t)| + a_{12}(t)|x_2^*(t)-y_2(t)| + m_1a_{12}(t)M_2|x_1^*(t)-y_1(t)|$$
$$+\int_t^{t+\tau_{11}} a_{11}(s)ds\Big\{[r_1(t)+a_{11}(t)M_1+a_{12}(t)M_2]|x_1^*(t)-y_1(t)|$$
$$+ a_{11}(t)M_1|x_1^*(t-\tau_{11})-y_1(t-\tau_{11})| + \frac{a_{12}(t)}{m_1}|x_2^*(t)-y_2(t)|\Big\}.$$

We now define

$$(3.12)\quad V_1(t) = V_{11}(t)+V_{12}(t)+V_{13}(t),$$

where

$$(3.13)\quad V_{13}(t) = M_1\int_{t-\tau_{11}}^{t}\int_{l+\tau_{11}}^{l+2\tau_{11}} a_{11}(s)a_{11}(l+\tau_{11})|x_1^*(l)-y_1(l)|dsdl.$$

It then follows from (3.11) – (3.13) that for $t \geq T+\tau$

$$(3.14)\quad D^+V_1(t)$$
$$\leq -a_{11}(t)|x_1^*(t)-y_1(t)| + a_{12}(t)|x_2^*(t)-y_2(t)| + m_1a_{12}(t)M_2|x_1^*(t)-y_1(t)|$$
$$+\int_t^{t+\tau_{11}} a_{11}(s)ds\Big\{[r_1(t)+a_{11}(t)M_1+a_{12}(t)M_2]|x_1^*(t)-y_1(t)|$$
$$+ \frac{a_{12}(t)}{m_1}|x_2^*(t)-y_2(t)|\Big\} + a_{11}(t+\tau_{11})M_1\int_{t+\tau_{11}}^{t+2\tau_{11}}|x_1^*(t)-y_1(t)|.$$

Define

$$(3.15)\quad V_j(t) = |\ln x_j^*(t) - \ln y_j(t)| + \int_{t-\tau_{j,j-1}}^{t} a_{j,j-1}(s+\tau_{j,j-1})|x_{j-1}^*(s)-y_{j-1}(s)|ds$$
$$+\int_t^{t+\tau_{jj}}\int_{s-\tau_{jj}}^{t} a_{jj}(s)\left\{\Big[r_j(u)\right.$$

$$
\begin{aligned}
&+ \frac{a_{j,j-1}(u)}{m_{j-1}} + a_{jj}(u)M_j + a_{j,j+1}(u)M_{j+1}\Bigg] |x_j^*(u) - y_j(u)| \\
&+ a_{jj}(u)M_j|x_j^*(u-\tau_{jj}) - y_j(u-\tau_{jj})| \\
&+ a_{j,j-1}(u)M_j|x_{j-1}^*(u-\tau_{j,j-1}) - y_{j-1}(u-\tau_{j,j-1})| \\
&+\frac{a_{j,j+1}(u)}{m_j}|x_{j+1}^*(u) - y_{j+1}(u))|\Bigg\} duds \\
&+ M_j\int_{t-\tau_{jj}}^{t}\int_{l+\tau_{jj}}^{l+2\tau_{jj}} a_{jj}(s)a_{jj}(l+\tau_{jj})|x_j^*(l) - y_j(l)|dsdl \\
&+ M_j\int_{t-\tau_{j,j-1}}^{t}\int_{l+\tau_{j,j-1}}^{l+\tau_{j,j-1}+\tau_{jj}} a_{jj}(s)a_{j,j-1}(l+\tau_{j,j-1})|x_{j-1}^*(l) - y_{j-1}(l)|dsdl, \\
&\qquad j = 2, 3, \ldots, n-1.
\end{aligned}
$$

$$
\begin{aligned}
V_n(t) &= |\ln x_n^*(t) - \ln y_n(t)| \\
&\quad + \int_{t-\tau_{n,n-1}}^{t} a_{n,n-1}(s+\tau_{n,n-1})|x_{n-1}^*(s) - y_{n-1}(s)|ds \\
&\quad + \int_{t}^{t+\tau_{nn}}\int_{s-\tau_{nn}}^{t} a_{nn}(s)\Bigg\{\Bigg[r_n(u) \\
&\quad + \frac{a_{n,n-1}(u)}{m_{n-1}} + a_{nn}(u)M_n\Bigg] |x_n^*(u) - y_n(u)| \\
&\quad + M_n a_{nn}(u)|x_n^*(u-\tau_{nn}) - y_n(u-\tau_{nn})| \\
&\quad + M_n a_{n,n-1}(u)|x_{n-1}^*(u-\tau_{n,n-1}) - y_{n-1}(u-\tau_{n,n-1})|\Bigg\} duds \\
&\quad + M_n\int_{t-\tau_{nn}}^{t}\int_{l+\tau_{nn}}^{l+2\tau_{nn}} a_{nn}(s)a_{nn}(l+\tau_{nn})|x_n(l) - y_n(l)|dsdl \\
&\quad + M_n\int_{t-\tau_{n,n-1}}^{t}\int_{l+\tau_{n,n-1}}^{l+\tau_{n,n-1}+\tau_{nn}} a_{nn}(s)a_{n,n-1}(l+\tau_{n,n-1})) \\
&\qquad\qquad \times |x_{n-1}^*(l) - y_{n-1}(l)|dsdl.
\end{aligned} \tag{3.16}
$$

Finally, we define

$$V(t) = \sum_{i=1}^{n} V_i(t).$$

We derive from (3.14) – (3.16) that for $t \geq T + \tau$

$$\frac{dV(t)}{dt} \leq -\sum_{i=1}^{n} A_i(t)|x_i^*(t) - y_i(t)|, \tag{3.17}$$

where $A_i(t)(i = 1, \ldots, n)$ are as defined in (3.3) – (3.5).

By hypothesis (H3), there exist constants $\alpha_i > 0$ $(i = 1, \ldots, n)$ and $T^* \geq T + \tau$, such that

$$A_i(t) \geq \alpha_i > 0 \text{ for } t \geq T^*. \tag{3.18}$$

Integrating both sides of (3.17) on $[T^*, t]$, we derive

$$V(t) + \sum_{i=1}^{n}\int_{T^*}^{t} A_i(t)|x_i^*(s) - y_i(s)|ds \leq V(T^*). \tag{3.19}$$

It follows from (3.18) and (3.19) that

$$V(t) + \sum_{i=1}^{n} \alpha_i \int_{T^*}^{t} |x_i^*(s) - y_i(s)| ds \leq V(T^*) \text{ for } t \geq T^*$$

Therefore, $V(t)$ is bounded on $[T^*, \infty]$ and also $\int_{T^*}^{t} |x_i^*(s) - y_i(s)| ds < \infty, \ \ i = 1, \dots, n.$ On the other hand, by Lemma 3.1, $|x_i^*(t) - y_i(t)|$ are bounded on $[T^*, \infty)$. According to system (1.1), we see that $\dot{x}_i^*(t)$ and $\dot{y}_i(t)$ are also bounded. Hence, $|x_i^*(t) - y_i(t)|$ $(i = 1, \dots, n)$ are uniformly continuous on $[T^*, \infty)$. By Barbalat's Lemma (see [15]), we can conclude that

$$\lim_{t \to +\infty} |x_i^*(t) - y_i(t)| = 0, \ \ i = 1, \dots, n.$$

The proof is complete. □

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