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INEQUALITIES INVOLVING A LOGARITHMICALLY CONVEX FUNCTION AND THEIR APPLICATIONS TO SPECIAL FUNCTIONS

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ABSTRACT. It has been shown that if f is a differentiable, logarithmically convex function on nonnegative semi-axis, then the function $[f(x)]^a/f(ax)$, $(a \geq 1)$ is decreasing on its domain. Applications to inequalities involving gamma function, Riemann's zeta function, and the complete elliptic integrals of the first kind are included.

Key words and phrases: Logarithmically convex functions, inequalities, gamma function, Riemann's zeta function, complete elliptic integrals of the first kind.

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1. INTRODUCTION AND NOTATION

Logarithmically convex (log-convex) functions are of interest in many areas of mathematics and science. They have been found to play an important role in the theory of special functions and mathematical statistics (see, e.g., [3], [4], [7]).

In what follows the symbols $\mathbb{R}_+$ and $\mathbb{R}_>$ will stand for the nonnegative semi-axis and positive semi-axis, respectively.

Recall that a function $f : [c, d] \to \mathbb{R}_>$ is said to be log-convex if $f[ux + (1 - u)y] \leq [f(x)]^u[f(y)]^{1-u}$ $(0 \leq u \leq 1)$ holds for all $x, y \in [c, d]$. It is well-known that a family of log-convex functions is closed under both addition and multiplication.

In the next section we shall establish a monotonicity property and some inequalities involving a function which is defined in terms of a log-convex function. Applications to inequalities for the gamma function, Riemann's zeta function, and the complete elliptic integrals of the first kind are also included in Section 2.

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2. MAIN RESULT AND ITS APPLICATIONS

We are in a position to prove the following.

Theorem 2.1. *Let $f : \mathbb{R}_+ \to \mathbb{R}_>$ be a differentiable, log-convex function and let $a \geq 1$. Then the function*

$$g(x) = \frac{[f(x)]^a}{f(ax)} \tag{2.1}$$

decreases on its domain. In particular, if $0 \leq x \leq y$, then the following inequalities

$$\frac{[f(y)]^a}{f(ay)} \leq \frac{[f(x)]^a}{f(ax)} \leq [f(0)]^{a-1} \tag{2.2}$$

hold true. If $0 < a \leq 1$, then the function g is an increasing function on $\mathbb{R}_+$ and the inequalities (2.2) *are reversed.*

Proof. We shall prove the theorem when $a \geq 1$. Logarithmic convexity of f implies that its logarithmic derivative $\alpha(x) := f'(x)/f(x)$ is an increasing function on $\mathbb{R}_+$ i.e., that

$$\alpha(x) \leq \alpha(ax). \tag{2.3}$$

Logarithmic differentiation of (2.1) gives

$$\frac{g'(x)}{g(x)} = a\left[\frac{f'(x)}{f(x)} - \frac{f'(ax)}{f(ax)}\right] = a\big[\alpha(x) - \alpha(ax)\big].$$

This in conjunction with (2.3) yields $g'(x) \leq 0$ because $g(x) > 0$ for all $x \in \mathbb{R}_+$. This proves the monotonicity property of the function g. Inequalities (2.2) now follow because for $0 \leq x \leq y$, $g(y) \leq g(x) \leq g(0)$. The proof is complete. □

The remaining part of this section deals with applications of the above result to some special functions. In what follows we shall always assume that $a \geq 1$.

2.1. **Inequalities involving the gamma function.** Let $f(x) = \Gamma(1+x)$ ($x \geq 0$). It is well known that the function f is log-convex (see, e.g., [3, Theorem 3.5-3]). Making use of Theorem 2.1 we conclude that the function $\big[\Gamma(1+x)\big]^a \big/ \Gamma(1+ax)$ decreases for all $x \geq 0$ and the inequalities

$$\frac{\big[\Gamma(1+y)\big]^a}{\Gamma(1+ay)} \leq \frac{\big[\Gamma(1+x)\big]^a}{\Gamma(1+ax)} \leq 1 \tag{2.4}$$

hold true for $0 \leq x \leq y$. Inequalities (2.4), when $y = 1$, have been obtained in [8, (2.3)]. Letting, in (2.4), $a = n$ (n-positive integer) and $y = 1$ we rediscover inequalities established in [2].

2.2. **Inequalities for the Riemann zeta function.** A beautiful formula which connects Euler's gamma function and Riemann's zeta function

$$\Gamma(1+x)\zeta(1+x) = \int_0^{\imath} \frac{t^x}{e^t - 1}\,dt \quad (x > 0) \tag{2.5}$$

is well known (see, e.g., [1, 23.2.7]). Applying Theorem B.6 in [3, pp. 296–297]) to the integral in (2.5) we conclude that the function $f(x) := \Gamma(1+x)\zeta(1+x)$ is log-convex for all $x \in \mathbb{R}_>$. Making use of the first inequality in (2.2) we arrive at

$$\frac{\big[\Gamma(1+y)\zeta(1+y)\big]^a}{\Gamma(1+ay)\zeta(1+ay)} \leq \frac{\big[\Gamma(1+x)\zeta(1+x)\big]^a}{\Gamma(1+ax)\zeta(1+ax)} \tag{2.6}$$

($0 < x \le y$). Application of the second inequality in (2.4) to the right side of (2.6) gives,

$$\frac{\big[\Gamma(1+y)\zeta(1+y)\big]^a}{\Gamma(1+ay)\zeta(1+ay)} \le \frac{\big[\zeta(1+x)\big]^a}{\zeta(1+ax)}\,. \tag{2.7}$$

Substituting $y = 1$ into (2.7) and taking into account that $\Gamma(2) = 1$ and $\zeta(2) = \pi^2/6$ we obtain

$$\left(\frac{\pi^2}{6}\right)^a \frac{1}{\Gamma(1+a)} \le \frac{\big[\zeta(1+x)\big]^a \zeta(1+a)}{\zeta(1+ax)}$$

($0 < x \le 1$).

Another inequality

$$\left(\frac{\pi^2}{6}\right)^a e^{a(1-x)} \frac{(1+ax)^{1/2+ax}}{(1+a)^{1/2+a}} \le \frac{\big[\zeta(1+x)\big]^a \zeta(1+a)}{\zeta(1+ax)} \tag{2.8}$$

($0 < x \le 1$), with equality if $x = 1$, also follows from (2.6). We let $y = 1$ to obtain

$$\left(\frac{\pi^2}{6}\right)^a \frac{\Gamma(1+ax)}{\Gamma(1+a)\big[\Gamma(1+x)\big]^a} \le \frac{\big[\zeta(1+x)\big]^a \zeta(1+a)}{\zeta(1+ax)}\,. \tag{2.9}$$

Taking into account that $1 \le 1/\Gamma(1+x)$ for $0 \le x \le 1$ and applying an inequality of J.D. Kečkić and P.M. Vasić [5]

$$e^{v-u}\frac{u^{u-1/2}}{v^{v-1/2}} \le \frac{\Gamma(u)}{\Gamma(v)}$$

($1 \le u \le v$) to $\Gamma(1+ax)/\Gamma(1+a)$ we conclude that the left-hand side of the inequality (2.9) is bounded from below by the first member of (2.8).

2.3. **Applications to elliptic integrals.** The complete elliptic integral of the first kind $R_K(x, y)$ ($x, y \in \mathbb{R}_>$) is defined by

$$R_K(x,y) = \frac{2}{\pi}\int_0^{\pi/2} (x\sin^2\theta + y\cos^2\theta)^{-1/2} d\theta \tag{2.10}$$

(see [3, Ch. 9]). It follows from Proposition 2.1 in [6] that $R_K(x, y)$ is log-convex in each of its variables. For $z > 0$ let $f(x) = R_K(x, z)$. Using the first inequality in (2.2) we have

$$\left[\frac{R_K(y,z)}{R_K(x,z)}\right]^a \le \frac{R_K(ay,z)}{R_K(ax,z)} \tag{2.11}$$

($0 < x \le y$).

The complete elliptic integral of the first kind in Legendre form, denoted by $K(k)$, is defined by

$$K(k) = \int_0^{\pi/2} (1 - k^2\sin^2\theta)^{-1/2} d\theta.$$

Making use of (2.10) we have $K(k) = \frac{\pi}{2}R_K(k'^2, 1)$, where $k'^2 = 1-k^2$. Assume that $0 < l \le k$ and let $l'^2 = 1 - l^2$. Letting, in (2.11), $x = k'^2$, $y = l'^2$, $z = 1$ we obtain

$$\left[\frac{K(l)}{K(k)}\right]^a \le \frac{K(m)}{K(r)}\,,$$

where $m^2 = 1 - al'^2$ and $r^2 = 1 - ak'^2$.

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