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ENERGY DECAY OF SOLUTIONS OF A WAVE EQUATION OF p-LAPLACIAN TYPE WITH A WEAKLY NONLINEAR DISSIPATION

ABBÈS BENAISSA AND SALIMA MIMOUNI

UNIVERSITÉ DJILLALI LIABÈS
FACULTÉ DES SCIENCES
DÉEPARTEMENT DE MATHÉMATIQUES
B. P. 89, SIDI BEL ABBÈS 22000, ALGERIA.
benaissa_abbes@yahoo.com

bbsalima@yahoo.fr

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ABSTRACT. In this paper we study decay properties of the solutions to the wave equation of $p-$Laplacian type with a weak nonlinear dissipative.

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1. INTRODUCTION

We consider the initial boundary problem for the nonlinear wave equation of $p-$Laplacian type with a weak nonlinear dissipation of the type

$$\text{(P)}\qquad \begin{cases} (|u'|^{l-2}u')' - \Delta_p u + \sigma(t)g(u') = 0 \text{ in } \Omega \times [0, +\infty[, \\ u = 0 \text{ on } \partial\Omega \times [0, +\infty[, \\ u(x, 0) = u_0(x), \;\; u'(x, 0) = u_1(x) \text{ in } \Omega. \end{cases}$$

where $\Delta_p u = div(|\nabla_x u|^{p-2}\nabla_x u)$, $p,\ l \geq 2$, $g : \mathbb{R} \to \mathbb{R}$ is a continuous non-decreasing function and σ is a positive function.

When $p = 2, l = 2$ and $\sigma \equiv 1$, for the case $g(x) = \delta x$ $(\delta > 0)$, Ikehata and Suzuki [5] investigated the dynamics, showing that for sufficiently small initial data (u_0, u_1), the trajectory $(u(t), u'(t))$ tends to $(0, 0)$ in $H_0^1(\Omega) \times L^2(\Omega)$ as $t \to +\infty$. When $g(x) = \delta|x|^{m-1}x$ $(m \geq 1)$, Nakao [8] investigated the decay property of the problem (P). In [8] the author has proved the existence of global solutions to the problem (P).

For the problem (P) with $\sigma \equiv 1, l = 2$, when $g(x) = \delta|x|^{m-1}x$ $(m \geq 1)$, Yao [1] proved that the energy decay rate is $E(t) \leq (1 + t)^{-\frac{p}{(mp-m-1)}}$ for $t \geq 0$ by using a general method

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introduced by Nakao [8]. Unfortunately, this method does not seem to be applicable in the case of more general functions σ and is more complicated.

Our purpose in this paper is to give energy decay estimates of the solutions to the problem (P) for a weak nonlinear dissipation. We extend the results obtained by Yao and prove in some cases an exponential decay when $p > 2$ and the dissipative term is not necessarily superlinear near the origin.

We use a new method recently introduced by Martinez [7] (see also [2]) to study the decay rate of solutions to the wave equation $u'' - \Delta_x u + g(u') = 0$ in $\Omega \times \mathbb{R}^+$, where Ω is a bounded domain of $\mathbb{R}^n$. This method is based on a new integral inequality that generalizes a result of Haraux [4].

Throughout this paper the functions considered are all real valued. We omit the space variable x of $u(t,x)$, $u_t(t,x)$ and simply denote $u(t,x)$, $u_t(t,x)$ by $u(t)$, $u'(t)$, respectively, when no confusion arises. Let l be a number with $2 \le l \le \infty$. We denote by $\|\cdot\|_l$ the L^l norm over Ω. In particular, the L^2 norm is denoted by $\|\cdot\|_2$. $(\cdot)$ denotes the usual L^2 inner product. We use familiar function spaces $W_0^{1,p}$.

2. Preliminaries and Main Results

First assume that the solution exists in the class

$$u \in C(\mathbb{R}_+, W_0^{1,p}(\Omega)) \cap C^1(\mathbb{R}_+, L^l(\Omega)). \tag{2.1}$$

$\lambda(x), \sigma(t)$ and g satisfy the following hypotheses:

(H1) $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ is a non increasing function of class C^1 on $\mathbb{R}_+$ satisfying

$$\int_0^{+\infty} \sigma(\tau) d\tau = +\infty. \tag{2.2}$$

(H2) Consider $g : \mathbb{R} \to \mathbb{R}$ a non increasing C^0 function such that

$$g(v)v > 0 \text{ for all } v \neq 0.$$

and suppose that there exist $c_i > 0$; $i = 1,2,3,4$ such that

$$c_1|v|^m \le |g(v)| \le c_2|v|^{\frac{1}{m}} \text{ if } |v| \le 1, \tag{2.3}$$

$$c_3|v|^s \le |g(v)| \le c_4|v|^r \text{ for all } |v| \ge 1, \tag{2.4}$$

where $m \ge 1$, $l - 1 \le s \le r \le \frac{n(p-1)+p}{n-p}$.

We define the energy associated to the solution given by (2.1) by the following formula

$$E(t) = \frac{l-1}{l}\|u'\|_l^l + \frac{1}{p}\|\nabla_x u\|_p^p.$$

We first state two well known lemmas, and then state and prove a lemma that will be needed later.

Lemma 2.1 (Sobolev-Poincaré inequality). *Let q be a number with $2 \le q < +\infty$ $(n = 1,2,\dots,p)$ or $2 \le q \le \frac{np}{(n-p)}$ $(n \ge p+1)$, then there is a constant $c_* = c(\Omega, q)$ such that*

$$\|u\|_q \le c_*\|\nabla u\|_p \quad \textit{for} \quad u \in W_0^{1,p}(\Omega).$$

Lemma 2.2 ([6]). *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and assume that there are two constants $q \ge 0$ and $A > 0$ such that*

$$\int_S^{+\infty} E^{q+1}(t)\, dt \le AE(S), \quad 0 \le S < +\infty,$$

then we have

$$E(t) \le cE(0)(1+t)^{\frac{-1}{q}} \ \ \forall t \ge 0, \quad \textit{if} \ \ q > 0$$

and

$$E(t) \le cE(0)e^{-\omega t} \ \ \forall t \ge 0, \quad \textit{if} \ \ q = 0,$$

where c and ω are positive constants independent of the initial energy $E(0)$.

Lemma 2.3 **([7]).** *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non increasing function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing C^2 function such that*

$$\phi(0) = 0 \quad \textit{and} \quad \phi(t) \to +\infty \ \ \textit{as} \ \ t \to +\infty.$$

Assume that there exist $q \ge 0$ and $A > 0$ such that

$$\int_S^{+\infty} E(t)^{q+1}(t)\phi'(t)\,dt \le AE(S), \quad 0 \le S < +\infty,$$

then we have

$$E(t) \le cE(0)(1+\phi(t))^{\frac{-1}{q}} \ \ \forall t \ge 0, \quad \textit{if} \ \ q > 0$$

and

$$E(t) \le cE(0)e^{-\omega\phi(t)} \ \ \forall t \ge 0, \quad \textit{if} \ \ q = 0,$$

where c and ω are positive constants independent of the initial energy $E(0)$.

Proof of Lemma 2.3. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $f(x) := E(\phi^{-1}(x))$. f is non-increasing, $f(0) = E(0)$ and if we set $x := \phi(t)$ we obtain

$$\begin{aligned}\int_{\phi(S)}^{\phi(T)} f(x)^{q+1}\,dx &= \int_{\phi(S)}^{\phi(T)} E\left(\phi^{-1}(x)\right)^{q+1}\,dx \\ &= \int_S^T E(t)^{q+1}\phi'(t)\,dt \\ &\le AE(S) = Af(\phi(S)) \quad 0 \le S < T < +\infty.\end{aligned}$$

Setting $s := \phi(S)$ and letting $T \to +\infty$, we deduce that

$$\int_s^{+\infty} f(x)^{q+1}\,dx \le Af(s) \quad 0 \le s < +\infty.$$

By Lemma 2.2, we can deduce the desired results. □

Our main result is the following

Theorem 2.4. *Let $(u_0, u_1) \in W_0^{1,p} \times L^l(\Omega)$ and suppose that (**H1**) and (**H2**) hold. Then the solution $u(x,t)$ of the problem (P) satisfies*

(1) *If $l \ge m+1$, we have*

$$E(t) \le C(E(0)) \exp\left(1 - \omega \int_0^t \sigma(\tau)\,d\tau\right) \quad \forall t > 0.$$

(2) *If $l < m+1$, we have*

$$E(t) \le \left(\frac{C(E(0))}{\int_0^t \sigma(\tau)\,d\tau}\right)^{\frac{p}{(mp-m-1)}} \quad \forall t > 0.$$

Examples

1) If $\sigma(t) = \dfrac{1}{t^\theta}$ $(0 \leq \theta \leq 1)$, by applying Theorem 2.4 we obtain

$$E(t) \leq C(E(0))e^{1-\omega t^{1-\theta}} \quad \text{if } \theta \in [0,1[,\ l \geq m+1,$$

$$E(t) \leq C(E(0))t^{-\frac{(1-\theta)p}{mp-m-1}} \quad \text{if } 0 \leq \theta < 1,\ l < m+1$$

and

$$E(t) \leq C(E(0))(\ln t)^{-\frac{p}{(mp-m-1)}} \quad \text{if } \theta = 1,\ l < m+1.$$

2) If $\sigma(t) = \dfrac{1}{t^\theta \ln t \ln_2 t \dots \ln_k t}$, where k is a positive integer and

$$\begin{cases} \ln_1(t) = \ln(t) \\ \ln_{k+1}(t) = \ln(\ln_k(t)), \end{cases}$$

by applying Theorem 2.4, we obtain

$$E(t) \leq C(E(0))(\ln_{k+1} t)^{-\frac{p}{(mp-m-1)}} \quad \text{if } \theta = 1,\ l < m+1,$$

$$E(t) \leq C(E(0))t^{-\frac{(1-\theta)p}{mp-m-1}} (\ln t \ln_2 t \dots \ln_k t)^{\frac{p}{mp-m-1}} \quad \text{if } 0 \leq \theta < 1,\ l < m+1.$$

3) If $\sigma(t) = \dfrac{1}{t^\theta (\ln t)^\gamma}$, by applying Theorem 2.4, we obtain

$$E(t) \leq C(E(0))t^{-\frac{(1-\theta)p}{mp-m-1}} (\ln t)^{\frac{\gamma p}{mp-m-1}} \quad \text{if } 0 \leq \theta < 1,\ l < m+1,$$

$$E(t) \leq C(E(0))(\ln t)^{-\frac{(1-\gamma)p}{mp-m-1}} \quad \text{if } \theta = 1, 0 \leq \gamma < 1,\ l < m+1,$$

$$E(t) \leq C(E(0))(\ln_2 t)^{-\frac{p}{mp-m-1}} \quad \text{if } \theta = 1, \gamma = 1,\ l < m+1.$$

Proof of Theorem 2.4.

First we have the following energy identity to the problem (P)

Lemma 2.5 (Energy identity)**.** *Let $u(t,x)$ be a local solution to the problem (P) on $[0,\infty)$ as in Theorem 2.4. Then we have*

$$E(t) + \int_\Omega \int_0^t \sigma(s) u'(s) g(u'(s))\, ds\, dx = E(0)$$

for all $t \in [0,\infty)$.

Proof of the energy decay. From now on, we denote by c various positive constants which may be different at different occurrences. We multiply the first equation of (P) by $E^q \phi' u$, where ϕ is a function satisfying all the hypotheses of Lemma 2.3 to obtain

$$\begin{aligned} 0 &= \int_S^T E^q \phi' \int_\Omega u((|u'|^{l-2}u')_t - \Delta_p u + \sigma(t) g(u'))\, dx\, dt \\ &= \left[E^q \phi' \int_\Omega u u' |u'|^{l-2}\, dx \right]_S^T - \int_S^T (qE'E^{q-1}\phi' + E^q \phi'') \int_\Omega u u' |u'|^{l-2}\, dxdt \\ &\qquad - \frac{3l-2}{l} \int_S^T E^q \phi' \int_\Omega |u'|^2\, dxdt + 2 \int_S^T E^q \phi' \int_\Omega \left(\frac{l-1}{l} u'^2 + \frac{1}{p} |\nabla u|^p \right) dxdt \\ &\qquad + \int_S^T E^q \phi' \int_\Omega \sigma(t) u g(u')\, dxdt + \left(1 - \frac{2}{p}\right) \int_S^T E^q \phi' \|\nabla u\|_p^p\, dxdt. \end{aligned}$$

We deduce that

$$(2.5)\quad 2\int_S^T E^{q+1}\phi'\,dt \le -\left[E^q\phi'\int_\Omega uu'|u'|^{l-2}\,dx\right]_S^T + \int_S^T (qE'E^{q-1}\phi' + E^q\phi'')\int_\Omega uu'|u'|^{l-2}\,dxdt + \frac{3l-2}{l}\int_S^T E^q\phi'\int_\Omega |u'|^l\,dxdt - \int_S^T E^q\phi'\int_\Omega \sigma(t)ug(u')\,dxdt.$$

Since E is nonincreasing, ϕ' is a bounded nonnegative function on $\mathbb{R}_+$ (and we denote by μ its maximum), using the Hölder inequality, we have

$$\left|E(t)^q\phi'\int_\Omega uu'|u'|^{l-2}\,dx\right| \le c\mu E(S)^{q+\frac{l-1}{l}+\frac{1}{p}} \qquad \forall t \ge S.$$

$$\int_S^T (qE'E^{q-1}\phi' + E^q\phi'')\int_\Omega uu'|u'|^{l-2}\,dxdt \le c\mu\int_S^T -E'(t)E(t)^{q-\frac{1}{l}+\frac{1}{p}}\,dt + c\int_S^T E(t)^{q+\frac{l-1}{l}+\frac{1}{p}}(-\phi''(t))\,dt \le c\mu E(S)^{q+\frac{l-1}{l}+\frac{1}{p}}.$$

Using these estimates we conclude from the above inequality that

$$(2.6)\quad 2\int_S^T E(t)^{1+q}\phi'(t)\,dt \le cE(S)^{q+\frac{l-1}{l}+\frac{1}{p}} + \frac{3l-2}{l}\int_S^T E^q\phi'\int_\Omega |u'|^l\,dxdt - \int_S^T E^q\phi'\int_\Omega \sigma(t)ug(u')\,dxdt \le cE(S)^{q+\frac{l-1}{l}+\frac{1}{p}} + \frac{3l-2}{l}\int_S^T E^q\phi'\int_\Omega |u'|^l\,dxdt - \int_S^T E^q\phi'\int_{|u'|\le 1}\sigma(t)ug(u')\,dxdt - \int_S^T E^q\phi'\int_{|u'|>1}\sigma(t)ug(u')\,dxdt.$$

Define

$$\phi(t) = \int_0^t \sigma(s)\,ds.$$

It is clear that ϕ is a non decreasing function of class C^2 on $\mathbb{R}_+$. The hypothesis (2.2) ensures that

$$(2.7)\qquad \phi(t) \to +\infty \text{ as } t \to +\infty.$$

Now, we estimate the terms of the right-hand side of (2.6) in order to apply the results of Lemma 2.3:

Using the Hölder inequality, we get for $l < m+1$

$$\int_S^T E^q\phi'\int_\Omega |u'|^l\,dxdt \le C\int_S^T E^q\phi'\int_\Omega \frac{1}{\sigma(t)}u'\rho(t,u')\,dx\,dt + C'\int_S^T E^q\phi'\int_\Omega \left(\frac{1}{\sigma(t)}u'\rho(t,u')\right)^{\frac{l}{(m+1)}}\,dx\,dt$$

$$\leq C \int_S^T E^q \frac{\phi'}{\sigma(t)}(-E')\,dt + C'(\Omega)\int_S^T E^q \frac{\phi'}{\sigma^{\frac{l}{m+1}}(t)}(-E')^{\frac{l}{m+1}}\,dt$$
$$\leq CE^{q+1}(S) + C'(\Omega)\int_S^T E^q \phi'^{\frac{m+1-l}{m+1}}\left(\frac{\phi'}{\sigma(t)}\right)^{\frac{l}{m+1}}(-E')^{\frac{l}{m+1}}\,dt.$$

Now, fix an arbitrarily small $\varepsilon > 0$ (to be chosen later). By applying Young's inequality, we obtain

$$(2.8)\quad \int_S^T E^q\phi' \int_\Omega |u'|^l\,dxdt \leq CE^{q+1}(S) + C'(\Omega)\frac{m+-l}{m+1}\varepsilon^{\frac{(m+1)}{(m+1-l)}}\int_S^T E^{q\frac{m+1}{m+1-l}}\phi'\,dt + C'(\Omega)\frac{l}{m+1}\frac{1}{\varepsilon^{\frac{(m+1)}{l}}}E(S).$$

If $l \geq m+1$, we easily obtain from (2.3) and (2.4)

$$(2.9)\quad \int_S^T E^q\phi'\int_\Omega |u'|^l\,dxdt \leq CE^{q+1}(S).$$

Next, we estimate the third term of the right-hand of (2.6). We get for $l < m+1$

$$(2.10)\quad \int_S^T E^q\phi'\int_{|u'|\leq 1}\sigma(t)ug(u')\,dxdt$$
$$\leq \varepsilon_1\int_S^T E^q\phi'\int_{|u'|\leq 1}\|u\|_p^p\,dt + C(\varepsilon_1)\int_S^T E^q\phi'\int_{|u'|\leq 1}(\sigma g(u'))^{\frac{p}{p-1}}\,dx$$
$$\leq c\varepsilon_1\int_S^T E^{q+1}\phi'\,dt + C(\varepsilon_1)\int_S^T E^q\phi'\int_{|u'|\leq 1}(\sigma g(u'))^{\frac{p}{p-1}}\,dx.$$

We now estimate the last term of the above inequality to get

$$(2.11)\quad \int_S^T E^q\phi'\int_{|u'|\leq 1}(\sigma g(u'))^{\frac{p}{p-1}}\,dx\,dt$$
$$\leq \int_S^T E^q\phi'\int_{|u'|\leq 1}(u'g(u'))^{\frac{p}{(m+1)(p-1)}}\,dx\,dt$$
$$\leq \int_S^T E^q\phi'\frac{1}{\sigma^{\frac{p}{(m+1)(p-1)}}}\int_{|u'|\leq 1}(\sigma u'g(u'))^{\frac{p}{(m+1)(p-1)}}\,dx\,dt$$
$$\leq C(\Omega)\int_S^T E^q\phi'\frac{1}{\sigma^{\frac{p}{(m+1)(p-1)}}}(-E')^{\frac{p}{(m+1)(p-1)}}\,dt.$$

Set $\varepsilon_2 > 0$; due to Young's inequality, we obtain

$$(2.12)\quad \int_S^T E^q\phi'\int_{|u'|\leq 1}(\sigma g(u'))^{\frac{p}{p-1}}dxdt$$
$$\leq C(\Omega)\frac{(m+1)(p-1)-p}{(m+1)(p-1)}\varepsilon_2^{\frac{(m+1)(p-1)}{(m+1)(p-1)-p}}\int_S^T E^{q\frac{(m+1)(p-1)}{(m+1)(p-1)-p}}\phi' dt + \frac{C(\Omega)\,p}{(m+1)(p-1)}\frac{1}{\varepsilon_2^{\frac{(m+1)(p-1)}{p}}}E(S),$$

we chose q such that

$$q\frac{(m+1)(p-1)}{(m+1)(p-1)-p} = q+1.$$

thus we find $q = \frac{mp-m-1}{p}$ and thus $q\frac{m+1}{m+1-l} = q+1+\alpha$ with $\alpha = \frac{(m+1)(p\,l-p-l)}{p(m+1-l)}$.

Using the Hölder inequality, the Sobolev imbedding and the condition (2.4), we obtain

$$\begin{aligned}
&\int_S^T E^q \phi' \int_{|u'|\geq 1} \sigma(t) u g(u')\,dxdt \\
&\quad\leq \int_S^T E^q \phi' \sigma(t) \left(\int_\Omega |u|^{r+1}\,dx\right)^{\frac{1}{(r+1)}} \left(\int_{|u'|>1} |g(u')|^{\frac{r+1}{r}}\,dx\right)^{\frac{r}{r+1}} dt \\
&\quad\leq c\int_S^T E^{q+\frac{1}{p}} \phi' \sigma^{\frac{1}{(r+1)}}(t) \left(\int_{|u'|>1} \sigma u' g(u')\,dx\right)^{\frac{r}{r+1}} dt \\
&\quad\leq c\int_S^T E^{q+\frac{1}{p}} \phi' \sigma^{\frac{1}{(r+1)}}(t)(-E')^{\frac{r}{r+1}}\,dt.
\end{aligned}$$

Applying Young's inequality, we obtain

$$\begin{aligned}
(2.13)\qquad &\int_S^T E^q \phi' \int_{|u'|\geq 1} \sigma(t) u g(u')\,dxdt \\
&\quad\leq \varepsilon_3 \int_S^T (E^{q+\frac{1}{p}} \phi' \sigma^{\frac{1}{(r+1)}}(t))^{r+1}\,dt + c(\varepsilon_3)\int_S^T (-E')\,dt \\
&\quad\leq \varepsilon_3 \mu^{r+1} E^{\frac{(p-1)(mr-1)}{p}}(0) \int_S^T E^{q+1}\phi'\,dt + c(\varepsilon_3)E(S).
\end{aligned}$$

If $l \geq m+1$, the last inequality is also valid in the domain $\{|u'|<1\}$ and with m instead of r.

Choosing $\varepsilon, \varepsilon_1, \varepsilon_2$ and ε_3 small enough, we deduce from (2.6), (2.8), (2.10), (2.12) and (2.13) for $l < m+1$

$$\begin{aligned}
\int_S^T E(t)^{1+q}\phi'(t)\,dt \leq CE(S)^{q+1} + C'E(S)^{q+\frac{l-1}{l}+\frac{1}{p}} + C''E(S) \\
+ C'''E(0)^{\frac{(p\,l-p-l)(m+1)}{p\,l}}E(S) + C''''E(0)^{\frac{(m\,r-1)(p-1)}{p\,r}}E(S),
\end{aligned}$$

where C, C', C'', C''', C'''' are different positive constants independent of $E(0)$.

Choosing ε_3 small enough, we deduce from (2.6), (2.9) and (2.13) for $l \geq m+1$

$$\int_S^T E(t)^{1+q}\phi'(t)\,dt \leq CE(S)^{q+1} + C'E(S)^{q+\frac{l-1}{l}+\frac{1}{p}} + C''E(0)^{\frac{(m^2-1)(p-1)}{p\,m}}E(S),$$

where C, C', C'' are different positive constants independent of $E(0)$, we may thus complete the proof by applying Lemma 2.3. □

Remark 2.6. We obtain the same results for the following problem

$$\begin{cases}
(|u'|^{l-2}u')' - e^{-\Phi(x)}div(e^{\Phi(x)}|\nabla_x u|^{p-2}\nabla_x u) + \sigma(t)g(u') = 0 \text{ in } \Omega\times[0,+\infty[, \\
u = 0 \text{ on } \partial\Omega\times[0,+\infty[, \\
u(x,0) = u_0(x),\ \ u'(x,0) = u_1(x) \text{ in } \Omega,
\end{cases}$$

where Φ is a positive function such that $\Phi \in L^\infty(\Omega)$, in this case $(u_0, u_1) \in W^{1,p}_{0,\Phi} \times L^l_\Phi$, where

$$W^{1,p}_{0,\Phi}(\Omega) = \left\{ u \in W^{1,p}_0(\Omega), \quad \int_\Omega e^{\Phi(x)} |\nabla_x u|^p \, dx < \infty \right\},$$

$$L^l_\Phi(\Omega) = \left\{ u \in L^l(\Omega), \quad \int_\Omega e^{\Phi(x)} |u|^l \, dx < \infty \right\}.$$

Thus the energy associated to the solution is given by the following formula

$$E(t) = \frac{l-1}{l} \| e^{\Phi(x)/l} u' \|_l^l + \frac{1}{p} \| e^{\Phi(x)/p} \nabla_x u \|_p^p.$$

REFERENCES

[1] YAO-JUN YE, On the decay of solutions for some nonlinear dissipative hyperbolic equations, *Acta Math. Appl. Sin. Engl. Ser.*, **20**(1) (2004), 93–100.

[2] A. BENAISSA AND S. MOKEDDEM, Global existence and energy decay of solutions to the Cauchy problem for a wave equation with a weakly nonlinear dissipation, *Abstr. Appl. Anal.*, **11** (2004), 935–955.

[3] Y. EBIHARA, M. NAKAO AND T. NAMBU, On the existence of global classical solution of initial boundary value problem for $u'' - \Delta u - u^3 = f$, *Pacific J. of Math.*, **60** (1975), 63–70.

[4] A. HARAUX, Two remarks on dissipative hyperbolic problems, in: *Research Notes in Mathematics*, Pitman, 1985, p. 161–179.

[5] R. IKEHATA AND T. SUZUKI, Stable and unstable sets for evolution equations of parabolic and hyperbolic type, *Hiroshima Math. J.*, **26** (1996), 475–491.

[6] V. KOMORNIK, *Exact Controllability and Stabilization. The Multiplier Method*, Masson-John Wiley, Paris, 1994.

[7] P. MARTINEZ, A new method to decay rate estimates for dissipative systems, *ESAIM Control Optim. Calc. Var.*, **4** (1999), 419–444.

[8] M. NAKAO, A difference inequality and its applications to nonlinear evolution equations, *J. Math. Soc. Japan*, **30** (1978), 747–762.

[9] M. NAKAO, On solutions of the wave equations with a sublinear dissipative term, *J. Diff. Equat.*, **69** (1987), 204–215.