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GENERALIZATIONS OF SOME NEW ČEBYŠEV TYPE INEQUALITIES

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ABSTRACT. We provide generalizations of some recently published Čebyšev type inequalities.

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1. INTRODUCTION

In a recent paper [1], B.G. Pachpatte proved the following Čebyšhev type inequalities:

Theorem 1.1. *Let $f, g : [a, b] \to \mathbb{R}$ be absolutely continuous functions on $[a, b]$ with $f', g' \in L_2[a, b]$, then,*

$$|P(F,G,f,g)| \le \frac{(b-a)^2}{12}\left[\frac{1}{b-a}\|f'\|_2^2 - ([f;a,b])^2\right]^{\frac{1}{2}} \times \left[\frac{1}{b-a}\|g'\|_2^2 - ([g;a,b])^2\right]^{\frac{1}{2}}, \tag{1.1}$$

$$|P(A,B,f,g)| \le \frac{(b-a)^2}{12}\left[\frac{1}{b-a}\|f'\|_2^2 - ([f;a,b])^2\right]^{\frac{1}{2}} \times \left[\frac{1}{b-a}\|g'\|_2^2 - ([g;a,b])^2\right]^{\frac{1}{2}}, \tag{1.2}$$

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where

$$P(\alpha,\beta,f,g)=\alpha\beta-\frac{1}{b-a}\left(\alpha\int_a^b g(t)\,dt+\beta\int_a^b f(t)\,dt\right)+\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(\frac{1}{b-a}g(t)\,dt\right), \tag{1.3}$$

$$[f;a,b]=\frac{f(b)-f(a)}{b-a}, \tag{1.4}$$

$$F=\frac{f(a)+f(b)}{2},\quad G=\frac{g(a)+g(b)}{2},\quad A=f\left(\frac{a+b}{2}\right),\quad B=g\left(\frac{a+b}{2}\right),$$

and

$$\|f\|_2:=\left[\int_a^b f^2(t)\,dt\right]^{\frac{1}{2}}.$$

Theorem 1.2. *Let $f,g:[a,b]\to\mathbb{R}$ be differentiable functions so that f',g' are absolutely continuous on $[a,b]$, then,*

$$|P(\overline{F},\overline{G},f,g)|\le\frac{(b-a)^4}{144}\|f''-[f';a,b]\|_\infty\|g''-[g';a,b]\|_\infty, \tag{1.5}$$

where

$$\overline{F}=\frac{f(a)+f(b)}{2}-\frac{(b-a)^2}{12}[f';a,b],$$
$$\overline{G}=\frac{g(a)+g(b)}{2}-\frac{(b-a)^2}{12}[g';a,b],$$

$P(\alpha,\beta,f,g)$ and $[f;a,b]$ are as defined in (1.3) and (1.4), and

$$\|f\|_\infty=\sup_{t\in[a,b]}|f(t)|<\infty.$$

In [2], B.G. Pachpatte presented an additional Čebyšev type inequality given in Theorem 1.3 below.

Theorem 1.3. *Let $f,g:[a,b]\to\mathbb{R}$ be absolutely continuous functions whose derivatives $f',g'\in L_p[a,b]$, $p>1$, then we have,*

$$|P(C,D,f,g)|\le\frac{1}{(b-a)^2}M^{\frac{2}{q}}\|f'\|_p\|g'\|_p, \tag{1.6}$$

where $P(\alpha,\beta,f,g)$ is as defined in (1.3),

$$C=\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right],$$
$$D=\frac{1}{3}\left[\frac{g(a)+g(b)}{2}+2g\left(\frac{a+b}{2}\right)\right],$$

$$M=\frac{(2^{q+1}+1)(b-a)^{q+1}}{3(q+1)6^q} \tag{1.7}$$

with $\frac{1}{p}+\frac{1}{q}=1$, and

$$\|f\|_p=\left(\int_a^b|f(t)|^p\,dt\right)^{\frac{1}{p}}<\infty.$$

In this paper, we provide some generalizations of the above three theorems.

2. Statement of Results

We use the following notation to simplify the detail of presentation. For suitable functions $f, g : [a,b] \to \mathbb{R}$ and real number $\theta \in [0,1]$ we set,

$$\Gamma_\theta = \frac{\theta}{2}[f(a)+f(b)] + (1-\theta) f\left(\frac{a+b}{2}\right),$$
$$\Delta_\theta = \frac{\theta}{2}[g(a)+g(b)] + (1-\theta) g\left(\frac{a+b}{2}\right),$$
$$\overline{\Gamma}_\theta = \Gamma_\theta + \frac{(1-3\theta)(b-a)^2}{24}[f', a, b],$$
$$\overline{\Delta}_\theta = \Delta_\theta + \frac{(1-3\theta)(b-a)^2}{24}[f', a, b],$$

where $[f; a, b]$ is as defined in (1.4).

We also use $P(\alpha, \beta, f, g)$ as defined in (1.3), where α and β are real constants.

The results are stated as Theorems 2.1, 2.2 and 2.3.

Theorem 2.1. *Let the assumptions of Theorem 1.1 hold, then for any* $\theta \in [0,1]$,

$$|P(\Gamma_\theta, \Delta_\theta, f, g)| \le \frac{(b-a)^2}{12}[\theta^3 + (1-\theta)^3] \times \left[\frac{1}{b-a}\|f'\|_2^2 - ([f;a,b])^2\right]^{\frac{1}{2}} \left[\frac{1}{b-a}\|g'\|_2^2 - ([g;a,b])^2\right]^{\frac{1}{2}}. \tag{2.1}$$

Theorem 2.2. *Let the assumptions of Theorem 1.2 hold, then for any* $\theta \in [0,1]$,

$$|P(\overline{\Gamma}_\theta, \overline{\Delta}_\theta, f, g)| \le (b-a)^4 I^2(\theta) \|f'' - [f';a,b]\|_\infty \|g'' - [g';a,b]\|_\infty, \tag{2.2}$$

where

$$I(\theta) = \begin{cases} \frac{\theta^3}{3} - \frac{\theta}{8} + \frac{1}{24}, & 0 \le \theta \le \frac{1}{2}, \\ \frac{1}{8}(\theta - \frac{1}{3}), & \frac{1}{2} < \theta \le 1. \end{cases} \tag{2.3}$$

Theorem 2.3. *Let the assumptions of Theorem 1.3 hold, then for any* $\theta \in [0,1]$,

$$|P(\Gamma_\theta, \Delta_\theta, f, g)| \le \frac{1}{(b-a)^2} M_\theta^{\frac{2}{q}} \|f'\|_p \|g'\|_p, \tag{2.4}$$

where

$$M_\theta = \frac{\theta^{q+1} + (1-\theta)^{q+1}}{(q+1)2^q}(b-a)^{q+1}, \tag{2.5}$$

and $\frac{1}{p} + \frac{1}{q} = 1$.

3. Proof of Theorem 2.1

Define the function,

$$K(\theta, t) = \begin{cases} t - (a + \theta\frac{b-a}{2}), & t \in [a, \frac{a+b}{2}], \\ t - (b - \theta\frac{b-a}{2}), & t \in (\frac{a+b}{2}, b], \end{cases} \tag{3.1}$$

and we obtain the following identities:

$$\Gamma_\theta - \frac{1}{b-a}\int_a^b f(t)\,dt = O(f;a,b;\theta), \tag{3.2}$$

$$\Delta_\theta - \frac{1}{b-a}\int_a^b g(t)\,dt = O(g;a,b;\theta), \tag{3.3}$$

where

$$O(f;a,b;\theta) = \frac{1}{2(b-a)^2}\int_a^b\int_a^b (f'(t)-f'(s))(k(\theta,t)-k(\theta,s)\,dt\,ds.$$

Multiplying the left sides and right sides of (3.2) and (3.3) we get,

$$P(\Gamma_\theta,\Delta_\theta,f,g) = O(f;a,b;\theta)O(g;a,b;\theta). \tag{3.4}$$

From (3.4),

$$|P(\Gamma_\theta,\Delta_\theta,f,g)| = |O(f;a,b;\theta)||O(g;a,b;\theta)|. \tag{3.5}$$

Using the Cauchy-Schwarz inequality for double integrals,

$$\begin{aligned} |O(f;a,b;\theta)| &\le \frac{1}{2(b-a)^2}\int_a^b\int_a^b |f'(t)-f'(s)||k(\theta,t)-k(\theta,s)|\,dt\,ds \\ &\le \left[\frac{1}{2(b-a)^2}\int_a^b\int_a^b (f'(t)-f'(s))^2\,dt\,ds\right]^{\frac{1}{2}} \\ &\qquad \times \left[\frac{1}{2(b-a)^2}\int_a^b\int_a^b (k(\theta,t)-k(\theta,s))^2\,dt\,ds\right]^{\frac{1}{2}}. \end{aligned} \tag{3.6}$$

By simple computation,

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b (f'(t)-f'(s))^2\,dt\,ds = \frac{1}{b-a}\int_a^b (f'(t))^2\,dt - \left(\frac{1}{b-a}\int_a^b f'(t)dt\right)^2, \tag{3.7}$$

and

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b (k(\theta,t)-K(\theta,s))^2\,dt\,ds = \frac{(b-a)^2}{12}[\theta^3+(1-\theta)^3]. \tag{3.8}$$

Using (3.7), (3.8) in (3.6),

$$|O(f;a,b;\theta)| \le \frac{b-a}{2\sqrt{3}}[\theta^3+(1-\theta)^3]^{\frac{1}{2}}\left[\frac{1}{b-a}\|f'\|_2^2 - ([f;a,b])^2\right]^{\frac{1}{2}}. \tag{3.9}$$

Similarly,

$$|O(g;a,b;\theta)| \le \frac{b-a}{2\sqrt{3}}[\theta^3+(1-\theta)^3]^{\frac{1}{2}}\left[\frac{1}{b-a}\|g'\|_2^2 - ([g;a,b])^2\right]^{\frac{1}{2}}. \tag{3.10}$$

Using (3.9) and (3.10) in (3.5), (2.1) follows.

Remark 3.1. If $\theta = 1$ and $\theta = 0$ in (2.1), the inequalities (1.1) and (1.2) are recaptured. Thus Theorem 2.1 may be regarded as a generalization of Theorem 1.1.

4. PROOF OF THEOREM 2.2

Define the function

$$L(\theta,t)=\begin{cases}\frac{1}{2}(t-a)[t-(1-\theta)a-\theta b], & t\in[a,\frac{a+b}{2}],\\ \frac{1}{2}(t-b)[t-\theta a-(1-\theta)b], & t\in(\frac{a+b}{2},b].\end{cases}$$

It is not difficult to find the following identities:

$$\frac{1}{b-a}\int_a^b f(t)\,dt-\overline{\Gamma}_\theta=Q(f',f'';a,b), \tag{4.1}$$

$$\frac{1}{b-a}\int_a^b g(t)\,dt-\overline{\Delta}_\theta=Q(g',g'';a,b), \tag{4.2}$$

where

$$Q(f',f'';a,b)=\frac{1}{b-a}\int_a^b L(\theta,t)\{f''(t)-[f';a,b]\}\,dt.$$

Multiplying the left sides and right sides of (4.1) and (4.2), we get,

$$P(\overline{\Gamma}_\theta,\overline{\Delta}_\theta,f,g)=Q(f',f'';a,b)Q(g',g'';a,b). \tag{4.3}$$

From (4.3),

$$|P(\overline{\Gamma}_\theta,\overline{\Delta}_\theta,f,g)|=|Q(f',f'';a,b)||Q(g',g'';a,b)|. \tag{4.4}$$

By simple computation, we have,

$$\begin{aligned}|Q(f',f'';a,b)|&\le\frac{1}{b-a}\int_a^b|L(\theta,t)||[f''(t)-[f';a,b]|\,dt\\ &\le\frac{1}{b-a}\|f''(t)-[f';a,b]\|_\infty\int_a^b|L(\theta,t)|\,dt,\end{aligned} \tag{4.5}$$

and similarly,

$$|Q(f',f'';a,b)|\le\frac{1}{b-a}\|f''(t)-[f';a,b]\|_\infty\int_a^b|L(\theta,t)|\,dt, \tag{4.6}$$

where

$$\int_a^b|L(\theta,t)|\,dt=(b-a)^3\times\begin{cases}\frac{\theta^3}{3}-\frac{\theta}{8}+\frac{1}{24}, & 0\le\theta\le\frac{1}{2},\\ \frac{1}{8}(\theta-\frac{1}{3}), & \frac{1}{2}<\theta\le 1.\end{cases} \tag{4.7}$$

Consequently, the inequalities (2.2) and (2.3) follow from (4.4) – (4.7).

Remark 4.1. If $\theta=1$ in (2.2) with (2.3), the inequality (1.5) is recaptured. Thus Theorem 2.2 may be regarded as a generalization of Theorem 1.2.

5. PROOF OF THEOREM 2.3

From (3.1), we can also find the following identities:

$$\Gamma_\theta-\frac{1}{b-a}\int_a^b f(t)\,dt=\frac{1}{b-a}\int_a^b K(\theta,t)f'(t)\,dt, \tag{5.1}$$

$$\Delta_\theta-\frac{1}{b-a}\int_a^b g(t)\,dt=\frac{1}{b-a}\int_a^b K(\theta,t)g'(t)\,dt. \tag{5.2}$$

Multiplying the left sides and right sides of (5.1) and (5.2) we get,

$$P(\Gamma_\theta, \Delta_\theta, f, g) = \frac{1}{(b-a)^2}\left(\int_a^b k(\theta,t) f'(t)\, dt\right)\left(\int_a^b k(\theta,t) g'(t)\, dt\right). \tag{5.3}$$

From (5.3) and using the properties of modulus and Hölder's integral inequality, we have,

$$\begin{aligned} |P(\Gamma_\theta, \Delta_\theta, f, g)| &\le \frac{1}{(b-a)^2}\left(\int_a^b |k(\theta,t)||f'(t)|\, dt\right)\left(\int_a^b |k(\theta,t)||g'(t)|\, dt\right) \\ &\le \frac{1}{(b-a)^2}\left[\left(\int_a^b |k(\theta,t)|^q\, dt\right)^{\frac{1}{q}}\left(\int_a^b |f'|^p\, dt\right)^{\frac{1}{p}}\right] \\ &\quad \times \left[\left(\int_a^b |k(\theta,t)|^q\, dt\right)^{\frac{1}{q}}\left(\int_a^b |g'|^p\, dt\right)^{\frac{1}{p}}\right] \\ &= \frac{1}{(b-a)^2}\left(\int_a^b |k(\theta,t)|^q\, dt\right)^{\frac{2}{q}} \|f'\|_p \|g'\|_p. \end{aligned} \tag{5.4}$$

A simple computation gives,

$$\begin{aligned} &\int_a^b |k(\theta,t)|^q\, dt \\ &= \int_a^{\frac{a+b}{2}} \left| t - \left(a + \theta\frac{b-a}{2}\right)\right|^q dt + \int_{\frac{a+b}{2}}^b \left| t - \left(b - \theta\frac{b-a}{2}\right)\right|^q dt \\ &= \int_a^{a+\theta\frac{b-a}{2}} \left(a + \theta\frac{b-a}{2} - t\right)^q dt + \int_{a+\theta\frac{b-a}{2}}^{\frac{a+b}{2}} \left(t - a - \theta\frac{b-a}{2}\right)^q dt \\ &\quad + \int_{\frac{a+b}{2}}^{b-\theta\frac{b-a}{2}} \left(b - \theta\frac{b-a}{2} - t\right)^q dt + \int_{b-\theta\frac{b-a}{2}}^b \left(t - b + \theta\frac{b-a}{2}\right)^q dt \\ &= \frac{2}{q+1}\left[\left(\frac{\theta}{2}\right)^{q+1}(b-a)^{q+1} + \left(\frac{1-\theta}{2}\right)^{q+1}(b-a)^{q+1}\right] \\ &= \frac{\theta^{q+1} + (1-\theta)^{q+1}}{(q+1)2^q}(b-a)^{q+1} = M_\theta. \end{aligned} \tag{5.5}$$

Consequently, the inequality (2.4) with (2.5) follow from (5.4) and (5.5).

Remark 5.1. If we take $\theta = \frac{1}{3}$ in (2.4) with (2.5), we recapture the inequality (1.6) with (1.7). Thus Theorem 2.3 may be regarded as a generalization of Theorem 1.3.

Remark 5.2. If we take $p = 2$ in Theorem 2.3, and replace $f(t)$ and $g(t)$ by $f(t) - [f; a, b]t$ and $g(t) - [g; a, b]t$ in (2.4), respectively, then inequality (2.1) is recaptured.

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