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COEFFICIENT INEQUALITY FOR A FUNCTION WHOSE DERIVATIVE HAS A POSITIVE REAL PART

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ABSTRACT. Let $\mathcal{R}$ denote the subclass of normalised analytic univalent functions f defined by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and satisfy

$$\mathrm{Re}\{f'(z)\} > 0$$

where $z \in \mathcal{D} = \{z : |z| < 1\}$. The object of the present paper is to introduce the functional $|a_2 a_4 - a_3^2|$. For $f \in \mathcal{R}$, we give sharp upper bound for $|a_2 a_4 - a_3^2|$.

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of normalised analytic functions f of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{1.1}$$

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where $z \in \mathcal{D} = \{z : |z| < 1\}$. In [9], Noonan and Thomas stated that the qth Hankel determinant of f is defined for $q \geq 1$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q+1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q+2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

Now, let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.2}$$

which are univalent in $\mathcal{D}$.

A classical theorem of Fekete and Szegö [1] considered the Hankel determinant of $f \in \mathcal{S}$ for $q = 2$ and $n = 1$,

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}.$$

They made an early study for the estimates of $|a_3 - \mu a_2^2|$ when $a_1 = 1$ and μ real. The well-known result due to them states that if $f \in \mathcal{S}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 4\mu - 3, & \text{if} \quad \mu \geq 1, \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), & \text{if} \quad 0 \leq \mu \leq 1, \\ 3 - 4\mu, & \text{if} \quad \mu \leq 0. \end{cases}$$

Hummel [3, 4] proved the conjecture of V. Singh that $|a_3 - a_2^2| \leq \frac{1}{3}$ for the class $\mathcal{C}$ of convex functions. Keogh and Merkes [5] obtained sharp estimates for $|a_3 - \mu a_2^2|$ when f is close-to-convex, starlike and convex in $\mathcal{D}$.

Here, we consider the Hankel determinant of $f \in \mathcal{S}$ for $q = 2$ and $n = 2$,

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

Now, we are working on the functional $|a_2 a_4 - a_3^2|$. In this earlier work, we find a sharp upper bound for the functional $|a_2 a_4 - a_3^2|$ for $f \in \mathcal{R}$. The subclass $\mathcal{R}$ is defined as the following.

Definition 1.1. Let f be given by (1.2). Then $f \in \mathcal{R}$ if it satisfies the inequality

$$\operatorname{Re}\{f'(z)\} > 0, \quad (z \in \mathcal{D}). \tag{1.3}$$

The subclass $\mathcal{R}$ was studied systematically by MacGregor [8] who indeed referred to numerous earlier investigations involving functions whose derivative has a positive real part.

We first state some preliminary lemmas which shall be used in our proof.

2. PRELIMINARY RESULTS

Let $\mathcal{P}$ be the family of all functions p analytic in $\mathcal{D}$ for which $\operatorname{Re}\{p(z)\} > 0$ and

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots \tag{2.1}$$

for $z \in \mathcal{D}$.

Lemma 2.1 ([10]). *If $p \in \mathcal{P}$ then $|c_k| \leq 2$ for each k.*

Lemma 2.2 ([2]). *The power series for $p(z)$ given in (2.1) converges in $\mathcal{D}$ to a function in $\mathcal{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, \qquad n = 1, 2, 3, \dots \tag{2.2}$$

and $c_{-k} = \bar{c}_k$, are all nonnegative. They are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k} z)$, $\rho_k > 0,\ t_k$ real and $t_k \neq t_j$ for $k \neq j$; in this case $D_n > 0$ for $n < m - 1$ and $D_n = 0$ for $n \geq m$.

This necessary and sufficient condition is due to Carathéodory and Toeplitz and can be found in [2].

3. MAIN RESULT

Theorem 3.1. *Let $f \in \mathbb{R}$. Then*

$$|a_2 a_4 - a_3^2| \leq \frac{4}{9}.$$

The result obtained is sharp.

Proof. We refer to the method by Libera and Zlotkiewicz [6, 7]. Since $f \in \mathcal{R}$, it follows from (1.3) that

$$f'(z) = p(z) \tag{3.1}$$

for some $z \in \mathcal{D}$. Equating coefficients in (3.1) yields

$$\begin{cases} 2a_2 = c_1 \\ 3a_3 = c_2 \\ 4a_4 = c_3 \end{cases}. \tag{3.2}$$

From (3.2), it can be easily established that

$$|a_2 a_4 - a_3^2| = \left| \frac{c_1 c_3}{8} - \frac{c_2^2}{9} \right|.$$

We make use of Lemma 2.2 to obtain the proper bound on $\left| \frac{c_1 c_3}{8} - \frac{c_2^2}{9} \right|$. We may assume without restriction that $c_1 > 0$. We begin by rewriting (2.2) for the cases $n = 2$ and $n = 3$.

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ c_1 & 2 & c_1 \\ \bar{c}_2 & c_1 & 2 \end{vmatrix} = 8 + 2\,\mathrm{Re}\{c_1^2 c_2\} - 2|c_2|^2 - 4c_1^2 \geq 0,$$

which is equivalent to

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{3.3}$$

for some x, $|x| \leq 1$. Then $D_3 \geq 0$ is equivalent to

$$|(4c_3 - 4c_1 c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2\left(4 - c_1^2\right)^2 - 2\left|2c_2 - c_1^2\right|^2;$$

and this, with (3.3), provides the relation

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z, \tag{3.4}$$

for some value of $z,\ |z| \leq 1$.

Suppose, now, that $c_1 = c$ and $c \in [0,2]$. Using (3.3) along with (3.4) we get

$$\left|\frac{c_1c_3}{8} - \frac{c_2^2}{9}\right| = \left|\frac{c^4}{288} + \frac{c^2(4-c^2)x}{144} - \frac{(4-c^2)(32+c^2)x^2}{288} + \frac{c(4-c^2)(1-|x|^2)z}{16}\right|$$

and an application of the triangle inequality shows that

$$\begin{aligned} &\left|\frac{c_1c_3}{8} - \frac{c_2^2}{9}\right| \\ &\le \frac{c^4}{288} + \frac{c(4-c^2)}{16} + \frac{c^2(4-c^2)\rho}{144} + \frac{(c-2)(c-16)(4-c^2)\rho^2}{288} \\ &= F(\rho) \end{aligned} \tag{3.5}$$

with $\rho = |x| \le 1$. We assume that the upper bound for (3.5) attains at the interior point of $\rho \in [0,1]$ and $c \in [0,2]$, then

$$F'(\rho) = \frac{c^2(4-c^2)}{144} + \frac{(c-2)(c-16)(4-c^2)\rho}{144}.$$

We note that $F'(\rho) > 0$ and consequently F is increasing and $Max_\rho\ F(\rho) = F(1)$, which contadicts our assumption of having the maximum value at the interior point of $\rho \in [0,1]$. Now let

$$G(c) = F(1) = \frac{c^4}{288} + \frac{c(4-c^2)}{16} + \frac{c^2(4-c^2)}{144} + \frac{(c-2)(c-16)(4-c^2)}{288},$$

then

$$G'(c) = \frac{-c(5+c^2)}{36} = 0$$

implies $c = 0$ which is a contradiction. Observe that

$$G''(c) = \frac{-5-3c^2}{36} < 0.$$

Thus any maximum points of G must be on the boundary of $c \in [0,2]$. However, $G(c) \ge G(2)$ and thus G has maximum value at $c = 0$. The upper bound for (3.5) corresponds to $\rho = 1$ and $c = 0$, in which case

$$\left|\frac{c_1c_3}{8} - \frac{c_2^2}{9}\right| \le \frac{4}{9}.$$

Equality is attained for functions in $\mathcal{R}$ given by

$$f'(z) = \frac{1+z^2}{1-z^2}.$$

This concludes the proof of our theorem. □

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