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INTEGRAL INEQUALITIES OF THE OSTROWSKI TYPE

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ABSTRACT. Integral inequalities of Ostrowski type are developed for $n-$times differentiable mappings, with multiple branches, on the L_∞ norm. Some particular inequalities are also investigated, which include explicit bounds for perturbed trapezoid, midpoint, Simpson's, Newton-Cotes and left and right rectangle rules. The results obtained provide sharper bounds than those obtained by Dragomir [5] and Cerone, Dragomir and Roumeliotis [2].

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1. INTRODUCTION

In 1938 Ostrowski [18] obtained a bound for the absolute value of the difference of a function to its average over a finite interval. The theorem is as follows.

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on $[a,b]$ and let $|f'(t)| \leq M$ for all $t \in (a,b)$, then the following bound is valid*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq (b-a)\,M \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] \tag{1.1}$$

for all $x \in [a,b]$.
The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

Dragomir and Wang [12, 13] extended the result (1.1) and applied the extended result to numerical quadrature rules and to the estimation of error bounds for some special means.

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Dragomir [8, 9, 10] further extended the result (1.1) to incorporate mappings of bounded variation, Lipschitzian mappings and monotonic mappings.

Cerone, Dragomir and Roumeliotis [3] as well as Dedić, Matić and Pečarić [4] and Pearce, Pečarić, Ujević and Varošanec [19] further extended the result (1.1) by considering $n-$times differentiable mappings on an interior point $x \in [a, b]$.

In particular, Cerone and Dragomir [1] proved the following result.

Theorem 1.2. *Let $f : [a, b] \to \mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$, (AC$[a, b]$ for short). Then for all $x \in [a, b]$ the following bound is valid:*

$$\left| \int_a^b f(t)\,dt - \sum_{j=0}^{n-1} \left(\frac{(b-x)^{j+1} + (-1)^j (x-a)^{j+1}}{(j+1)!} \right) f^{(j)}(x) \right| \le \frac{\left\| f^{(n)} \right\|_\infty}{(n+1)!} \left((x-a)^{n+1} + (b-x)^{n+1} \right) \ \text{ if } f^{(n)} \in L_\infty [a, b], \tag{1.2}$$

where

$$\left\| f^{(n)} \right\|_\infty := \sup_{t \in [a,b]} \left| f^{(n)}(t) \right| < \infty.$$

Dragomir also generalised the Ostrowski inequality for k points, $x_1, \ldots, x_k$ and obtained the following theorem.

Theorem 1.3. *Let $I_k := a = x_0 < x_1 < ... < x_{k-1} < x_k = b$ be a division of the interval $[a, b]$, α_i $(i = 0, ..., k+1)$ be "$k+2$" points so that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, ..., k)$ and $\alpha_{k+1} = b$. If $f : [a, b] \to \mathbb{R}$ is AC$[a, b]$, then we have the inequality*

$$\begin{aligned} \left| \int_a^b f(x)\,dx - \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) \right| &\le \left[\frac{1}{4} \sum_{i=0}^{k-1} h_i^2 + \sum_{i=0}^{k-1} \left(\alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right)^2 \right] \|f'\|_\infty \\ &\le \frac{1}{2} \|f'\|_\infty \sum_{i=0}^{k-1} h_i^2 \\ &\le \frac{1}{2} (b-a) \|f'\|_\infty \nu(h), \end{aligned} \tag{1.3}$$

where $h_i := x_{i+1} - x_i$ $(i = 0, ..., k-1)$ and $\nu(h) := \max \{h_i | i = 0, ..., k-1\}$.
The constant $\frac{1}{4}$ in the first inequality and the constant $\frac{1}{2}$ in the second and third inequalities are the best possible.

The main aim of this paper is to develop the upcoming Theorem 3.1 (see page 10) of integral inequalities for $n-$times differentiable mappings. The motivation for this work has been to improve the order of accuracy of the results given by Dragomir [5], and Cerone and Dragomir [1]. In the case of Dragomir [5], the bound of (1.3) is of order 1, whilst in this paper we improve the bound of (1.3) to order n.

We begin the process by obtaining the following integral equalities.

2. Integral Identities

Theorem 2.1. *Let $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ be a division of the interval $[a, b]$ and α_i $(i = 0, \ldots, k+1)$ be '$k+2$' points so that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, \ldots, k)$ and*

$\alpha_{k+1} = b$. If $f : [a,b] \to \mathbb{R}$ is a mapping such that $f^{(n-1)}$ is AC$[a,b]$, then for all $x_i \in [a,b]$ we have the identity:

$$\int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!} \sum_{i=0}^{k-1} \Big\{ (x_{i+1} - \alpha_{i+1})^j f^{(j-1)}(x_{i+1}) - (x_i - \alpha_{i+1})^j f^{(j-1)}(x_i) \Big\} = (-1)^n \int_a^b K_{n,k}(t) f^{(n)}(t)\,dt, \tag{2.1}$$

where the Peano kernel

$$K_{n,k}(t) := \begin{cases} \dfrac{(t-\alpha_1)^n}{n!}, & t \in [a, x_1) \\ \dfrac{(t-\alpha_2)^n}{n!}, & t \in [x_1, x_2) \\ \vdots & \\ \dfrac{(t-\alpha_{k-1})^n}{n!}, & t \in [x_{k-2}, x_{k-1}) \\ \dfrac{(t-\alpha_k)^n}{n!}, & t \in [x_{k-1}, b], \end{cases} \tag{2.2}$$

n and k are natural numbers, $n \geq 1$, $k \geq 1$ and $f^{(0)}(x) = f(x)$.

Proof. The proof is by mathematical induction. For $n = 1$, from (2.1) we have the equality

$$\begin{aligned} \int_a^b f(t)\,dt &= \sum_{i=0}^{k-1} \Big[(x_{i+1} - \alpha_{i+1})^j f(x_{i+1}) - (x_i - \alpha_{i+1})^j f(x_i) \Big] \\ &\quad - \int_a^b K_{1,k}(t) f'(t)\,dt, \end{aligned} \tag{2.3}$$

where

$$K_{1,k}(t) := \begin{cases} (t-\alpha_1), & t \in [a, x_1) \\ (t-\alpha_2), & t \in [x_1, x_2) \\ \vdots & \\ (t-\alpha_{k-1}), & t \in [x_{k-2}, x_{k-1}) \\ (t-\alpha_k), & t \in [x_{k-1}, b]. \end{cases}$$

To prove (2.3), we integrate by parts as follows

$$\begin{aligned} &\int_a^b K_{1,k}(t) f'(t)\,dt \\ &\qquad = \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} (t - \alpha_{i+1}) f'(t)\,dt \\ &\qquad = \sum_{i=0}^{k-1} \left[(t - \alpha_{i+1}) f(t) \Big|_{x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} f(t)\,dt \right] \end{aligned}$$

$$= \sum_{i=0}^{k-1} \left[(x_{i+1} - \alpha_{i+1}) f(x_{i+1}) - (x_i - \alpha_{i+1}) f(x_i) \right] - \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} f(t)\, dt.$$

$$\int_a^b f(t)\, dt + \int_a^b K_{1,k}(t) f'(t)\, dt = \sum_{i=0}^{k-1} \left[(x_{i+1} - \alpha_{i+1}) f(x_{i+1}) - (x_i - \alpha_{i+1}) f(x_i) \right].$$

Hence (2.3) is proved.

Assume that (2.1) holds for 'n' and let us prove it for '$n+1$'. We need to prove the equality

$$\int_a^b f(t)\, dt + \sum_{i=0}^{k-1} \sum_{j=1}^{n+1} \frac{(-1)^j}{j!} \left\{ (x_{i+1} - \alpha_{i+1})^j f^{(j-1)}(x_{i+1}) - (x_i - \alpha_{i+1})^j f^{(j-1)}(x_i) \right\} = (-1)^{n+1} \int_a^b K_{n+1,k}(t) f^{(n+1)}(t)\, dt, \tag{2.4}$$

where from (2.2)

$$K_{n+1,k}(t) := \begin{cases} \frac{(t-\alpha_1)^{n+1}}{(n+1)!}, & t \in [a, x_1) \\ \frac{(t-\alpha_2)^{n+1}}{(n+1)!}, & t \in [x_1, x_2) \\ \vdots & \\ \frac{(t-\alpha_{k-1})^{n+1}}{(n+1)!}, & t \in [x_{k-2}, x_{k-1}) \\ \frac{(t-\alpha_k)^{n+1}}{(n+1)!}, & t \in [x_{k-1}, b]. \end{cases}$$

Consider

$$\int_a^b K_{n+1,k}(t) f^{(n+1)}(t)\, dt = \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \frac{(t-\alpha_{i+1})^{n+1}}{(n+1)!} f^{(n+1)}(t)\, dt$$

and upon integrating by parts we have

$$\begin{aligned} &\int_a^b K_{n+1,k}(t) f^{(n+1)}(t)\, dt \\ &= \sum_{i=0}^{k-1} \left[\frac{(t-\alpha_{i+1})^{n+1}}{(n+1)!} f^{(n)}(t) \Big|_{x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} \frac{(t-\alpha_{i+1})^n}{n!} f^{(n)}(t)\, dt \right] \\ &= \sum_{i=0}^{k-1} \left\{ \frac{(x_{i+1}-\alpha_{i+1})^{n+1}}{(n+1)!} f^{(n)}(x_{i+1}) - \frac{(x_i-\alpha_{i+1})^{n+1}}{(n+1)!} f^{(n)}(x_i) \right\} \\ &\qquad - \int_a^b K_{n,k}(t) f^{(n)}(t)\, dt. \end{aligned}$$

Upon rearrangement we may write

$$\begin{aligned} \int_a^b K_{n,k}(t) f^{(n)}(t)\, dt = \sum_{i=0}^{k-1} &\left\{ \frac{(x_{i+1}-\alpha_{i+1})^{n+1}}{(n+1)!} f^{(n)}(x_{i+1}) - \frac{(x_i-\alpha_{i+1})^{n+1}}{(n+1)!} f^{(n)}(x_i) \right\} \\ &- \int_a^b K_{n+1,k}(t) f^{(n+1)}(t)\, dt. \end{aligned}$$

Now substitute $\int_a^b K_{n,k}(t) f^{(n)}(t)\,dt$ from the induction hypothesis (2.1) such that

$$(-1)^n \int_a^b f(t)\,dt + (-1)^n \sum_{i=0}^{k-1} \left[\sum_{j=1}^{n} \frac{(-1)^j}{j!} \left\{ (x_{i+1} - \alpha_{i+1})^j f^{(j-1)}(x_{i+1}) - (x_i - \alpha_{i+1})^j f^{(j-1)}(x_i) \right\} \right]$$
$$= \sum_{i=0}^{k-1} \left\{ \frac{(x_{i+1} - \alpha_{i+1})^{n+1}}{(n+1)!} f^{(n)}(x_{i+1}) - \frac{(x_i - \alpha_{i+1})^{n+1}}{(n+1)!} f^{(n)}(x_i) \right\} - \int_a^b K_{n+1,k}(t) f^{(n+1)}(t)\,dt.$$

Collecting the second and third terms and rearranging, we can state

$$\int_a^b f(t)\,dt + \sum_{i=0}^{k-1} \sum_{j=1}^{n+1} \frac{(-1)^j}{j!} \left\{ (x_{i+1} - \alpha_{i+1})^j f^{(j-1)}(x_{i+1}) - (x_i - \alpha_{i+1})^j f^{(j-1)}(x_i) \right\}$$
$$= (-1)^{n+1} \int_a^b K_{n+1,k}(t) f^{(n+1)}(t)\,dt,$$

which is identical to (2.4), hence Theorem 2.1 is proved. □

The following corollary gives a slightly different representation of Theorem 2.1, which will be useful in the following work.

Corollary 2.2. *From Theorem 2.1, the equality (2.1) may be represented as*

$$\int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!} \left[\sum_{i=0}^{k} \left\{ (x_i - \alpha_i)^j - (x_i - \alpha_{i+1})^j \right\} f^{(j-1)}(x_i) \right] = (-1)^n \int_a^b K_{n,k}(t) f^{(n)}(t)\,dt. \tag{2.5}$$

Proof. From (2.1) consider the second term and rewrite it as

$$S_1 + S_2 := \sum_{i=0}^{k-1} \left\{ -(x_{i+1} - \alpha_{i+1}) f(x_{i+1}) + (x_i - \alpha_{i+1}) f(x_i) \right\} + \sum_{i=0}^{k-1} \left[\sum_{j=2}^{n} \frac{(-1)^j}{j!} \left\{ (x_{i+1} - \alpha_{i+1})^j f^{(j-1)}(x_{i+1}) - (x_i - \alpha_{i+1})^j f^{(j-1)}(x_i) \right\} \right]. \tag{2.6}$$

Now

$$\begin{aligned} S_1 &= (a-\alpha_1) f(a) + \sum_{i=1}^{k-1} (x_i - \alpha_{i+1}) f(x_i) \\ &\quad + \sum_{i=0}^{k-2} \{-(x_{i+1} - \alpha_{i+1}) f(x_{i+1})\} - (b-\alpha_k) f(b) \\ &= (a-\alpha_1) f(a) + \sum_{i=1}^{k-1} (x_i - \alpha_{i+1}) f(x_i) \\ &\quad + \sum_{i=1}^{k-1} \{-(x_i - \alpha_i) f(x_i)\} - (b-\alpha_k) f(b) \\ &= -(\alpha_1 - a) f(a) - \sum_{i=1}^{k-1} (\alpha_{i+1} - \alpha_i) f(x_i) - (b-\alpha_k) f(b). \end{aligned}$$

Also,

$$\begin{aligned} S_2 &= \sum_{i=0}^{k-2} \left[\sum_{j=2}^{n} \frac{(-1)^j}{j!} \left\{ (x_{i+1} - \alpha_{i+1})^j f^{(j-1)}(x_{i+1}) \right\} \right] \\ &\quad + \sum_{j=2}^{n} \frac{(-1)^j}{j!} \left\{ (x_k - \alpha_k)^j f^{(j-1)}(x_k) \right\} \\ &\quad - \sum_{j=2}^{n} \frac{(-1)^j}{j!} \left\{ (x_0 - \alpha_1)^j f^{(j-1)}(x_0) \right\} \\ &\quad - \sum_{i=1}^{k-1} \left[\sum_{j=2}^{n} \frac{(-1)^j}{j!} \left\{ (x_i - \alpha_{i+1})^j f^{(j-1)}(x_i) \right\} \right] \\ &= \sum_{j=2}^{n} \frac{(-1)^j}{j!} (b-\alpha_k)^j f^{(j-1)}(b) - \sum_{j=2}^{n} \frac{(-1)^j}{j!} (a-\alpha_1)^j f^{(j-1)}(a) \\ &\quad + \sum_{i=1}^{k-1} \left[\sum_{j=2}^{n} \frac{(-1)^j}{j!} \left\{ (x_i - \alpha_i)^j - (x_i - \alpha_{i+1})^j \right\} f^{(j-1)}(x_i) \right]. \end{aligned}$$

From (2.6)

$$\begin{aligned} S_1 + S_2 &:= -\left\{ (b-\alpha_k) f(b) + (\alpha_1 - a) f(a) + \sum_{i=1}^{k-1} (\alpha_{i+1} - \alpha_i) f(x_i) \right\} \\ &\quad + \sum_{j=2}^{n} \frac{(-1)^j}{j!} \left\{ (b-\alpha_k)^j f^{(j-1)}(b) - (a-\alpha_1)^j f^{(j-1)}(a) \right\} \\ &\quad + \sum_{i=1}^{k-1} \left[\sum_{j=2}^{n} \frac{(-1)^j}{j!} \left\{ (x_i - \alpha_i)^j - (x_i - \alpha_{i+1})^j \right\} f^{(j-1)}(x_i) \right] \end{aligned}$$

$$
\begin{aligned}
&= -\left\{(\alpha_1 - a) f(a) + \sum_{i=1}^{k-1} (\alpha_{i+1} - \alpha_i) f(x_i) + (b - \alpha_k) f(b)\right\} \\
&\quad + \sum_{j=2}^{n} \frac{(-1)^j}{j!} \Bigg[-(a - \alpha_1)^j f^{(j-1)}(a) \\
&\quad + \sum_{i=1}^{k-1} \left\{(x_i - \alpha_i)^j - (x_i - \alpha_{i+1})^j\right\} f^{(j-1)}(x_i) + (b - \alpha_k)^j f^{(j-1)}(b) \Bigg].
\end{aligned}
$$

Keeping in mind that $x_0 = a$, $\alpha_0 = 0$, $x_k = b$ and $\alpha_{k+1} = b$ we may write

$$
\begin{aligned}
S_1 + S_2 &= -\sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) \\
&\quad + \sum_{j=2}^{n} \frac{(-1)^j}{j!} \left[\sum_{i=0}^{k} \left\{(x_i - \alpha_i)^j - (x_i - \alpha_{i+1})^j\right\} f^{(j-1)}(x_i)\right] \\
&= \sum_{j=1}^{n} \frac{(-1)^j}{j!} \left[\sum_{i=0}^{k} \left\{(x_i - \alpha_i)^j - (x_i - \alpha_{i+1})^j\right\} f^{(j-1)}(x_i)\right].
\end{aligned}
$$

And substituting $S_1 + S_2$ into the second term of (2.1) we obtain the identity (2.5). □

If we now assume that the points of the division I_k are fixed, we obtain the following corollary.

Corollary 2.3. *Let $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ be a division of the interval $[a, b]$. If $f : [a, b] \to \mathbb{R}$ is as defined in Theorem 2.1, then we have the equality*

$$
\int_a^b f(t)\, dt + \sum_{j=1}^{n} \frac{1}{2^j j!} \left[\sum_{i=0}^{k} \left\{-h_i^j + (-1)^j h_{i-1}^j\right\} f^{(j-1)}(x_i)\right] = (-1)^n \int_a^b K_{n,k}(t) f^{(n)}(t)\, dt, \tag{2.7}
$$

where $h_i := x_{i+1} - x_i$, $h_{-1} := 0$ and $h_k := 0$.

Proof. Choose

$$
\begin{aligned}
\alpha_0 &= a, \quad \alpha_1 = \frac{a + x_1}{2}, \quad \alpha_2 = \frac{x_1 + x_2}{2}, \ \ldots, \\
\alpha_{k-1} &= \frac{x_{k-2} + x_{k-1}}{2}, \alpha_k = \frac{x_{k-1} + x_k}{2} \ \text{ and } \ \alpha_{k+1} = b.
\end{aligned}
$$

From Corollary 2.2, the term

$$
\begin{aligned}
&(b - \alpha_k) f(b) + (\alpha_1 - a) f(a) + \sum_{i=1}^{k-1} (\alpha_{i+1} - \alpha_i) f(x_i) \\
&\qquad = \frac{1}{2} \left\{h_0 f(a) + \sum_{i=1}^{k-1} (h_i + h_{i-1}) f(x_i) + h_{k-1} f(b)\right\},
\end{aligned}
$$

the term

$$\sum_{j=2}^{n}\frac{(-1)^j}{j!}\left\{(b-\alpha_k)^j f^{(j-1)}(b)-(a-\alpha_1)^j f^{(j-1)}(a)\right\}$$
$$=\sum_{j=2}^{n}\frac{(-1)^j}{j!2^j}\left\{h_{k-1}^j f^{(j-1)}(b)-(-1)^j h_0^j f^{(j-1)}(a)\right\}$$

and the term

$$\sum_{i=1}^{k-1}\left[\sum_{j=2}^{n}\frac{(-1)^j}{j!}\left\{(x_i-\alpha_i)^j-(x_i-\alpha_{i+1})^j\right\}f^{(j-1)}(x_i)\right]$$
$$=\sum_{i=1}^{k-1}\left[\sum_{j=2}^{n}\frac{(-1)^j}{j!2^j}\left\{h_{i-1}^j-(-1)^j h_i^j\right\}f^{(j-1)}(x_i)\right].$$

Putting the last three terms in (2.5) we obtain

$$\int_a^b f(t)\,dt-\frac{1}{2}\left\{h_0 f(a)+\sum_{i=1}^{k-1}(h_i+h_{i-1})f(x_i)+h_{k-1}f(b)\right\}$$
$$+\sum_{j=2}^{n}\frac{(-1)^j}{j!2^j}\left\{h_{k-1}^j f^{(j-1)}(b)-(-1)^j h_0^j f^{(j-1)}(a)\right\}$$
$$+\sum_{i=1}^{k-1}\left[\sum_{j=2}^{n}\frac{(-1)^j}{j!2^j}\left\{h_{i-1}^j-(-1)^j h_i^j\right\}f^{(j-1)}(x_i)\right]$$
$$=(-1)^n\int_a^b K_{n,k}(t)\,f^{(n)}(t)\,dt.$$

Collecting the inner three terms of the last expression, we have

$$\int_a^b f(t)\,dt+\sum_{j=1}^{n}\frac{(-1)^j}{j!2^j}\sum_{i=0}^{k}\left\{h_{i-1}^j-(-1)^j h_i^j\right\}f^{(j-1)}(x_i)=(-1)^n\int_a^b K_{n,k}(t)\,f^{(n)}(t)\,dt,$$

which is equivalent to the identity (2.7). □

The case of equidistant partitioning is important in practice, and with this in mind we obtain the following corollary.

Corollary 2.4. *Let*

$$I_k : x_i = a+i\left(\frac{b-a}{k}\right),\quad i=0,\ldots,k \tag{2.8}$$

be an equidistant partitioning of $[a,b]$*, and* $f : [a,b] \to \mathbb{R}$ *be a mapping such that* $f^{(n-1)}$ *is* $AC[a,b]$ *, then we have the equality*

$$\int_a^b f(t)\,dt + \sum_{j=1}^{n} \left(\frac{b-a}{2k}\right)^j \frac{1}{j!} \Bigg[-f^{(j-1)}(a) + \sum_{i=1}^{k-1} \left\{ (-1)^j - 1 \right\} f^{(j-1)}(x_i) + (-1)^j f^{(j-1)}(b) \Bigg] = (-1)^n \int_a^b K_{n,k}(t)\, f^{(n)}(t)\,dt. \tag{2.9}$$

It is of some interest to note that the second term of (2.9) involves only even derivatives at all interior points x_i*,* $i = 1, \dots, k-1$*.*

Proof. Using (2.8) we note that

$$\begin{aligned} h_0 &= x_1 - x_0 = \frac{b-a}{k}, \quad h_{k-1} = (x_k - x_{k-1}) = \frac{b-a}{k}, \\ h_i &= x_{i+1} - x_i = \frac{b-a}{k} \text{ and } h_{i-1} = x_i - x_{i-1} = \frac{b-a}{k}, \quad (i = 1, \dots, k-1) \end{aligned}$$

and substituting into (2.7) we have

$$\int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{1}{j!2^j} \Bigg[-\left(\frac{b-a}{2k}\right)^j f^{(j-1)}(a) + \sum_{i=0}^{k-1} \left\{ -\left(\frac{b-a}{k}\right)^j + (-1)^j \left(\frac{b-a}{k}\right)^j \right\} f^{(j-1)}(x_i) + (-1)^j \left(\frac{b-a}{k}\right)^j f^{(j-1)}(b) \Bigg] = (-1)^n \int_a^b K_{n,k}(t)\, f^{(n)}(t)\,dt,$$

which simplifies to (2.9) after some minor manipulation. □

The following Taylor-like formula with integral remainder also holds.

Corollary 2.5. *Let* $g : [a,y] \to \mathbb{R}$ *be a mapping such that* $g^{(n)}$ *is* $AC[a,y]$*. Then for all* $x_i \in [a,y]$ *we have the identity*

$$g(y) = g(a) - \sum_{i=0}^{k-1} \left[\sum_{j=1}^{n} \frac{(-1)^j}{j!} \left\{ (x_{i+1} - \alpha_{i+1})^j g^{(j)}(x_{i+1}) - (x_i - \alpha_{i+1})^j g^{(j)}(x_i) \right\} \right] + (-1)^n \int_a^y K_{n,k}(y,t)\, g^{(n+1)}(t)\,dt \tag{2.10}$$

or

$$g(y) = g(a) - \sum_{j=1}^{n} \frac{(-1)^j}{j!} \left[\sum_{i=0}^{k} \left\{ (x_i - \alpha_i)^j - (x_i - \alpha_{i+1})^j \right\} g^{(j)}(x_i) \right] + (-1)^n \int_a^y K_{n,k}(y,t)\, g^{(n+1)}(t)\,dt. \tag{2.11}$$

The proof of (2.10) and (2.11) follows directly from (2.1) and (2.5) respectively upon choosing $b = y$ and $f = g'$.

3. INTEGRAL INEQUALITIES

In this section we utilise the equalities of Section 2 and develop inequalities for the representation of the integral of a function with respect to its derivatives at a multiple number of points within some interval.

Theorem 3.1. *Let $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ be a division of the interval $[a, b]$ and α_i $(i = 0, \ldots, k+1)$ be '$k + 2$' points so that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, \ldots, k)$ and $\alpha_{k+1} = b$. If $f : [a, b] \to \mathbb{R}$ is a mapping such that $f^{(n-1)}$ is AC$[a, b]$, then for all $x_i \in [a, b]$ we have the inequality:*

$$\left|\int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!}\left[\sum_{i=0}^{k}\left\{(x_i - \alpha_i)^j - (x_i - \alpha_{i+1})^j\right\} f^{(j-1)}(x_i)\right]\right| \tag{3.1}$$
$$\le \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\sum_{i=0}^{k-1}\left\{(\alpha_{i+1} - x_i)^{n+1} + (x_{i+1} - \alpha_{i+1})^{n+1}\right\}$$
$$\le \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\sum_{i=0}^{k-1} h_i^{n+1}$$
$$\le \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}(b-a)\,\nu^n(h) \quad \text{if } f^{(n)} \in L_\infty[a, b],$$

where

$$\begin{aligned}\left\|f^{(n)}\right\|_\infty &:= \sup_{t\in[a,b]}\left|f^{(n)}(t)\right| < \infty,\\ h_i &:= x_{i+1} - x_i \text{ and}\\ \nu(h) &:= \max\{h_i | i = 0, \ldots, k-1\}.\end{aligned}$$

Proof. From Corollary 2.2 we may write

$$\left|\int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!}\left[\sum_{i=0}^{k}\left\{(x_i - \alpha_i)^j - (x_i - \alpha_{i+1})^j\right\} f^{(j-1)}(x_i)\right]\right| \tag{3.2}$$
$$= \left|(-1)^n \int_a^b K_{n,k}(t)\, f^{(n)}(t)\,dt\right|,$$

and

$$\left|(-1)^n \int_a^b K_{n,k}(t)\, f^{(n)}(t)\,dt\right| \le \left\|f^{(n)}\right\|_\infty \int_a^b \left|K_{n,k}(t)\right| dt,$$

$$\begin{aligned}\int_a^b \left|K_{n,k}(t)\right| dt &= \sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}} \frac{\left|t - \alpha_{i+1}\right|^n}{n!}dt\\ &= \sum_{i=0}^{k-1}\left[\int_{x_i}^{\alpha_{i+1}} \frac{(\alpha_{i+1} - t)^n}{n!}dt + \int_{\alpha_{i+1}}^{x_{i+1}} \frac{(t - \alpha_{i+1})^n}{n!}dt\right]\\ &= \frac{1}{(n+1)!}\sum_{i=0}^{k-1}\left\{(\alpha_{i+1} - x_i)^{n+1} + (x_{i+1} - \alpha_{i+1})^{n+1}\right\}\end{aligned}$$

and thus

$$\left|(-1)^n \int_a^b K_{n,k}(t) f^{(n)}(t)\, dt\right| \leq \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!} \sum_{i=0}^{k-1} \left\{(\alpha_{i+1} - x_i)^{n+1} + (x_{i+1} - \alpha_{i+1})^{n+1}\right\}.$$

Hence, from (3.2), the first part of the inequality (3.1) is proved. The second and third lines follow by noting that

$$\frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!} \sum_{i=0}^{k-1} \left\{(\alpha_{i+1} - x_i)^{n+1} + (x_{i+1} - \alpha_{i+1})^{n+1}\right\} \leq \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!} \sum_{i=0}^{k-1} h_i^{n+1},$$

since for $0 < B < A < C$ it is well known that

$$(A - B)^{n+1} + (C - A)^{n+1} \leq (C - B)^{n+1}. \tag{3.3}$$

Also

$$\frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!} \sum_{i=0}^{k-1} h_i^{n+1} \leq \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!} \nu^n(h) \sum_{i=0}^{k-1} h_i = \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!} (b-a)\, \nu^n(h),$$

where $\nu(h) := \max\{h_i | i = 0, \ldots, k-1\}$ and therefore the third line of the inequality (3.1) follows, hence Theorem 3.1 is proved. □

When the points of the division I_k are fixed, we obtain the following inequality.

Corollary 3.2. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that $f^{(n-1)}$ is AC$[a,b]$, and I_k be defined as in Corollary 2.3, then*

$$\left|\int_a^b f(t)\, dt + \sum_{j=1}^{n} \frac{(-1)^j}{2^j j!} \left[\sum_{i=0}^{k} \left\{-h_i^j + (-1)^j h_{i+1}^j\right\} f^{(j-1)}(x_i)\right]\right| \leq \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!2^n} \sum_{i=0}^{k-1} h_i^{n+1} \tag{3.4}$$

for $f^{(n)} \in L_\infty[a,b]$.

Proof. From Corollary 2.3 we choose

$$\begin{aligned} \alpha_0 &= a, \ \ \alpha_1 = \frac{a + x_1}{2}, \ \ldots, \\ \alpha_{k-1} &= \frac{x_{k-2} + x_{k-1}}{2}, \ \ \alpha_k = \frac{x_{k-1} + x_k}{2} \text{ and } \alpha_{k+1} = b. \end{aligned}$$

Now utilising the first line of the inequality (3.1), we may evaluate

$$\sum_{i=0}^{k-1} \left\{(\alpha_{i+1} - x_i)^{n+1} + (x_{i+1} - \alpha_{i+1})^{n+1}\right\} = \sum_{i=0}^{k-1} 2 \left(\frac{h_i}{2}\right)^{n+1}$$

and therefore the inequality (3.4) follows. □

For the equidistant partitioning case we have the following inequality.

Corollary 3.3. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that $f^{(n-1)}$ is AC$[a,b]$ and let I_k be defined by (2.8). Then*

$$\left| \int_a^b f(t)\,dt + \sum_{j=1}^{n} \left(\frac{b-a}{2k}\right)^j \frac{1}{j!} \times \left[-f^{(j-1)}(a) + \sum_{i=1}^{k-1} \left\{ (-1)^j - 1 \right\} f^{(j-1)}(x_i) + (-1)^j f^{(j-1)}(b) \right] \right| \le \frac{\left\| f^{(n)} \right\|_\infty}{(n+1)!\,(2k)^n} (b-a)^{n+1} \tag{3.5}$$

for $f^{(n)} \in L_\infty[a,b]$.

Proof. We may utilise (2.9) and from (3.1), note that

$$h_0 = x_1 - x_0 = \frac{b-a}{k} \text{ and } h_i = x_{i+1} - x_i = \frac{b-a}{k},\ i = 1, \ldots, k-1$$

in which case (3.5) follows. □

The following inequalities for Taylor-like expansions also hold.

Corollary 3.4. *Let g be defined as in Corollary 2.5. Then we have the inequality*

$$\left| g(y) - g(a) + \sum_{j=1}^{n} \frac{(-1)^j}{j!} \left[\sum_{i=0}^{k} \left\{ (x_i - \alpha_i)^j - (x_i - \alpha_{i+1})^j \right\} g^{(j)}(x_i) \right] \right| \le \frac{\left\| g^{(n+1)} \right\|_\infty}{(n+1)!} \sum_{i=0}^{k-1} \left\{ (\alpha_{i+1} - x_i)^{n+1} + (x_{i+1} - \alpha_{i+1})^{n+1} \right\} \text{ if } g^{(n+1)} \in L_\infty[a,b] \tag{3.6}$$

for all $x_i \in [a,y]$ where

$$\left\| g^{(n+1)} \right\|_\infty := \sup_{t \in [a,y]} \left| g^{(n+1)}(t) \right| < \infty.$$

Proof. Follows directly from (2.11) and using the norm as in (3.1). □

When the points of the division I_k are fixed we obtain the following.

Corollary 3.5. *Let g be defined as in Corollary 2.5 and $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = y$ be a division of the interval $[a,y]$. Then we have the inequality*

$$\left| g(y) - g(a) + \sum_{j=1}^{n} \frac{1}{2^j j!} \left[\sum_{i=0}^{k} \left\{ -h_i^j + (-1)^j h_{i+1}^j \right\} g^{(j)}(x_i) \right] \right| \le \frac{\left\| g^{(n+1)} \right\|_\infty}{(n+1)!2^n} \sum_{i=0}^{k-1} h_i^{n+1} \text{ if } g^{(n+1)} \in L_\infty[a,y]. \tag{3.7}$$

Proof. The proof follows directly from using (2.7). □

For the equidistant partitioning case we have:

Corollary 3.6. *Let g be defined as in Corollary 2.5 and*

$$I_k : x_i = a + i \cdot \left(\frac{y-a}{k}\right), \quad i = 0, \ldots, k$$

be an equidistant partitioning of $[a, y]$, *then we have the inequality:*

$$\begin{aligned}(3.8)\quad &\left| g(y) - g(a) + \sum_{j=1}^{n} \left(\frac{y-a}{2k}\right)^{j} \frac{1}{j!} \right.\\ &\left.\times \left[-g^{(j)}(a) + \sum_{i=1}^{k-1} \left\{ (-1)^{j} - 1 \right\} g^{(j)}(x_i) + (-1)^{j} g^{(j)}(y) \right] \right| \\ &\qquad \le \frac{\left\| g^{(n+1)} \right\|_{\infty}}{(n+1)!\,(2k)^{n}} (y-a)^{n+1} \text{ if } g^{(n+1)} \in L_{\infty}[a, y].\end{aligned}$$

Proof. The proof follows directly upon using (2.9) with $f' = g$ and $b = y$. □

4. THE CONVERGENCE OF A GENERAL QUADRATURE FORMULA

Let

$$\Delta_m : a = x_0^{(m)} < x_1^{(m)} < \cdots < x_{m-1}^{(m)} < x_m^{(m)} = b$$

be a sequence of division of $[a, b]$ and consider the sequence of real numerical integration formula

$$\begin{aligned}(4.1)\quad & I_m\left(f, f', \dots, f^{(n)}, \Delta_m, w_m\right) \\ &:= \sum_{j=0}^{m} w_j^{(m)} f\left(x_j^{(m)}\right) - \sum_{r=2}^{n} \frac{(-1)^r}{r!} \left[\sum_{j=0}^{m} \left\{ \left(x_j^{(m)} - a - \sum_{s=0}^{j-1} w_s^{(m)} \right)^{r} \right.\right. \\ &\qquad \left.\left. - \left(x_j^{(m)} - a - \sum_{s=0}^{j} w_s^{(m)} \right)^{r} \right\} f^{(r-1)}\left(x_j^{(m)}\right) \right],\end{aligned}$$

where w_j $(j = 0, \dots, m)$ are the quadrature weights and assume that $\sum_{j=0}^{m} w_j^{(m)} = b - a$.

The following theorem contains a sufficient condition for the weights $w_j^{(m)}$ so that $I_m\left(f, f', \dots, f^{(n)}, \Delta_m, w_m\right)$ approximates the integral $\int_a^b f(x)\,dx$ with an error expressed in terms of $\left\| f^{(n)} \right\|_{\infty}$.

Theorem 4.1. *Let* $f : [a, b] \to \mathbb{R}$ *be a continuous mapping on* $[a, b]$ *such that* $f^{(n-1)}$ *is AC*$[a, b]$. *If the quadrature weights,* $w_j^{(m)}$ *satisfy the condition*

$$(4.2)\qquad x_i^{(m)} - a \le \sum_{j=0}^{i} w_j^{(m)} \le x_{i+1}^{(m)} - a \text{ for all } i = 0, \dots, m-1$$

then we have the estimation

$$(4.3)\qquad \left| I_m\left(f, f', \dots, f^{(n)}, \Delta_m, w_m\right) - \int_a^b f(t)\,dt \right|$$

$$\leq \frac{\left\|f^{(n)}\right\|_{\infty}}{(n+1)!} \sum_{i=0}^{m-1} \left[\left(a + \sum_{j=0}^{i} w_j^{(m)} - x_i^{(m)} \right)^{n+1} - \left(x_{i+1}^{(m)} - a - \sum_{j=0}^{i} w_j^{(m)} \right)^{n+1} \right]$$
$$\leq \frac{\left\|f^{(n)}\right\|_{\infty}}{(n+1)!} \sum_{i=0}^{m-1} \left(h_i^{(m)} \right)^{n+1}$$
$$\leq \frac{\left\|f^{(n)}\right\|_{\infty}}{(n+1)!} \left[\nu\left(h^{(m)}\right)\right](b-a), \quad \textit{where } \ f^{(n)} \in L_{\infty}[a,b].$$

Also

$$\nu\left(h^{(m)}\right) := \max_{i=0,\dots,m-1} \left\{h_i^{(m)}\right\}$$
$$h_i^{(m)} := x_{i+1}^{(m)} - x_i^{(m)}.$$

In particular, if $\left\|f^{(n)}\right\|_{\infty} < \infty$*, then*

$$\lim_{\nu\left(h^{(m)}\right)\to 0} I_m\left(f, f', \dots, f^{(n)}, \Delta_m, w_m\right) = \int_a^b f(t)\, dt$$

uniformly by the influence of the weights w_m.

Proof. Define the sequence of real numbers

$$\alpha_{i+1}^{(m)} := a + \sum_{j=0}^{i} w_j^{(m)}, \quad i = 0, \dots, m.$$

Note that

$$\alpha_{i+1}^{(m)} = a + \sum_{j=0}^{m} w_j^{(m)} = a + b - a = b.$$

By the assumption (4.2), we have

$$\alpha_{i+1}^{(m)} \in \left[x_i^{(m)}, x_{i+1}^{(m)}\right] \quad \text{for all } \ i = 0, \dots, m-1.$$

Define $\alpha_0^{(m)} = a$ and compute

$$\alpha_1^{(m)} - \alpha_0^{(m)} = w_0^{(m)}$$
$$\alpha_{i+1}^{(m)} - \alpha_i^{(m)} = a + \sum_{j=0}^{i} w_j^{(m)} - a - \sum_{j=0}^{i-1} w_j^{(m)} = w_i^{(m)} \quad (i = 0, \dots, m-1)$$

and

$$\alpha_{m+1}^{(m)} - \alpha_m^{(m)} = a + \sum_{j=0}^{m} w_j^{(m)} - a - \sum_{j=0}^{m-1} w_j^{(m)} = w_m^{(m)}.$$

Consequently

$$\sum_{i=0}^{m} \left(\alpha_{i+1}^{(m)} - \alpha_i^{(m)}\right) f\left(x_i^{(m)}\right) = \sum_{i=0}^{m} w_i^{(m)} f\left(x_i^{(m)}\right),$$

and let

$$\sum_{j=0}^{m} w_j^{(m)} f\left(x_j^{(m)}\right) - \sum_{r=2}^{n} \frac{(-1)^r}{r!}\left[\sum_{j=0}^{m}\left\{\left(x_j^{(m)} - a - \sum_{s=0}^{j-1} w_s^{(m)}\right)^r - \left(x_j^{(m)} - a - \sum_{s=0}^{j} w_s^{(m)}\right)^r\right\} f^{(r-1)}\left(x_j^{(m)}\right)\right] := I_m\left(f, f', \dots, f^{(n)}, \Delta_m, w_m\right).$$

Applying the inequality (3.1) we obtain the estimate (4.3). □

The case when the partitioning is equidistant is important in practice. Consider the equidistant partition

$$E_m := x_i^{(m)} := a + i\frac{b-a}{m}, \quad (i = 0, \dots, m)$$

and define the sequence of numerical quadrature formulae

$$I_m\left(f, f', \dots, f^{(n)}, \Delta_m, w_m\right) := \sum_{j=0}^{m} w_j^{(m)} f\left(a + \frac{j(b-a)}{n}\right) - \sum_{r=2}^{n} \frac{(-1)^r}{r!}\left[\sum_{j=0}^{m}\left\{\left(\frac{j(b-a)}{n} - \sum_{s=0}^{j-1} w_s^{(m)}\right)^r - \left(\frac{j(b-a)}{n} - \sum_{s=0}^{j} w_s^{(m)}\right)^r\right\} f^{(r-1)}\left(a + \frac{j(b-a)}{n}\right)\right].$$

The following corollary holds.

Corollary 4.2. *Let $f : [a,b] \to \mathbb{R}$ be AC$[a,b]$. If the quadrature weights $w_j^{(m)}$ satisfy the condition*

$$\frac{i}{m} \le \frac{1}{b-a}\sum_{j=0}^{i} w_j^{(m)} \le \frac{i+1}{m}, \quad i = 0, 1, \dots, n-1,$$

then the following bound holds:

$$\left|I_m\left(f, f', \dots, f^{(n)}, \Delta_m, w_m\right) - \int_a^b f(t)\,dt\right| \le \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\sum_{i=0}^{m-1}\left[\left(\sum_{j=0}^{i} w_j^{(m)} - i\left(\frac{b-a}{m}\right)\right)^{n+1} - \left((i+1)\left(\frac{b-a}{m}\right) - \sum_{j=0}^{i} w_j^{(m)}\right)^{n+1}\right] \le \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left(\frac{b-a}{m}\right)^{n+1}, \quad \text{where } f^{(n)} \in L_\infty[a,b].$$

In particular, if $\left\|f^{(n)}\right\|_\infty < \infty$, then

$$\lim_{m\to\infty} I_m\left(f, f', \dots, f^{(n)}, w_m\right) = \int_a^b f(t)\,dt$$

uniformly by the influence of the weights w_m.

The proof of Corollary 4.2 follows directly from Theorem 4.1.

5. Grüss Type Inequalities

The Grüss inequality [15], is well known in the literature. It is an integral inequality which establishes a connection between the integral of a product of two functions and the product of the integrals of the two functions.

Theorem 5.1. *Let $h, g : [a, b] \to \mathbb{R}$ be two integrable functions such that $\phi \leq h(x) \leq \Phi$ and $\gamma \leq g(x) \leq \Gamma$ for all $x \in [a, b]$, ϕ, Φ, γ and Γ are constants. Then we have*

$$|T(h, g)| \leq \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma), \tag{5.1}$$

where

$$T(h, g) := \frac{1}{b-a}\int_a^b h(x)\, g(x)\, dx - \frac{1}{b-a}\int_a^b h(x)\, dx \cdot \frac{1}{b-a}\int_a^b g(x)\, dx \tag{5.2}$$

and the inequality (5.1) is sharp, in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one.

For a simple proof of this fact as well as generalisations, discrete variants, extensions and associated material, see [17]. The Grüss inequality is also utilised in the papers [6, 7, 14] and the references contained therein.

A *premature* Grüss inequality is the following.

Theorem 5.2. *Let f and g be integrable functions defined on $[a, b]$ and let $\gamma \leq g(x) \leq \Gamma$ for all $x \in [a, b]$. Then*

$$|T(h, g)| \leq \frac{\Gamma - \gamma}{2}(T(f, f))^{\frac{1}{2}}, \tag{5.3}$$

where $T(f, f)$ is as defined in (5.2).

Theorem 5.2 was proved in 1999 by Matić, Pečarić and Ujević [16] and it provides a sharper bound than the Grüss inequality (5.1). The term *premature* is used to highlight the fact that the result (5.3) is obtained by not fully completing the proof of the Grüss inequality. The premature Grüss inequality is completed if one of the functions, f or g, is explicitly known.

We now give the following theorem based on the premature Grüss inequality (5.3).

Theorem 5.3. *Let $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ be a division of the interval $[a, b]$, α_i $(i = 0, \ldots, k+1)$ be '$k+1$' points such that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, \ldots, k)$ and $\alpha_k = b$. If $f : [a, b] \to \mathbb{R}$ is AC$[a, b]$ and n time differentiable on $[a, b]$, then assuming that the n^{th} derivative $f^{(n)} : (a, b) \to \mathbb{R}$ satisfies the condition*

$$m \leq f^{(n)} \leq M \quad \text{for all } x \in (a, b),$$

we have the inequality

$$\begin{aligned} \Bigg| (-1)^n \int_a^b f(t)\, dt + (-1)^n \sum_{j=1}^{n} \frac{(-1)^j}{j!} & \\ \times \left[\sum_{i=0}^{k-1} \left\{ \left(\frac{h_i}{2} - \delta_i\right)^j f^{(j-1)}(x_{i+1}) - \left(\frac{h_i}{2} + \delta_i\right)^j f^{(j-1)}(x_i) \right\} \right] & \\ - \left(\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(b-a)(n+1)!} \right) \sum_{i=0}^{k-1} \left(\frac{h_i}{2}\right)^{n+1} \times \left[\sum_{r=0}^{n+1} \binom{n+1}{r} \left(\frac{2\delta_i}{h_i}\right)^r \{1 + (-1)^r\} \right] \Bigg| & \end{aligned} \tag{5.4}$$

$$\leq \frac{M-m}{2}\left[\frac{b-a}{(2n+1)(n!)^2}\sum_{i=0}^{k-1}\left(\frac{h_i}{2}\right)^{2n+1}\times\left[\sum_{r=0}^{2n+1}\binom{2n+1}{r}\left(\frac{2\delta_i}{h_i}\right)^r\{1+(-1)^r\}\right]\right.$$
$$\left.-\left(\frac{1}{(n+1)!}\sum_{i=0}^{k-1}\left(\frac{h_i}{2}\right)^{n+1}\left[\sum_{r=0}^{n+1}\binom{n+1}{r}\left(\frac{2\delta_i}{h_i}\right)^r\{1+(-1)^r\}\right]\right)^2\right]^{\frac{1}{2}}$$

where

$$\begin{aligned} h_i &:= x_{i+1}-x_i \text{ and} \\ \delta_i &:= \alpha_{i+1}-\frac{x_{i+1}+x_i}{2}, \quad i=0,\dots,k-1. \end{aligned}$$

Proof. We utilise (5.2) and (5.3), multiply through by $(b-a)$ and choose $h(t) := K_{n,k}(t)$ as defined by (2.2) and $g(t) := f^{(n)}(t)$, $t\in[a,b]$ such that

$$(5.5)\quad \left|\int_a^b K_{n,k}(t)\,f^{(n)}(t)\,dt-\frac{1}{b-a}\int_a^b f^{(n)}(t)\,dt\cdot\int_a^b K_{n,k}(t)\,dt\right| \\ \leq\frac{\Gamma-\gamma}{2}\left[(b-a)\int_a^b K_{n,k}^2(t)\,dt-\left(\int_a^b K_{n,k}(t)\,dt\right)^2\right]^{\frac{1}{2}}.$$

Now we may evaluate

$$\int_a^b f^{(n)}(t)\,dt = f^{(n-1)}(b)-f^{(n-1)}(a)$$

and

$$\begin{aligned} G_1 &:= \int_a^b K_{n,k}(t)\,dt \\ &= \sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}}\frac{1}{n!}(t-\alpha_{i+1})^n\,dt \\ &= \frac{1}{(n+1)!}\sum_{i=0}^{k-1}\left\{(x_{i+1}-\alpha_{i+1})^{n+1}+(\alpha_{i+1}-x_i)^{n+1}\right\}. \end{aligned}$$

Using the definitions of h_i and δ_i we have

$$x_{i+1}-\alpha_{i+1}=\frac{h_i}{2}-\delta_i \qquad \text{and} \qquad \alpha_{i+1}-x_i=\frac{h_i}{2}+\delta_i$$

such that

$$\begin{aligned} G_1 &= \frac{1}{(n+1)!}\sum_{i=0}^{k-1}\left\{\left(\frac{h_i}{2}-\delta_i\right)^{n+1}+\left(\frac{h_i}{2}+\delta_i\right)^{n+1}\right\} \\ &= \frac{1}{(n+1)!}\sum_{i=0}^{k-1}\left[\sum_{r=0}^{n+1}\binom{n+1}{r}(-\delta_i)^r\left(\frac{h_i}{2}\right)^{n+1-r}+\sum_{r=0}^{n+1}\binom{n+1}{r}\delta_i^r\left(\frac{h_i}{2}\right)^{n+1-r}\right] \\ &= \frac{1}{(n+1)!}\sum_{i=0}^{k-1}\left(\frac{h_i}{2}\right)^{n+1}\left[\sum_{r=0}^{n+1}\binom{n+1}{r}\left(\frac{2\delta_i}{h_i}\right)^r\{1+(-1)^r\}\right]. \end{aligned}$$

Also,

$$\begin{aligned} G_2 &:= \int_a^b K_{n,k}^2(t)\,dt \\ &= \sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}} \frac{(t-\alpha_{i+1})^{2n}}{(n!)^2}dt \\ &= \frac{1}{(2n+1)(n!)^2}\sum_{i=0}^{k-1}\left\{(x_{i+1}-\alpha_{i+1})^{2n+1}+(\alpha_{i+1}-x_i)^{2n+1}\right\} \\ &= \frac{1}{(2n+1)(n!)^2}\sum_{i=0}^{k-1}\left\{\left(\frac{h_i}{2}-\delta_i\right)^{2n+1}+\left(\frac{h_i}{2}+\delta_i\right)^{2n+1}\right\} \\ &= \frac{1}{(2n+1)(n!)^2}\sum_{i=0}^{k-1}\left(\frac{h_i}{2}\right)^{2n+1}\left[\sum_{r=0}^{2n+1}\binom{2n+1}{r}\left(\frac{2\delta_i}{h_i}\right)^r\{1+(-1)^r\}\right]. \end{aligned}$$

From identity (2.1), we may write

$$\begin{aligned} \int_a^b K_{n,k}(t)\,f^{(n)}(t)\,dt &= (-1)^n\int_a^b f(t)\,dt+(-1)^n\sum_{j=1}^{n}\frac{(-1)^j}{j!} \\ &\quad\times\left[\sum_{i=0}^{k-1}\left\{(x_{i+1}-\alpha_{i+1})^j f^{(j-1)}(x_{i+1})-(\alpha_{i+1}-x_i)^j f^{(j-1)}(x_i)\right\}\right] \end{aligned}$$

and from the left hand side of (5.5) we obtain

$$\begin{aligned} G_3 &:= (-1)^n\int_a^b f(t)\,dt+(-1)^n\sum_{j=1}^{n}\frac{(-1)^j}{j!} \\ &\quad\times\left[\sum_{i=0}^{k-1}\left\{\left(\frac{h_i}{2}-\delta_i\right)^j f^{(j-1)}(x_{i+1})-\left(\frac{h_i}{2}+\delta_i\right)^j f^{(j-1)}(x_i)\right\}\right] \\ &\quad-\left(\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)(n+1)!}\right)\sum_{i=0}^{k-1}\left(\frac{h_i}{2}\right)^{n+1}\times\left[\sum_{r=0}^{n+1}\binom{n+1}{r}\left(\frac{2\delta_i}{h_i}\right)^r\{1+(-1)^r\}\right] \end{aligned}$$

after substituting for G_1.

From the right hand side of (5.5) we substitute for G_1 and G_2 so that

$$\begin{aligned} G_4 &:= (b-a)\int_a^b K_{n,k}^2(t)\,dt-\left(\int_a^b K_{n,k}(t)\,dt\right)^2 \\ &= \frac{b-a}{(2n+1)(n!)^2}\sum_{i=0}^{k-1}\left(\frac{h_i}{2}\right)^{2n+1}\left[\sum_{r=0}^{2n+1}\binom{2n+1}{r}\left(\frac{2\delta_i}{h_i}\right)^r\{1+(-1)^r\}\right] \\ &\quad-\left(\frac{1}{(n+1)!}\sum_{i=0}^{k-1}\left(\frac{h_i}{2}\right)^{n+1}\left[\sum_{r=0}^{n+1}\binom{n+1}{r}\left(\frac{2\delta_i}{h_i}\right)^r\{1+(-1)^r\}\right]\right)^2. \end{aligned}$$

Hence,

$$|G_3|\le\frac{M-m}{2}(G_4)^{\frac{1}{2}},$$

and Theorem 5.3 has been proved. □

Corollary 5.4. *Let f, I_k and α_k be defined as in Theorem 5.3 and further define*

$$\delta = \alpha_{i+1} - \frac{x_{i+1} + x_i}{2} \tag{5.6}$$

for all $i = 0, \ldots, k-1$ such that

$$|\delta| \le \frac{1}{2} \min \{h_i | i = 1, \ldots, k\}. \tag{5.7}$$

The following inequality applies:

$$\begin{aligned} &\Bigg| (-1)^n \int_a^b f(t)\,dt + (-1)^n \sum_{j=1}^{n} \frac{1}{j!} \\ &\times \left[\sum_{i=0}^{k-1} \left\{ (-i)^j \left(\frac{h_i}{2} - \delta_i \right)^j f^{(j-1)}(x_{i+1}) - \left(\frac{h_i}{2} + \delta_i \right)^j f^{(j-1)}(x_i) \right\} \right] \\ &- \left(\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(b-a)(n+1)!} \right) \sum_{i=0}^{k-1} \sum_{r=0}^{n+1} \delta^r \left(\frac{h_i}{2} \right)^{n+1-r} \{1 + (-1)^r\} \Bigg| \\ \le\ & \frac{M-m}{2} \left[\frac{b-a}{(2n+1)(n!)^2} \sum_{i=0}^{k-1} \sum_{r=0}^{2n+1} \left[\binom{2n+1}{r} \delta^r \left(\frac{h_i}{2} \right)^{2n+1-r} \{1 + (-1)^r\} \right] \right. \\ &\left. - \left(\frac{1}{(n+1)!} \sum_{i=0}^{k-1} \sum_{r=0}^{n+1} \binom{n+1}{r} \delta^r \left(\frac{h_i}{2} \right)^{n+1-r} \{1 + (-1)^r\} \right)^2 \right]^{\frac{1}{2}}. \end{aligned} \tag{5.8}$$

The proof follows directly from (5.4) upon the substitution of (5.6) and some minor simplification.

Remark 5.5. If for any division $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ of the interval $[a, b]$, we choose $\delta = 0$ in (5.6), we have the inequality

$$\begin{aligned} &\Bigg| (-1)^n \int_a^b f(t)\,dt + (-1)^n \sum_{j=1}^{n} \frac{1}{j!} \times \sum_{i=0}^{k-1} \left(\frac{h_i}{2} \right)^j \left\{ (-1)^j f^{(j-1)}(x_{i+1}) - f^{(j-1)}(x_i) \right\} \\ &\qquad -2 \left(\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(b-a)(n+1)!} \right) \sum_{i=0}^{k-1} \left(\frac{h_i}{2} \right)^{n+1} \Bigg| \\ &\le \frac{M-m}{2} \left[\frac{2(b-a)}{(2n+1)(n!)^2} \sum_{i=0}^{k-1} \left(\frac{h_i}{2} \right)^{2n+1} - \left(\frac{2}{(n+1)!} \sum_{i=0}^{k-1} \left(\frac{h_i}{2} \right)^{n+1} \right)^2 \right]^{\frac{1}{2}}. \end{aligned} \tag{5.9}$$

The proof follows directly from (5.8).

Remark 5.6. Let $f^{(n)}$ be defined as in Theorem 5.3 and consider an equidistant partitioning E_k of the interval $[a, b]$, where

$$E_k := x_i = a + i \left(\frac{b-a}{k} \right), \quad i = 0, \ldots, k.$$

The following inequality applies

$$\begin{aligned}(5.10)\quad &\Bigg|(-1)^n \int_a^b f(t)\,dt + (-1)^n \sum_{j=1}^{n} \frac{1}{j!}\left(\frac{b-a}{2k}\right)^j \\ &\times \left\{(-1)^j f^{(j-1)}\left(\frac{a(k-i-1)+b(i-1)}{k}\right) - f^{(j-1)}\left(\frac{a(k-i)+ib}{k}\right)\right\} \\ &-2\left(\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)(n+1)!}\right)\left(\frac{b-a}{2k}\right)^{n+1}\Bigg| \\ &\le (M-m)\cdot\frac{nk}{(n+1)!\sqrt{2n+1}}\left(\frac{b-a}{2k}\right)^{n+1}.\end{aligned}$$

Proof. The proof follows upon noting that $h_i = x_{i+1} - x_i = \left(\frac{b-a}{k}\right)$, $i = 0, \dots, k$. □

6. Some Particular Integral Inequalities

In this subsection we point out some special cases of the integral inequalities in Section 3. In doing so, we shall recover, subsume and extend the results of a number of previous published papers [2, 5].

We shall recover the left and right rectangle inequalities, the perturbed trapezoid inequality, the midpoint and Simpson's inequalities and the Newton-Cotes three eighths inequality, and a Boole type inequality.

In the case when $n = 1$, for the kernel $K_{1,k}(t)$ of (2.3), the inequality (3.1), reduces to the results obtained by Dragomir [5] for the cases when $f : [a, b] \to \mathbb{R}$ is absolutely continuous and f' belongs to the $L_\infty[a, b]$ space.

Similarly, for $n = 1$, Dragomir [11] extended Theorem 3.1 for the case when $f\,[a, b] \to \mathbb{R}$ is a function of bounded variation on $[a, b]$.

For the two branch Peano kernel,

$$(6.1)\qquad K_{n,2}(t) := \begin{cases} \dfrac{1}{n!}(t-a)^n, & t \in [a, x) \\[2ex] \dfrac{1}{n!}(t-b)^n, & t \in (x, b] \end{cases}$$

the inequality (3.1) reduces to the result (1.2) obtained by Cerone and Dragomir [1] and [3]. A number of other particular cases are now investigated.

Theorem 6.1. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous mapping on $[a, b]$ and $a \le x_1 \le b$, $a \le \alpha_1 \le x_1 \le \alpha_2 \le b$. Then we have*

$$\begin{aligned}(6.2)\quad &\Bigg|\int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!}\Big[-(a-\alpha_1)^j f^{(j-1)}(a) \\ &+ \left\{(x_1-\alpha_1)^j - (x_1-\alpha_2)^j\right\} f^{(j-1)}(x_1) + (b-\alpha_2)^j f^{(j-1)}(b)\Big]\Bigg|\end{aligned}$$

$$\leq \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left((\alpha_1-a)^{n+1}+(x_1-\alpha_1)^{n+1}+(\alpha_2-x_1)^{n+1}+(b-\alpha_2)^{n+1}\right)$$
$$\leq \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left((x_1-a)^{n+1}+(b-x_1)^{n+1}\right)$$
$$\leq \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}(b-a)^{n+1}, \qquad f^{(n)}\in L_\infty[a,b].$$

Proof. Consider the division $a=x_0\leq x_1\leq x_2=b$ and the numbers $\alpha_0=a$, $\alpha_1\in[a,x_1)$, $\alpha_2\in(x_1,b]$ and $\alpha_3=b$.

From the left hand side of (3.1) we obtain

$$\sum_{j=1}^{n}\frac{(-1)^j}{j!}\sum_{i=0}^{2}\left\{(x_i-\alpha_i)^j-(x_i-\alpha_{i+1})^j\right\}f^{(j-1)}(x_i)$$
$$=\sum_{j=1}^{n}\frac{(-1)^j}{j!}\Big[-(a-\alpha_1)^j f^{(j-1)}(a)+\left\{(x_1-\alpha_1)^j-(x_1-\alpha_2)^j\right\}f^{(j-1)}(x_1)$$
$$+(b-\alpha_2)^j f^{(j-1)}(b)\Big].$$

From the right hand side of (3.1) we obtain

$$\frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\sum_{i=0}^{1}\left\{(\alpha_{i+1}-x_i)^{n+1}+(x_{i+1}-\alpha_{i+1})^{n+1}\right\}$$
$$=\frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left((\alpha_1-a)^{n+1}+(x_1-\alpha_1)^{n+1}+(\alpha_2-x_1)^{n+1}+(b-\alpha_2)^{n+1}\right)$$

and hence the first line of the inequality (6.2) follows.

Notice that if we choose $\alpha_1=a$ and $\alpha_2=b$ in Theorem 6.1 we obtain the inequality (1.2). □

The following proposition embodies a number of results, including the Ostrowski inequality, the midpoint and Simpson's inequalities and the three-eighths Newton-Cotes inequality including its generalisation.

Proposition 6.2. *Let f be defined as in Theorem 6.1 and let $a\leq x_1\leq b$, and $a\leq\frac{(m-1)a+b}{m}\leq x_1\leq\frac{a+(m-1)b}{m}\leq b$ for m a natural number, $m\geq 2$, then we have the inequality*

(6.3) $|P_{m,n}|$

$$:=\left|\int_a^b f(t)\,dt+\sum_{j=1}^{n}\frac{(-1)^j}{j!}\left[\left(\frac{b-a}{m}\right)^j\left\{f^{(j-1)}(b)-(-1)^j f^{(j-1)}(a)\right\}\right.\right.$$
$$\left.\left.+\left\{\left((x_1-a)-\frac{b-a}{m}\right)^j-\left(\frac{b-a}{m}-(b-x_1)\right)^j\right\}f^{(j-1)}(x_1)\right]\right|$$
$$\leq\frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left(2\left(\frac{b-a}{m}\right)^{n+1}+\left(x_1-a-\left(\frac{b-a}{m}\right)\right)^{n+1}\right.$$
$$\left.+\left(b-x_1-\left(\frac{b-a}{m}\right)\right)^{n+1}\right)\quad \textit{if}\quad f^{(n)}\in L_\infty[a,b].$$

Proof. From Theorem 6.1 we note that

$$\alpha_1 = \frac{(m-1)\,a+b}{m} \text{ and } \alpha_2 = \frac{a+(m-1)\,b}{m}$$

so that

$$\begin{aligned} a-\alpha_1 &= -\left(\frac{b-a}{m}\right),\ b-\alpha_2 = \frac{b-a}{m}, \\ x_1-\alpha_1 &= x_1-a-\left(\frac{b-a}{m}\right) \text{ and } x_1-\alpha_2 = \left(\frac{b-a}{m}\right)-(b-x_1)\,. \end{aligned}$$

From the left hand side of (6.2) we have

$$\begin{aligned} &-(a-\alpha_1)^j f^{(j-1)}(a) + \left\{(x_1-\alpha_1)^j - (x_1-\alpha_2)^j\right\} f^{(j-1)}(x_1) + (b-\alpha_2)^j f^{(j-1)}(b) \\ &\qquad = \left(\frac{b-a}{m}\right)^j \left\{f^{(j-1)}(b) - f^{(j-1)}(a)\right\} \\ &\qquad\quad + \left\{\left(x_1-a-\left(\frac{b-a}{m}\right)\right)^j - \left(\left(\frac{b-a}{m}\right)-(b-x_1)\right)^j\right\} f^{(j-1)}(x_1)\,. \end{aligned}$$

From the right hand side of (6.2),

$$\begin{aligned} &(\alpha_1-a)^{n+1} + (x_1-\alpha_1)^{n+1} + (\alpha_2-x_1)^{n+1} + (b-\alpha_2)^{n+1} \\ &\qquad = 2\left(\frac{b-a}{m}\right)^{n+1} + \left(x_1-a-\left(\frac{b-a}{m}\right)\right)^{n+1} + \left(b-x_1-\left(\frac{b-a}{m}\right)\right)^{n+1} \end{aligned}$$

and the inequality (6.3) follows, hence the proof is complete. □

The following corollary points out that the optimum of Proposition 6.2 occurs at $x_1 = \frac{\alpha_1+\alpha_2}{2} = \frac{a+b}{2}$ in which case we have:

Corollary 6.3. *Let f be defined as in Proposition 6.2 and let $x_1 = \frac{a+b}{2}$ in which case we have the inequality*

$$\begin{aligned} (6.4)\qquad &\left|P_{m,n}\left(\frac{a+b}{2}\right)\right| \\ &:= \left|\int_a^b f(t)\,dt + \sum_{j=1}^n \frac{(-1)^j}{j!}\left[\left(\frac{b-a}{m}\right)^j \left\{f^{(j-1)}(b) - (-1)^j f^{(j-1)}(a)\right\}\right.\right. \\ &\quad \left.\left. + \left(\frac{(m-2)(b-a)}{2m}\right)^j \left(1-(-1)^j\right) f^{(j-1)}\left(\frac{a+b}{2}\right)\right]\right| \\ &\le \frac{2\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left(\frac{b-a}{m}\right)^{n+1}\left(1+\left(\frac{m-2}{2}\right)^{n+1}\right) \quad \text{if} \quad f^{(n)} \in L_\infty[a,b]\,. \end{aligned}$$

The proof follows directly from (6.3) upon substituting $x_1 = \frac{a+b}{2}$.

A number of other corollaries follow naturally from Proposition 6.2 and Corollary 6.3 and will now be investigated.

The following two corollaries generalise the Simpson inequality and follow directly from (6.3) and (6.4) for $m = 6$.

Corollary 6.4. *Let the conditions of Corollary 6.3 hold and put $m = 6$. Then we have the inequality*

$$(6.5)\qquad \left|P_{6,n}\right|$$
$$:= \left|\int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!}\left[\left(\frac{b-a}{6}\right)^j \left\{f^{(j-1)}(b) - (-1)^j f^{(j-1)}(a)\right\} + \left\{\left(x_1 - \frac{5a+b}{6}\right)^j - \left(x_1 - \frac{a+5b}{6}\right)^j\right\} f^{(j-1)}(x_1)\right]\right|$$
$$\leq \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left(2\left(\frac{b-a}{6}\right)^{n+1} + \left(x_1 - \frac{5a+b}{6}\right)^{n+1} + \left(-x_1 + \frac{a+5b}{6}\right)^{n+1}\right), \quad f^{(n)} \in L_\infty[a,b],$$

which is the generalised Simpson inequality.

Corollary 6.5. *Let the conditions of Corollary 6.3 hold and put $m = 6$,. Then at the midpoint $x_1 = \frac{a+b}{2}$ we have the inequality*

$$(6.6)\qquad \left|P_{6,n}\left(\frac{a+b}{2}\right)\right|$$
$$:= \left|\int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!}\left[\left(\frac{b-a}{6}\right)^j \left\{f^{(j-1)}(b) - (-1)^j f^{(j-1)}(a)\right\} + \left(\frac{b-a}{3}\right)^j \left(1 - (-1)^j\right) f^{(j-1)}\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq \frac{2\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left(\frac{b-a}{6}\right)^{n+1}\left(1 + 2^{n+1}\right), \quad f^{(n)} \in L_\infty[a,b].$$

Remark 6.6. Choosing $n = 1$ in (6.6) we have

$$(6.7)\qquad \left|P_{6,1}\left(\frac{a+b}{2}\right)\right| := \left|\int_a^b f(t)\,dt - \left(\frac{b-a}{6}\right)\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)\right|$$
$$\leq \frac{5}{36}\left\|f'\right\|_\infty (b-a)^2, \quad f' \in L_\infty[a,b].$$

Remark 6.7. Choosing $n = 2$ in (6.6) we have a perturbed Simpson type inequality,

$$(6.8)\qquad \left|P_{6,2}\left(\frac{a+b}{2}\right)\right|$$
$$:= \left|\int_a^b f(t)\,dt - \left(\frac{b-a}{6}\right)\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right) + \left(\frac{b-a}{6}\right)^2\left(\frac{f'(b) - f'(a)}{2}\right)\right|$$
$$\leq \frac{(b-a)^3}{72}\left\|f''\right\|_\infty, \; f'' \in L_\infty[a,b].$$

Corollary 6.8. *Let f be defined as in Corollary 6.3 and let $m = 4$, then we have the inequality*

$$\left|P_{4,n}\left(\frac{a+b}{2}\right)\right| := \left|\int_a^b f(t)\,dt + \sum_{j=1}^{n}\frac{(-1)^j}{j!}\left(\frac{b-a}{4}\right)^j \times\left[\left(f^{(j-1)}(b) - (-1)^j f^{(j-1)}(a)\right) + \left(1-(-1)^j\right) f^{(j-1)}\left(\frac{a+b}{2}\right)\right]\right| \le \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!4^n}(b-a)^{n+1}, \quad f^{(n)} \in L_\infty[a,b]. \tag{6.9}$$

Remark 6.9. From (6.9) we choose $n = 2$ and we have the inequality

$$\left|P_{4,2}\left(\frac{a+b}{2}\right)\right| := \left|\int_a^b f(t)\,dt - \left(\frac{b-a}{4}\right)\left(f(b) + f(a) + 2f\left(\frac{a+b}{2}\right)\right) + \left(\frac{b-a}{4}\right)^2\left(\frac{f'(b)-f'(a)}{2}\right)\right| \le \frac{\|f''\|_\infty}{96}(b-a)^3, \quad f'' \in L_\infty[a,b]. \tag{6.10}$$

Theorem 6.10. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous mapping on $[a,b]$ and let $a < x_1 \le x_2 \le b$ and $\alpha_1 \in [a, x_1)$, $\alpha_2 \in [x_1, x_2)$ and $\alpha_3 \in [x_2, b]$. Then we have the inequality*

$$\left|\int_a^b f(t)\,dt + \sum_{j=1}^{n}\frac{(-1)^j}{j!}\left[-(a-\alpha_1)^j f^{(j-1)}(a) + \left((x_1-\alpha_1)^j - (x_1-\alpha_2)^j\right) f^{(j-1)}(x_1) + \left((x_2-\alpha_2)^j - (x_2-\alpha_3)^j\right) f^{(j-1)}(x_2) + (b-\alpha_3)^j f^{(j-1)}(b)\right]\right| \le \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left((\alpha_1-a)^{n+1} + (x_1-\alpha_1)^{n+1} + (\alpha_2-x_1)^{n+1} + (x_2-\alpha_2)^{n+1} + (\alpha_3-x_2)^{n+1} + (b-\alpha_3)^{n+1}\right), \quad f^{(n)} \in L_\infty[a,b]. \tag{6.11}$$

Proof. Consider the division $a = x_0 < x_1 < x_2 = b$, $\alpha_1 \in [a, x_1)$, $\alpha_2 \in [x_1, x_2)$, $\alpha_3 \in [x_2, b]$, $\alpha_0 = a$, $x_0 = a$, $x_3 = b$ and put $\alpha_4 = b$. From the left hand side of (3.1) we obtain

$$\sum_{j=1}^{n}\frac{(-1)^j}{j!}\sum_{i=0}^{3}\left\{(x_i-\alpha_i)^j - (x_i-\alpha_{i+1})^j\right\} f^{(j-1)}(x_i) = \sum_{j=1}^{n}\frac{(-1)^j}{j!}\left[-(a-\alpha_1)^j f^{(j-1)}(a) + \left((x_1-\alpha_1)^j - (x_1-\alpha_2)^j\right) f^{(j-1)}(x_1) + \left((x_2-\alpha_2)^j - (x_2-\alpha_3)^j\right) f^{(j-1)}(x_2) + (b-\alpha_3)^j f^{(j-1)}(b)\right].$$

From the right hand side of (3.1) we obtain

$$\begin{aligned}\frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\sum_{i=0}^{2}\left\{(\alpha_{i+1}-x_i)^{n+1}+(x_{i+1}-\alpha_{i+1})^{n+1}\right\}\\ =\frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\Big((\alpha_1-a)^{n+1}+(x_1-\alpha_1)^{n+1}+(\alpha_2-x_1)^{n+1}\\ +(x_2-\alpha_2)^{n+1}+(\alpha_3-x_2)^{n+1}+(b-\alpha_3)^{n+1}\Big)\end{aligned}$$

and hence the first line of the inequality (6.11) follows and Theorem 6.10 is proved. □

Corollary 6.11. *Let f be defined as in Theorem 6.10 and consider the division*

$$a\le\alpha_1\le\frac{(m-1)\,a+b}{m}\le\alpha_2\le\frac{a+(m-1)\,b}{m}\le\alpha_3\le b$$

for m a natural number, $m\ge 2$. Then we have the inequality

$$\begin{aligned}(6.12)\quad |Q_{m,n}| :=& \left|\int_a^b f(t)\,dt+\sum_{j=1}^{n}\frac{(-1)^j}{j!}\Big[-(a-\alpha_1)^j f^{(j-1)}(a)\right.\\ &+\left\{\left(\frac{(m-1)\,a+b}{m}-\alpha_1\right)^j-\left(\frac{(m-1)\,a+b}{m}-\alpha_2\right)^j\right\}f^{(j-1)}(x_1)\\ &+\left\{\left(\frac{a+(m-1)\,b}{m}-\alpha_2\right)^j-\left(\frac{a+(m-1)\,b}{m}-\alpha_3\right)^j\right\}f^{(j-1)}(x_2)\\ &\left.+\;(b-\alpha_3)^j f^{(j-1)}(b)\Big]\right|\\ \le& \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}R_{m,n}\\ :=& \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left((\alpha_1-a)^{n+1}+\left(\frac{(m-1)\,a+b}{m}-\alpha_1\right)^{n+1}\right.\\ &+\left(\alpha_2-\frac{(m-1)\,a+b}{m}\right)^{n+1}+\left(\frac{a+(m-1)\,b}{m}-\alpha_2\right)^{n+1}\\ &\left.+\left(\alpha_3-\frac{a+(m-1)\,b}{m}\right)^{n+1}+(b-\alpha_3)^{n+1}\right),\quad f^{(n)}\in L_\infty[a,b].\end{aligned}$$

Proof. Choose in Theorem 6.10, $x_1=\frac{(m-1)a+b}{m}$, and $x_2=\frac{a+(m-1)b}{m}$, hence the theorem is proved. □

Remark 6.12. For particular choices of the parameters m and n, Corollary 6.11 contains a generalisation of the three-eighths rule of Newton and Cotes.

The following corollary is a consequence of Corollary 6.11.

Corollary 6.13. *Let f be defined as in Theorem 6.10 and choose $\alpha_2 = \frac{a+b}{2} = \frac{x_1+x_2}{2}$, then we have the inequality*

$$\begin{aligned}(6.13)\qquad \left|\bar{Q}_{m,n}\right| &:= \Bigg|\int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!}\Bigg[-(a-\alpha_1)^j f^{(j-1)}(a) \\ &\quad + \left\{(x_1-\alpha_1)^j - (-1)^j \left(\frac{(m-2)(b-a)}{2m}\right)^j\right\} f^{(j-1)}(x_1) \\ &\quad + \left\{\left(\frac{(m-2)(b-a)}{2m}\right)^j - (x_2-\alpha_3)^j\right\} f^{(j-1)}(x_2) \\ &\quad + (b-\alpha_3)^j f^{(j-1)}(b)\Bigg]\Bigg| \\ &\le \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\bar{R}_{m,n} \\ &:= \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left((\alpha_1-a)^{n+1} + \left(\frac{(m-1)a+b}{m} - \alpha_1\right)^{n+1}\right. \\ &\quad +2\left((b-a)\frac{(m-2)}{2m}\right)^{n+1} + \left(\alpha_3 - \frac{a+(m-1)b}{m}\right)^{n+1} \\ &\quad \left. + (b-\alpha_3)^{n+1}\right), \qquad f^{(n)} \in L_\infty[a,b].\end{aligned}$$

Proof. If we put $\alpha_2 = \frac{a+b}{2} = \frac{x_1+x_2}{2}$ into (6.12), we obtain the inequality (6.13) and the corollary is proved. □

The following corollary contains an optimum estimate for the inequality (6.13).

Corollary 6.14. *Let f be defined as in Theorem 6.10 and make the choices*

$$\begin{aligned}\alpha_1 &= \left(\frac{3m-4}{2m}\right)a + \left(\frac{4-m}{2m}\right)b \quad \textit{and} \\ \alpha_3 &= \left(\frac{4-m}{2m}\right)a + \left(\frac{3m-4}{2m}\right)b\end{aligned}$$

then we have the best estimate

$$\begin{aligned}(6.14)\qquad &\left|\hat{Q}_{m,n}\right| \\ &:= \Bigg|\int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!} \\ &\qquad \times \Bigg[\left(\frac{(b-a)(4-m)}{2m}\right)^j \left\{f^{(j-1)}(b) - (-1)^j f^{(j-1)}(a)\right\} \\ &\qquad + \left(\frac{(b-a)(m-2)}{2m}\right)^j \left\{1-(-1)^j\right\}\left(f^{(j-1)}(x_1) + f^{(j-1)}(x_2)\right)\Bigg]\Bigg| \\ &\le \frac{2\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left(\frac{b-a}{2m}\right)^{n+1}\left((4-m)^{n+1} + 2(m-2)^{n+1}\right), \quad f^{(n)} \in L_\infty[a,b].\end{aligned}$$

Proof. Using the choice $\alpha_2 = \frac{a+b}{2} = \frac{x_1+x_2}{2}$, $x_1 = \frac{(m-1)a+b}{m}$ and $x_2 = \frac{a+(m-1)b}{m}$ we may calculate

$$(\alpha_1 - a) = \frac{(4-m)(b-a)}{2m} = (b - \alpha_3)$$

and

$$\begin{aligned}(x_1 - \alpha_1) &= (\alpha_2 - x_1) = (x_2 - \alpha_2) = (\alpha_3 - x_2) \\ &= \frac{(m-2)(b-a)}{2m}.\end{aligned}$$

Substituting in the inequality (6.13) we obtain the proof of (6.14). □

Remark 6.15. For $m = 3$, we have the best estimation of (6.14) such that

$$\begin{aligned}\left|\hat{Q}_{3,n}\right| &:= \left|\int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!}\right. \\ &\quad \times \left[\left(\frac{b-a}{6}\right)^j \left\{f^{(j-1)}(b) - (-1)^j f^{(j-1)}(a)\right\} + \left(\frac{b-a}{6}\right)^j\right. \\ &\quad \left.\left.\times \left\{1-(-1)^j\right\}\left(f^{(j-1)}\left(\frac{2a+b}{2}\right) + f^{(j-1)}\left(\frac{a+2b}{2}\right)\right)\right]\right| \\ &\le \frac{\left\|f^{(n)}\right\|_\infty}{6^n (n+1)!}(b-a)^{n+1}, \quad f^{(n)} \in L_\infty[a,b]. \end{aligned} \tag{6.15}$$

Proof. From the right hand side of (6.14), consider the mapping

$$M_{m,n} := \left(\frac{4-m}{2m}\right)^{n+1} + 2\left(\frac{m-2}{2m}\right)^{n+1}$$

then

$$M'_{m,n} = \frac{2(n+1)}{m^2}\left(-\left(\frac{2}{m} - \frac{1}{2}\right)^n + \left(\frac{1}{2} - \frac{1}{m}\right)^n\right)$$

and $M_{m,n}$ attains its optimum when

$$\frac{2}{m} - \frac{1}{2} = \frac{1}{2} - \frac{1}{m},$$

in which case $m = 3$. Substituting $m = 3$ into (6.14), we obtain (6.15) and the corollary is proved. □

When $n = 2$, then from (6.15) we have

$$\begin{aligned}\left|\hat{Q}_{3,2}\right| &:= \left|\int_a^b f(t)\,dt - \left[\left(\frac{b-a}{6}\right)(f(b) + f(a))\right.\right. \\ &\qquad + \frac{1}{2}\left(\frac{b-a}{6}\right)^2 (f'(b) - f'(a)) \\ &\qquad \left.\left.+ \left(\frac{b-a}{3}\right)\left(f\left(\frac{2a+b}{2}\right) + f\left(\frac{a+2b}{2}\right)\right)\right]\right| \\ &\le \frac{\|f''\|_\infty}{216}(b-a)^3, \quad f'' \in L_\infty[a,b].\end{aligned}$$

The next corollary encapsulates the generalised Newton-Cotes inequality.

Corollary 6.16. *Let f be defined as in Theorem 6.10 and choose*

$$\begin{aligned} x_1 &= \frac{2a+b}{3},\ x_2 = \frac{a+2b}{3},\ \alpha_2 = \frac{a+b}{2}, \\ \alpha_1 &= \frac{(r-1)\,a+b}{r} \text{ and } \alpha_3 = \frac{a+(r-1)\,b}{r}. \end{aligned}$$

Then for r a natural number, $r \geq 3$, we have the inequality

$$\begin{aligned} &|T_{r,n}| \\ &:= \left| \int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!} \left[\left(\frac{b-a}{r}\right)^j \left\{ f^{(j-1)}(b) - (-1)^j f^{(j-1)}(a) \right\} \right.\right. \\ &\quad + \left\{ \left(\frac{(b-a)(r-3)}{3r}\right)^j - (-1)^j \left(\frac{b-a}{6}\right)^j \right\} f^{(j-1)}(x_1) \\ &\quad \left.\left. + \left\{ \left(\frac{b-a}{6}\right)^j - (-1)^j \left(\frac{(b-a)(r-3)}{3r}\right)^j \right\} f^{(j-1)}(x_2) \right] \right| \\ &\leq \frac{2\left\|f^{(n)}\right\|_\infty}{(n+1)!} (b-a)^{n+1} \left(\frac{1}{r^{n+1}} + \left(\frac{r-3}{3r}\right)^{n+1} + \frac{1}{6^{n+1}} \right), \quad f^{(n)} \in L_\infty[a,b]. \end{aligned} \tag{6.16}$$

Proof. From Theorem 6.10, we put $x_1 = \frac{2a+b}{3}$, $x_2 = \frac{a+2b}{3}$, $\alpha_2 = \frac{a+b}{2}$, $\alpha_1 = \frac{(r-1)a+b}{r}$ and $\alpha_3 = \frac{a+(r-1)b}{r}$. Then (6.16) follows. □

Remark 6.17. The optimum estimate of the inequality (6.16) occurs when $r = 6$.
from (6.16) consider the mapping

$$M_{r,n} := \frac{1}{r^{n+1}} + \left(\frac{r-3}{3r}\right)^{n+1} + \frac{1}{6^{n+1}}$$

the $M'_{r,n} = -(n+1)\,r^{-n-2} + \left(\frac{n+1}{r^2}\right)\left(\frac{1}{3} - \frac{1}{r}\right)^n$ and $M_{r,n}$ attains its optimum when $\frac{1}{r} = \frac{1}{3} - \frac{1}{r}$, in which case $r = 6$. In this case, we obtain the inequality (6.15) and specifically for $n = 1$, we obtain a Simpson type inequality

$$\begin{aligned} \left|\hat{Q}_{3,1}\right| &:= \left| \int_a^b f(t)\,dt - \left(\frac{b-a}{6}\right)(f(a) + f(b)) \right. \\ &\quad \left. - \left(\frac{b-a}{3}\right)\left(f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right) \right| \\ &\leq \frac{\|f'\|_\infty}{12} (b-a)^2, \quad f' \in L_\infty[a,b], \end{aligned} \tag{6.17}$$

which is better than that given by (6.7).

Corollary 6.18. *Let f be defined as in Theorem 6.10 and choose $m = 8$ such that $\alpha_1 = \frac{7a+b}{8}$, $\alpha_2 = \frac{a+b}{8}$ and $\alpha_3 = \frac{a+7b}{8}$ with $x_1 = \frac{2a+b}{3}$ and $x_2 = \frac{a+2b}{3}$. Then we have the inequality*

$$(6.18)\quad |T_{8,n}| := \left| \int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!} \left[\left(\frac{b-a}{8}\right)^j \left(f^{(j-1)}(b) - (-1)^j f^{(j-1)}(a)\right) + \left(\frac{b-a}{6}\right)^j \left(\left(\frac{5}{4}\right)^j - (-1)^j\right) \left(f^{(j-1)}(x_1) - (-1)^j f^{(j-1)}(x_2)\right)\right]\right| \le \frac{2\left\|f^{(n)}\right\|_\infty}{(n+1)!} \left(\frac{b-a}{24}\right)^{n+1} \left(3^{n+1} + 4^{n+1} + 5^{n+1}\right), \quad f^{(n)} \in L_\infty[a,b].$$

Proof. From Theorem 6.10 we put

$$x_1 = \frac{2a+b}{3},\quad x_2 = \frac{a+2b}{3},\quad \alpha_1 = \frac{7a+b}{8},\quad \alpha_3 = \frac{a+7b}{8}$$

and $\alpha_2 = \frac{a+b}{2}$ and the inequality (6.18) is obtained. □

When $n = 1$ we obtain from (6.18) the 'three-eighths rule' of Newton-Cotes.

Remark 6.19. From (6.16) with $r = 3$ we have

$$|T_{3,n}| := \left| \int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!} \left[\left(\frac{b-a}{3}\right)^j \left(f^{(j-1)}(b) - (-1)^j f^{(j-1)}(a)\right) + \left(\frac{b-a}{6}\right)^j \left\{f^{(j-1)}(x_2) - f^{(j-1)}(x_1)\right\}\right]\right| \le \frac{2\left\|f^{(n)}\right\|_\infty}{(n+1)!} (b-a)^{n+1} \left(\frac{1}{3^{n+1}} + \frac{1}{6^{n+1}}\right),\ f^{(n)} \in L_\infty[a,b].$$

In particular, for $n = 2$, we have the inequality

$$|T_{3,2}| := \left| \int_a^b f(t)\,dt - \left(\frac{b-a}{3}\right)(f(b) + f(a)) + \left(\frac{b-a}{3}\right)^2 \left(\frac{f'(b) - f'(a)}{2}\right) - \left(\frac{b-a}{6}\right)\left(f\left(\frac{a+2b}{3}\right) + f\left(\frac{2a+b}{3}\right)\right) + \left(\frac{b-a}{6}\right)^2 \left(\frac{f'\left(\frac{a+2b}{3}\right) - f'\left(\frac{2a+b}{3}\right)}{2}\right)\right| \le \frac{\|f''\|_\infty}{72} (b-a)^3,\quad f'' \in L_\infty[a,b].$$

The following theorem encapsulates Boole's rule.

Theorem 6.20. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous mapping on $[a,b]$ and let $a < x_1 < x_2 < x_3 < b$ and $\alpha_1 \in [a, x_1)$, $\alpha_2 \in [x_1, x_2)$, $\alpha_3 \in [x_2, x_3)$ and $\alpha_4 \in [x_3, b]$. Then we* *have the inequality*

$$\begin{aligned}(6.19)\quad & \left| \int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!} \Big[-(a-\alpha_1)^j f^{(j-1)}(a) \right. \\ & + \left((x_1-\alpha_1)^j - (x_1-\alpha_2)^j \right) f^{(j-1)}(x_1) \\ & + \left((x_2-\alpha_2)^j - (x_2-\alpha_3)^j \right) f^{(j-1)}(x_2) \\ & \left. + \left((x_3-\alpha_3)^j - (x_3-\alpha_4)^j \right) f^{(j-1)}(x_3) + (b-\alpha_4)^j f^{(j-1)}(b) \Big] \right| \\ & \le \frac{\left\| f^{(n)} \right\|_\infty}{(n+1)!} \Big((\alpha_1-a)^{n+1} + (x_1-\alpha_1)^{n+1} + (\alpha_2-x_1)^{n+1} \\ & + (x_2-\alpha_2)^{n+1} + (\alpha_3-x_2)^{n+1} + (x_3-\alpha_3)^{n+1} \\ & + (\alpha_4-x_3)^{n+1} + (b-\alpha_4)^{n+1} \Big) \quad \textit{if} \quad f^{(n)} \in L_\infty[a,b].\end{aligned}$$

Proof. Follows directly from (3.1) with the points $\alpha_0 = x_0 = a$, $x_4 = \alpha_5 = b$ and the division $a = x_0 < x_1 < x_2 < x_3 = b$, $\alpha_1 \in [a, x_1)$, $\alpha_2 \in [x_1, x_2)$, $\alpha_3 \in [x_2, x_3)$ and $\alpha_4 \in [x_3, b]$. □

The following inequality arises from Theorem 6.20.

Corollary 6.21. *Let f be defined as in Theorem 6.10 and choose $\alpha_1 = \frac{11a+b}{12}$, $\alpha_2 = \frac{11a+7b}{18}$, $\alpha_3 = \frac{7a+11b}{18}$, $\alpha_4 = \frac{a+11b}{12}$, $x_1 = \frac{7a+2b}{9}$, $x_3 = \frac{2a+7b}{9}$ and $x_2 = \frac{x_1+x_3}{2} = \frac{a+b}{2}$, then we can state:*

$$\begin{aligned} & \left| \int_a^b f(t)\,dt + \sum_{j=1}^{n} \frac{(-1)^j}{j!} \left[\left(\frac{b-a}{12} \right)^j \left\{ f^{(j-1)}(b) - (-1)^j f^{(j-1)}(a) \right\} \right. \right. \\ & + \left(\frac{b-a}{6} \right)^j \left\{ \left(\frac{5}{6} \right)^j - (-1)^j \right\} \left\{ f^{(j-1)}\left(\frac{7a+2b}{9} \right) - (-1)^j f^{(j-1)}\left(\frac{2a+7b}{9} \right) \right\} \\ & \left. \left. + \left(\frac{b-a}{9} \right)^j \left\{ 1 - (-1)^j \right\} f^{(j-1)}\left(\frac{a+b}{2} \right) \right] \right| \\ & \le \frac{2\left\| f^{(n)} \right\|_\infty}{(n+1)!} \left(\frac{b-a}{36} \right)^{n+1} \left(3^{n+1} + 4^{n+1} + 5^{n+1} + 6^{n+1} \right), \ \textit{if } f^{(n)} \in L_\infty[a,b].\end{aligned}$$

7. APPLICATIONS FOR NUMERICAL INTEGRATION

In this section we utilise the particular inequalities of the previous sections and apply them to numerical integration.

Consider the partitioning of the interval $[a,b]$ given by $\Delta_m : a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b$, put $h_i := x_{i+1} - x_i$ $(i = 0, \ldots, m-1)$ and put $\nu(h) := \max(h_i | i = 0, \ldots, m-1)$. The following theorem holds.

Theorem 7.1. *Let $f : [a,b] \to \mathbb{R}$ be AC$[a,b]$, $k \ge 1$ and $m \ge 1$. Then we have the composite quadrature formula*

$$\int_a^b f(t)\,dt = A_k(\Delta_m, f) + R_k(\Delta_m, f) \tag{7.1}$$

where

$$A_k(\Delta_m, f) := -T_k(\Delta_m, f) - U_k(\Delta_m, f), \tag{7.2}$$

$$(7.3)\qquad T_k(\Delta_m, f) := \sum_{i=0}^{m-1}\sum_{j=1}^{n}\left(\frac{h_i}{2k}\right)^j \frac{1}{j!}\left[-f^{(j-1)}(x_i) + (-1)^j f^{(j-1)}(x_{i+1})\right]$$

and

$$(7.4)\quad U_k(\Delta_m, f) := \sum_{i=0}^{m-1}\sum_{j=1}^{n}\left(\frac{h_i}{2k}\right)^j \frac{1}{j!}\times\left[\sum_{r=1}^{k-1}\left\{(-1)^j - 1\right\} f^{(j-1)}\left(\frac{(k-r)x_i + rx_{i+1}}{k}\right)\right]$$

is a perturbed quadrature formula. The remainder $R_k(\Delta_m, f)$ *satisfies the estimation*

$$(7.5)\qquad |R_k(\Delta_m, f)| \le \frac{\left\|f^{(n)}\right\|_\infty}{2^n (n+1)! k^{n+1}} \sum_{i=0}^{m-1} h_i^{n+1},\ \textit{if } f^{(n)} \in L_\infty[a,b],$$

where $\nu(h) := \max(h_i | i = 0, \dots, m-1)$.

Proof. We shall apply Corollary 3.3 on the interval $[x_i, x_{i+1}]$, $(i = 0, \dots, m-1)$. Thus we obtain

$$\left|\int_{x_i}^{x_{i+1}} f(t)\,dt + \sum_{j=1}^{n}\left(\frac{h_i}{2k}\right)^j \frac{1}{j!}\left[-f^{(j-1)}(x_i) + (-1)^j f^{(j-1)}(x_{i+1})\right.\right.$$
$$\left.\left. + \sum_{r=1}^{k-1}\left\{(-1)^j - 1\right\} f^{(j-1)}\left(\frac{(k-r)x_i + rx_{i+1}}{k}\right)\right]\right|$$
$$\le \frac{1}{(n+1)!2^n}\sup_{t\in[x_i,x_{i+1}]}\left|f^{(n)}(t)\right|\left(\frac{x_{i+1}-x_i}{k}\right)^{n+1}.$$

Summing over i from 0 to $m-1$ and using the generalised triangle inequality, we have

$$\sum_{i=0}^{m-1}\left|\int_{x_i}^{x_{i+1}} f(t)\,dt + \sum_{j=1}^{n}\left(\frac{h_i}{2k}\right)^j \frac{1}{j!}\left[-f^{(j-1)}(x_i) + (-1)^j f^{(j-1)}(x_{i+1})\right.\right.$$
$$\left.\left. + \sum_{r=1}^{k-1}\left\{(-1)^j - 1\right\} f^{(j-1)}\left(\frac{(k-r)x_i + rx_{i+1}}{k}\right)\right]\right|$$
$$\le \frac{1}{(n+1)!2^n}\sum_{i=0}^{m-1}\frac{h_i^{n+1}}{k^{n+1}}\sup_{t\in[x_i,x_{i+1}]}\left|f^{(n)}(t)\right|.$$

Now,

$$(7.6)\quad \left|\int_a^b f(t)\,dt + \sum_{i=0}^{m-1}\sum_{j=1}^{n}\left(\frac{h_i}{2k}\right)^j \frac{1}{j!}\left[-f^{(j-1)}(x_i) + (-1)^j f^{(j-1)}(x_{i+1})\right]\right.$$
$$\left. + \sum_{i=0}^{m-1}\sum_{j=1}^{n}\left(\frac{h_i}{2k}\right)^j \frac{1}{j!}\left[\sum_{r=1}^{k-1}\left\{(-1)^j - 1\right\} f^{(j-1)}\left(\frac{(k-r)x_i + rx_{i+1}}{k}\right)\right]\right|$$
$$\le R_k(\Delta_m, f).$$

As $\sup_{t\in[x_i,x_{i+1}]}\left|f^{(n)}(t)\right| \le \left\|f^{(n)}\right\|_\infty$, the inequality in (7.5) follows and the theorem is proved. □

The following corollary holds.

Corollary 7.2. *Let $f : [a, b] \to \mathbb{R}$ be a mapping such that $f^{(n-1)}$ is AC$[a, b]$, then we have the equality*

$$\int_a^b f(t)\,dt = -T_2(\Delta_m, f) - U_2(\Delta_m, f) + R_2(\Delta_m, f), \tag{7.7}$$

where

$$T_2(\Delta_m, f) := \sum_{i=0}^{m-1}\sum_{j=1}^{n}\left(\frac{h_i}{4}\right)^j \frac{1}{j!}\left[-f^{(j-1)}(x_i) + (-1)^j f^{(j-1)}(x_{i+1})\right]$$

$U_2(\Delta_m, f)$ *is the perturbed midpoint quadrature rule, containing only even derivatives*

$$U_2(\Delta_m, f) := \sum_{i=0}^{m-1}\sum_{j=1}^{n}\left(\frac{h_i}{4}\right)^j \frac{1}{j!}\left\{(-1)^j - 1\right\} f^{(j-1)}\left(\frac{x_i + x_{i+1}}{2}\right)$$

and the remainder, $R_2(\Delta_m, f)$ *satisfies the estimation*

$$|R_2(\Delta_m, f)| \le \frac{\left\|f^{(n)}\right\|_\infty}{2^{2n+1}(n+1)!}\sum_{i=0}^{m-1} h_i^{n+1}, \quad \text{if } f^{(n)} \in L_\infty[a, b].$$

Corollary 7.3. *Let f and Δ_m be defined as above. Then we have the equality*

$$\int_a^b f(t)\,dt = -T_3(\Delta_m, f) - U_3(\Delta_m, f) + R_3(\Delta_m, f), \tag{7.8}$$

where

$$T_3(\Delta_m, f) := \sum_{i=0}^{m-1}\sum_{j=1}^{n}\left(\frac{h_i}{6}\right)^j \frac{1}{j!}\left[-f^{(j-1)}(x_i) + (-1)^j f^{(j-1)}(x_{i+1})\right]$$

and

$$U_3(\Delta_m, f) := \sum_{i=0}^{m-1}\sum_{j=1}^{n}\frac{\left((-1)^j - 1\right)}{j!}\left(\frac{h_i}{6}\right)^j \times\left[f^{(j-1)}\left(\frac{2x_i + x_{i+1}}{3}\right) + f^{(j-1)}\left(\frac{x_i + 2x_{i+1}}{3}\right)\right]$$

and the remainder satisfies the bound

$$|R_3(\Delta_m, f)| \le \frac{\left\|f^{(n)}\right\|_\infty}{3 \cdot 6^n (n+1)!}\sum_{i=0}^{m-1} h_i^{n+1}, \quad \text{if } f^{(n)} \in L_\infty[a, b].$$

Theorem 7.4. *Let f and Δ_m be defined as in Theorem 7.1 and suppose that $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, \ldots, m-1)$. Then we have the quadrature formula:*

$$\int_a^b f(t)\,dt = \sum_{i=0}^{m-1}\sum_{j=1}^{n}\frac{(-1)^j}{j!}\Big\{(\xi_i - x_i)^j f^{(j-1)}(x_i) - (-1)^j (x_{i+1} - \xi_i)^j f^{(j-1)}(x_{i+1})\Big\} + R(\xi, \Delta_m, f) \tag{7.9}$$

and the remainder, $R(\xi, \Delta_m, f)$ *satisfies the inequality*

$$\text{(7.10)}\quad |R(\xi, \Delta_m, f)| \leq \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!} \sum_{i=0}^{m-1} \left((\xi_i - x_i)^{n+1} + (x_{i+1} - \xi_i)^{n+1}\right), \quad \textit{if} \;\; f^{(n)} \in L_\infty[a,b].$$

Proof. From Theorem 3.1 we put $\alpha_0 = a$, $x_0 = a$, $x_1 = b$, $\alpha_2 = b$ and $\alpha_1 = \alpha \in [a,b]$ such that

$$\int_a^b f(t)\,dt + \sum_{j=1}^n \frac{(-1)^j}{j!}\left[-(a-\alpha)^j f^{(j-1)}(a) + (b-\alpha)^j f^{(j-1)}(b)\right] = R(\xi, \Delta_m, f).$$

Over the interval $[x_i, x_{i+1}]$ $(i = 0, \ldots, m-1)$, we have

$$\int_{x_i}^{x_{i+1}} f(t)\,dt + \sum_{j=1}^n \frac{(-1)^j}{j!}\left[(x_{i+1} - \xi_i)^j f^{(j-1)}(x_{i+1}) - (-1)^j (\xi_i - x_i)^j f^{(j-1)}(x_i)\right] = R(\xi, \Delta_m, f)$$

and therefore, using the generalised triangle inequality

$$\begin{aligned}
&|R(\xi, \Delta_m, f)| \\
&\leq \sum_{i=0}^{m-1} \left| \int_{x_i}^{x_{i+1}} f(t)\,dt + \sum_{i=0}^{m-1}\sum_{j=1}^n \frac{(-1)^j}{j!}\left[(x_{i+1} - \xi_i)^j f^{(j-1)}(x_{i+1}) \right.\right. \\
&\qquad \left.\left. - (-1)^j (\xi_i - x_i)^j f^{(j-1)}(x_i)\right] \right| \\
&\leq \frac{1}{(n+1)!} \sum_{i=0}^{m-1} \sup_{t \in [x_i, x_{i+1}]} \left|f^{(n)}(t)\right| \left((\xi_i - x_i)^{n+1} + (x_{i+1} - \xi_i)^{n+1}\right).
\end{aligned}$$

The inequality in (7.11) follows, since we have $\sup_{t \in [x_i, x_{i+1}]} \left|f^{(n)}(t)\right| \leq \left\|f^{(n)}\right\|_\infty$, and Theorem 7.4 is proved. □

The following corollary is a consequence of Theorem 7.4.

Corollary 7.5. *Let* f *and* Δ_m *be defined as in Theorem 7.1. The following estimates apply.*

(i) *The* n^{th} *order left rectangle rule*

$$\int_a^b f(t)\,dt = \sum_{i=0}^{m-1}\sum_{j=1}^n \frac{(-h_i)^j}{j!} f^{(j-1)}(x_i) + R_l(\Delta_m, f).$$

(ii) *The* n^{th} *order right rectangle rule*

$$\int_a^b f(t)\,dt = -\sum_{i=0}^{m-1}\sum_{j=1}^n \frac{(h_i)^j}{j!} f^{(j-1)}(x_{i+1}) + R_r(\Delta_m, f).$$

(iii) *The* n^{th} *order trapezoidal rule*

$$\int_a^b f(t)\,dt = \sum_{i=0}^{m-1}\sum_{j=1}^n \left(-\frac{h_i}{2}\right)^j \frac{1}{j!}\left\{f^{(j-1)}(x_i) - (-1)^j f^{(j-1)}(x_{i+1})\right\} + R_T(\Delta_m, f),$$

where

$$|R_l(\Delta_m,f)| = |R_r(\Delta_m,f)| \le \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\sum_{i=0}^{m-1} h_i^{n+1}, \text{ if } f^{(n)} \in L_\infty[a,b]$$

and

$$|R_T(\Delta_m,f)| \le \frac{\left\|f^{(n)}\right\|_\infty}{2^n (n+1)!}\sum_{i=0}^{m-1} h_i^{n+1}, \text{ if } f^{(n)} \in L_\infty[a,b].$$

Theorem 7.6. *Consider the interval* $x_i \le \alpha_i^{(1)} \le \xi_i \le \alpha_i^{(2)} \le x_{i+1}$, $i = 0,\dots,m-1$, *and let* f *and* Δ_m *be defined as above. Then we have the equality*

$$\begin{aligned}(7.11)\quad \int_a^b f(t)\,dt = \sum_{i=0}^{m-1}\sum_{j=1}^{n}\frac{(-1)^j}{j!}\Big\{\left(x_i-\alpha_i^{(i)}\right)^j f^{(j-1)}(x_i) \\ -\left\{\left(\xi_i-\alpha_i^{(1)}\right)^j - \left(\xi_i-\alpha_i^{(2)}\right)^j\right\} f^{(j-1)}(\xi_i) \\ -\left(x_{i+1}-\alpha_i^{(2)}\right)^j f^{(j-1)}(x_{i+1})\Big\} + R\left(\xi,\alpha_i^{(1)},\alpha_i^{(2)},\Delta_m,f\right)\end{aligned}$$

and the remainder satisfies the estimation

$$\begin{aligned}\left|R\left(\xi,\alpha_i^{(1)},\alpha_i^{(2)},\Delta_m,f\right)\right| \le \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\sum_{i=0}^{m-1}\Big\{\left(\alpha_i^{(1)}-x_i\right)^{n+1} + \left(\xi_i-\alpha_i^{(1)}\right)^{n+1} \\ + \left(\alpha_i^{(2)}-\xi_i\right)^{n+1} + \left(x_{i+1}-\alpha_i^{(2)}\right)^{n+1}\Big\}, \text{ if } f^{(n)} \in L_\infty[a,b].\end{aligned}$$

The proof follows directly from Theorem 6.1 on the intervals $[x_i, x_{i+1}]$, $(i = 0,\dots,m-1)$.

The following Riemann type formula also holds.

Corollary 7.7. *Let* f *and* Δ_m *be defined as in Theorem 7.1 and choose* $\xi_i \in [x_i, x_{i+1}]$, $(i = 0,\dots,m-1)$. *Then we have the equality*

$$\begin{aligned}(7.12)\quad \int_a^b f(t)\,dt = \sum_{i=0}^{m-1}\sum_{j=1}^{n}\frac{(-1)^j}{j!}\left\{(\xi_i-x_i)^{n+1} - (\xi_i-x_{i+1})^{n+1}\right\} f^{(j-1)}(\xi_i) \\ + R_R(\xi,\Delta_m,f)\end{aligned}$$

and the remainder satisfies the estimation

$$|R_R(\xi,\Delta_m,f)| \le \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\sum_{i=0}^{m-1}\left((\xi_i-x_i)^{n+1} + (x_{i+1}-\xi_i)^{n+1}\right), \quad \text{if } f^{(n)} \in L_\infty[a,b].$$

The proof follows from (7.11) where $\alpha_i^{(1)} = x_i$ and $\alpha_i^{(2)} = x_{i+1}$.

Remark 7.8. If in (7.12) we choose the midpoint $2\xi_i = x_{i+1} + x_i$ we obtain the generalised midpoint quadrature formula

(7.13)

$$\int_a^b f(t)\,dt = \sum_{i=0}^{m-1}\sum_{j=1}^{n}\frac{(-1)^{j+1}}{j!}\left(\frac{h_i}{2}\right)^j \left\{1-(-1)^j\right\} f^{(j-1)}\left(\frac{x_i+x_{i+1}}{2}\right) + R_M(\Delta_m,f)$$

and $R_M(\Delta_m, f)$ is bounded by

$$|R_M(\Delta_m, f)| \le \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!2^n} \sum_{i=0}^{m-1} h_i^{n+1}, \quad \text{if } f^{(n)} \in L_\infty[a,b].$$

Corollary 7.9. *Consider a set of points*

$$\xi_i \in \left[\frac{5x_i + x_{i+1}}{6}, \frac{x_i + 5x_{i+1}}{6}\right] \quad (i = 0, \dots, m-1)$$

and let f and Δ_m be defined as in Theorem 7.1. Then we have the equality

$$\int_a^b f(t)\,dt = \sum_{i=0}^{m-1} \sum_{j=1}^{n} \frac{(-1)^j}{j!} \left[\left(\frac{h_i}{6}\right)^j \left\{(-1)^j f^{(j-1)}(x_i) - f^{(j-1)}(x_{i+1})\right\}\right.$$
$$\left. - \left\{\left(\xi_i - \frac{5x_i + x_{i+1}}{6}\right)^j - \left(\xi_i - \frac{x_i + 5x_{i+1}}{6}\right)^j\right\} f^{(j-1)}(\xi_i)\right] + R_s(\Delta_m, f) \tag{7.14}$$

and the remainder, $R_s(\Delta_m, f)$ satisfies the bound

$$|R_s(\Delta_m, f)| \le \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!} \sum_{i=0}^{m-1} \left\{2\left(\frac{h_i}{6}\right)^{n+1} + \left(\xi_i - \frac{5x_i + x_{i+1}}{6}\right)^{n+1}\right.$$
$$\left. + \left(\frac{x_i + 5x_{i+1}}{6} - \xi_i\right)^{n+1}\right\}, \quad \textit{if } f^{(n)} \in L_\infty[a,b].$$

Remark 7.10. If in (7.14) we choose the midpoint $\xi_i = \frac{x_{i+1}+x_i}{2}$ we obtain a generalised Simpson formula:

$$\int_a^b f(t)\,dt = \sum_{i=0}^{m-1} \sum_{j=1}^{n} \frac{(-1)^j}{j!} \left[\left(\frac{h_i}{6}\right)^j \left\{(-1)^j f^{(j-1)}(x_i) - f^{(j-1)}(x_{i+1})\right\}\right.$$
$$\left. - \left(\frac{h_i}{3}\right)^j \left\{1 - (-1)^j\right\} f^{(j-1)}\left(\frac{x_{i+1} + x_i}{2}\right)\right] + R_s(\Delta_m, f)$$

and $R_s(\Delta_m, f)$ is bounded by

$$|R_s(\Delta_m, f)| \le \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!} 2\left(1 + 2^{n+1}\right) \sum_{i=0}^{m-1} \left(\frac{h_i}{6}\right)^{n+1}, \quad \text{if } f^{(n)} \in L_\infty[a,b].$$

The following is a consequence of Theorem 7.6.

Corollary 7.11. *Consider the interval*

$$x_i \le \alpha_i^{(1)} \le \frac{x_{i+1} + x_i}{2} \le \alpha_i^{(2)} \le x_{i+1} \quad (i = 0, \dots, m-1),$$

and let f and Δ_m be defined as in Theorem 7.1.
The following equality is obtained:

$$\int_a^b f(t)\,dt = \sum_{i=0}^{m-1}\sum_{j=1}^{n}\frac{(-1)^j}{j!}\left\{\left(x_i-\alpha_i^{(i)}\right)^j f^{(j-1)}(x_i)\right. \\ -\left\{\left(\frac{x_{i+1}+x_i}{2}-\alpha_i^{(1)}\right)^j-\left(\frac{x_{i+1}+x_i}{2}-\alpha_i^{(2)}\right)^j\right\}f^{(j-1)}\left(\frac{x_{i+1}+x_i}{2}\right) \\ \left.-\left(x_{i+1}-\alpha_i^{(2)}\right)^j f^{(j-1)}(x_{i+1})\right\}+R_B\left(\alpha_i^{(1)},\alpha_i^{(2)},\Delta_m,f\right), \tag{7.15}$$

where the remainder satisfies the bound

$$\left|R_B\left(\alpha_i^{(1)},\alpha_i^{(2)},\Delta_m,f\right)\right| \\ \le \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\sum_{i=0}^{m-1}\left\{\left(\alpha_i^{(1)}-x_i\right)^{n+1}+\left(\frac{x_{i+1}+x_i}{2}-\alpha_i^{(1)}\right)^{n+1}\right. \\ \left.+\left(\alpha_i^{(2)}-\frac{x_{i+1}+x_i}{2}\right)^{n+1}+\left(x_{i+1}-\alpha_i^{(2)}\right)^{n+1}\right\}.$$

The following remark applies to Corollary 7.11.

Remark 7.12. If in (7.15) we choose

$$\alpha_i^{(1)}=\frac{3x_i+x_{i+1}}{4}\ \text{ and }\ \alpha_i^{(2)}=\frac{x_i+3x_{i+1}}{4},$$

we have the formula:

$$\int_a^b f(t)\,dt = \sum_{i=0}^{m-1}\sum_{j=1}^{n}\frac{(-1)^j}{j!}\left(\frac{h_i}{4}\right)^j\left[(-1)^j f^{(j-1)}(x_i)-f^{(j-1)}(x_{i+1})\right. \\ \left.-\left\{1-(-1)^j\right\}f^{(j-1)}\left(\frac{x_{i+1}+x_i}{2}\right)\right]+R_B(\Delta_m,f). \tag{7.16}$$

The remainder, $R_B(\Delta_m,f)$ satisfies the bound

$$|R_B(\Delta_m,f)|\le\frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\times 4\sum_{i=0}^{m-1}\left(\frac{h_i}{4}\right)^{n+1},\ \text{ if }\ f^{(n)}\in L_\infty[a,b].$$

The following theorem incorporates the Newton-Cotes formula.

Theorem 7.13. *Consider the interval*

$$x_i\le\alpha_i^{(1)}\le\xi_i^{(1)}\le\alpha_i^{(2)}\le\xi_i^{(2)}\le\alpha_i^{(3)}\le x_{i+1}\quad(i=0,\dots,m-1),$$

and let Δ_m and f be defined as in Theorem 7.1. This consideration gives us the equality

$$\int_a^b f(t)\,dt = \sum_{i=0}^{m-1}\sum_{j=1}^{n}\frac{(-1)^j}{j!}\Bigg[\left(x_i-\alpha_i^{(1)}\right)^j f^{(j-1)}(x_i) - \left(x_{i+1}-\alpha_i^{(3)}\right)^j f^{(j-1)}(x_{i+1}) - \left\{\left(\xi_i^{(1)}-\alpha_i^{(1)}\right)^j - \left(\xi_i^{(1)}-\alpha_i^{(2)}\right)^j\right\} f^{(j-1)}\left(\xi_i^{(1)}\right) - \left\{\left(\xi_i^{(2)}-\alpha_i^{(2)}\right)^j - \left(\xi_i^{(2)}-\alpha_i^{(3)}\right)^j\right\} f^{(j-1)}\left(\xi_i^{(2)}\right)\Bigg] + R\left(\alpha_i^{(1)},\alpha_i^{(2)},\alpha_i^{(3)},\xi_i^{(1)},\xi_i^{(2)},\Delta_m,f\right). \tag{7.17}$$

The remainder satisfies the bound

$$\left|R\left(\alpha_i^{(1)},\alpha_i^{(2)},\alpha_i^{(3)},\xi_i^{(1)},\xi_i^{(2)},\Delta_m,f\right)\right| \le \frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\sum_{i=0}^{m-1}\Bigg\{\left(\alpha_i^{(1)}-x_i\right)^{n+1} + \left(\xi_i^{(1)}-\alpha_i^{(1)}\right)^{n+1} + \left(\alpha_i^{(2)}-\xi_i^{(1)}\right)^{n+1} + \left(\xi_i^{(2)}-\alpha_i^{(2)}\right)^{n+1} + \left(\alpha_i^{(3)}-\xi_i^{(2)}\right)^{n+1} + \left(x_{i+1}-\alpha_i^{(3)}\right)^{n+1}\Bigg\}, \quad \text{if } f^{(n)}\in L_\infty[a,b].$$

The following is a consequence of Theorem 7.13.

Corollary 7.14. *Let f and Δ_m be defined as above and make the choices $\alpha_i^{(1)}=\frac{7x_i+x_{i+1}}{8}$, $\alpha_i^{(2)}=\frac{x_i+x_{i+1}}{2}$, $\alpha_i^{(3)}=\frac{x_i+7x_{i+1}}{8}$, $\xi_i^{(1)}=\frac{2x_i+x_{i+1}}{3}$ and $\xi_i^{(2)}=\frac{x_i+2x_{i+1}}{3}$, then we have the equality:*

$$\int_a^b f(t)\,dt = \sum_{i=0}^{m-1}\sum_{j=1}^{n}\frac{(-1)^j}{j!}\Bigg[\left(\frac{h_i}{8}\right)^j\left\{(-1)^j f^{(j-1)}(x_i) - f^{(j-1)}(x_{i+1})\right\} - \left(\frac{h_i}{24}\right)^j\left\{5^j-(-4)^j\right\} f^{(j-1)}\left(\frac{2x_i+x_{i+1}}{3}\right) - \left(\frac{h_i}{24}\right)^j\left\{4^j-(-5)^j\right\} f^{(j-1)}\left(\frac{x_i+2x_{i+1}}{3}\right)\Bigg] + R_N(\Delta_m,f), \tag{7.18}$$

where the remainder satisfies the bound

$$|R_N(\Delta_m,f)| \le \frac{2\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left(3^{n+1}+4^{n+1}+5^{n+1}\right)\sum_{i=0}^{m-1}\left(\frac{h_i}{24}\right)^{n+1}, \quad \text{if } f^{(n)}\in L_\infty[a,b].$$

When $n=1$, we obtain from (7.18) the three-eighths rule of Newton-Cotes.

Remark 7.15. For $n = 2$ from (7.18), we obtained a perturbed three-eighths Newton-Cotes formula:

$$\int_a^b f(t)\,dt = \sum_{i=0}^{m-1}\left(\left(\frac{h_i}{8}\right)(f(x_i) + f(x_{i+1})) + \left(\frac{h_i}{8}\right)^2\left(\frac{f'(x_i) - f'(x_{i+1})}{2}\right)\right.$$
$$+ \left(\frac{3h_i}{8}\right)\left(f\left(\frac{2x_i + x_{i+1}}{3}\right) + f\left(\frac{x_i + 2x_{i+1}}{3}\right)\right)$$
$$\left. - \left(\frac{3h_i}{8}\right)^2\left\{\frac{f'\left(\frac{2x_i+x_{i+1}}{3}\right) - f'\left(\frac{x_i+2x_{i+1}}{3}\right)}{2}\right\}\right] + R_N(\Delta_m, f),$$

where the remainder satisfies the bound

$$|R_N(\Delta_m, f)| \le \frac{\|f''\|_\infty}{192}\sum_{i=0}^{m-1} h_i^3, \text{ if } f'' \in L_\infty[a, b].$$

8. Conclusion

This paper has extended many previous Ostrowski type results. Integral inequalities for $n-$times differentiable mappings have been obtained by the use of a generalised Peano kernel. Some particular integral inequalities, including the trapezoid, midpoint, Simpson and Newton-Cotes rules have been obtained and further developed into composite quadrature rules.

Further work in this area may be undertaken by considering the Chebychev and Lupaş inequalities. Similarly, the following alternate Grüss type results may be used to examine all the interior point rules of this paper.

Let $\sigma(h(x)) = h(x) - M(g)$ where

$$M(h) = \frac{1}{b-a}\int_a^b h(t)\,dt.$$

Then from (5.2)

$$T(h, g) = M(hg) - M(h)M(g).$$

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