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HARDY'S INTEGRAL INEQUALITY FOR COMMUTATORS OF HARDY OPERATORS

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ABSTRACT. The authors establish the Hardy integral inequality for commutators generated by Hardy operators and Lipschitz functions.

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1. INTRODUCTION AND MAIN RESULTS

Let f be a non-negative and integral function on $\mathbb{R}^+$, Hardy operators are defined by

$$(\mathcal{H}f)(x) = \frac{1}{x}\int_0^x f(t)\,dt, \quad x > 0,$$

and

$$(\mathcal{V}f)(x) = \int_x^\infty f(t)\,dt, \quad x > 0.$$

The Hardy integral inequality results are well known [9, 10, 11]; in particular

$$\left(\int_0^\infty (\mathcal{H}f(x))^p dx\right)^{\frac{1}{p}} \leq \frac{p}{p-1}\left(\int_0^\infty (f(x))^p\,dx\right)^{\frac{1}{p}}, \tag{1.1}$$

and

$$\left(\int_0^\infty (\mathcal{V}f(x))^p dx\right)^{\frac{1}{p}} \leq p\left(\int_0^\infty (xf(x))^p\,dx\right)^{\frac{1}{p}}. \tag{1.2}$$

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In (1.1), the constant $\frac{p}{p-1}$ is the best possible. In (1.2), the constant p is also the best possible. The inequality (1.1) was first proved by Hardy in an attempt to give a simple proof of Hilbert's double series theorem [12].

Let f be a non-negative and integral function on $\mathbb{R}^+$, and the fractional Hardy operator be defined by

$$(\mathcal{H}^\alpha f)(x) = \frac{1}{x^{1-\alpha}} \int_0^x f(t)\,dt, \quad x > 0,$$

for $\frac{1}{p} - \frac{1}{q} = \alpha$, $0 < \alpha < 1$. There are fractional order Hardy integral inequalities which correspond to (1.1) and (1.2):

$$\left(\int_0^\infty (\mathcal{H}^\alpha f(x))^q dx \right)^{\frac{1}{q}} \le C \left(\int_0^\infty (f(x))^p \, dx \right)^{\frac{1}{p}}, \tag{1.3}$$

and

$$\left(\int_0^\infty (\mathcal{V} f(x))^q dx \right)^{\frac{1}{q}} \le C \left(\int_0^\infty (x^{1-\alpha} f(x))^p \, dx \right)^{\frac{1}{p}}. \tag{1.4}$$

The Hardy inequality (1.3) can be found in [2], (1.4) in [13].

The Hardy integral inequalities have received considerable attention and a large number of papers have appeared which deal with its alternative proofs, various generalizations, numerous variants and applications. For earlier developments of this kind of inequality and many important applications in analysis, see [11]. Among numerous papers dealing with such inequalities, we choose to refer to the papers [3], [5], [9], [10], [16] – [21] and some of the references cited therein.

Definition 1.1. Let $0 \le \alpha < 1$ and $b(x)$ be a measurable, locally integrable function. Then the commutator of the Hardy operator U_b^α is defined by

$$\mathcal{H}_b^\alpha f(x) = \frac{1}{x^{1-\alpha}} \int_0^x f(t)(b(x) - b(t))dt, \quad x > 0.$$

Fu [7] obtained the following results.

Theorem 1.1. *For $b \in \dot{\Lambda}_\beta(\mathbb{R}^+)$, $0 < \beta < 1$, $\mathcal{H}_b^\alpha$ is a bounded operator from $L^p(\mathbb{R}^+)$ to $L^q(\mathbb{R}^+)$, $1 < p < q < \infty$, $0 \le \alpha < 1$, $0 < \alpha + \beta < 1$, $\frac{1}{p} - \frac{1}{q} = \alpha + \beta$.*

Remark 1.2. In Theorem 1.1, If we let $\alpha = 0$, Then the result corresponds to Hardy inequality (1.3).

Definition 1.2. Let $b(x)$ be a measurable, locally integrable function. Then the commutator of the Hardy operator $\mathcal{V}_b$ is defined by

$$\mathcal{V}_b f(x) = \int_x^\infty f(t)(b(x) - b(t))dt, \quad x > 0.$$

In Definition 1.1, if we let $\alpha = 0$, then we denote $\mathcal{H}_b^\alpha$ by $\mathcal{H}_b$.

It can be seen that if $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, $0 < \beta < 1$, then $\mathcal{H}_b$ has a similar boundedness property to $\mathcal{H}^\beta$. A natural question regarding the boundedness property of $\mathcal{V}_b$, can be answered in the affirmative by the following inequality (1.5).

Theorem 1.3. *If $b \in \dot{\Lambda}_\beta(\mathbb{R}^+)$, $\frac{1}{p} - \frac{1}{q} = \beta$, $0 < \beta < 1$, $p > 1$. Then*

$$\|\mathcal{V}_b f\|_q \le C \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^+)} \|(\cdot) f(\cdot)\|_p. \tag{1.5}$$

Theorem 1.4. *If* $b \in \dot{\Lambda}_\beta(\mathbb{R}^+)$, $0 < \beta < 1$, $\frac{1}{p} - \frac{1}{q} = \alpha + \beta$, $0 < \alpha + \beta < 1$, $0 \le \alpha < 1$, $p > 1$. *Then*

$$\|\mathcal{V}_b f\|_q \le C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^+)}\|(\cdot)^{1-\alpha} f(\cdot)\|_p. \tag{1.6}$$

If we let $\alpha = 0$ in Theorem 1.4, then Theorem 1.3 can be obtained without difficulty. Thus we just need to prove Theorem 1.4. Before we prove the main theorem, let us state some lemmas and notations.

The Besov-Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^+)$ is the space of functions f satisfying

$$\|f\|_{\dot{\Lambda}_\beta(\mathbb{R}^+)} = \sup_{x,h\in\mathbb{R}^+, h\neq 0} \frac{|f(x+h) - f(x)|}{|h|^\beta} < \infty.$$

Lemma 1.5 ([8, 15]). *For any* $x, y \in \mathbb{R}^+$, *if* $f \in \dot{\Lambda}_\beta(\mathbb{R}^+)$, $0 < \beta < 1$, *then*

$$|f(x) - f(y)| \le |x - y|^\beta \|f\|_{\dot{\Lambda}_\beta(\mathbb{R}^+)}, \tag{1.7}$$

and given any interval I *in* $\mathbb{R}^+$

$$\sup_{x\in I} |f(x) - f_I| \le C|I|^\beta \|f\|_{\dot{\Lambda}_\beta(\mathbb{R}^+)},$$

where

$$f_I = \frac{1}{|I|}\int_I f.$$

Lemma 1.6 ([7, 14]). *Let* $s > 0, q \ge p > 1$ *, then*

$$\sum_{i=-\infty}^{\infty} \left| \sum_{k=i}^{\infty} 2^{(i-k)/s} \left(\int_{2^k}^{2^{k+1}} |f(t)|^p dt \right)^{\frac{1}{p}} \right|^q \le C \left(\int_0^\infty |f(t)|^p dt \right)^{\frac{q}{p}},$$

where

$$C = \left(\frac{2^{p/2s}}{2^{p/2s} - 1} \right) \left(\frac{2^{q'/2s}}{2^{q'/2s} - 1} \right)^{\frac{q}{q'}}, \qquad \frac{1}{q'} + \frac{1}{q} = 1.$$

There are two different methods to prove Theorem 1.4.

2. PROOF OF THEOREM 1.4

First Proof.

$$\begin{aligned}
\int_0^\infty |\mathcal{V}_b f(x)|^q dx &= \sum_{i=-\infty}^{\infty} \int_{2^i}^{2^{i+1}} \left| \int_x^\infty (b(x) - b(t)) f(t) dt \right|^q dx \\
&\le \sum_{i=-\infty}^{\infty} \int_{2^i}^{2^{i+1}} \left(\int_{2^i}^\infty |(b(x) - b(t)) f(t)|\, dt \right)^q dx \\
&= \sum_{i=-\infty}^{\infty} \int_{2^i}^{2^{i+1}} \left(\sum_{k=i}^{\infty} \int_{2^k}^{2^{k+1}} |(b(x) - b(t)) f(t)|\, dt \right)^q dx \\
&\le 2^{q/q'} \sum_{i=-\infty}^{\infty} \int_{2^i}^{2^{i+1}} \left(\sum_{k=i}^{\infty} \int_{2^k}^{2^{k+1}} \left|(b(x) - b_{(2^i, 2^{i+1}]}) f(t)\right| dt \right)^q dx \\
&\quad + 2^{q/q'} \sum_{i=-\infty}^{\infty} \int_{2^i}^{2^{i+1}} \left(\sum_{k=i}^{\infty} \int_{2^k}^{2^{k+1}} \left|(b(t) - b_{(2^i, 2^{i+1}]}) f(t)\right| dt \right)^q dx \\
&:= I + J.
\end{aligned}$$

By Lemma 1.5 and the Hölder inequality,

$$\begin{aligned}
I &= 2^{q/q'} \sum_{i=-\infty}^{\infty} \int_{2^i}^{2^{i+1}} \left|b(x) - b_{(2^i,2^{i+1}]}\right|^q dx \left(\sum_{k=i}^{\infty} \int_{2^k}^{2^{k+1}} |f(t)|\, dt \right)^q \\
&\leq 2^{q/q'} \sum_{i=-\infty}^{\infty} 2^i \left(\sup_{x\in(2^i,2^{i+1}]} \left|b(x) - b_{(2^i,2^{i+1}]}\right| \right)^q \\
&\qquad \times \left\{ \sum_{k=i}^{\infty} \left(\int_{2^k}^{2^{k+1}} |f(t)|^p dt \right)^{\frac{1}{p}} \left(\int_{2^k}^{2^{k+1}} dt \right)^{\frac{1}{p}'} \right\}^q \\
&\leq C 2^{q/q'} \sum_{i=-\infty}^{\infty} 2^{i(q\beta+1)} \left(\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^+)} \right)^q \left\{ \sum_{k=i}^{\infty} \left(\int_{2^k}^{2^{k+1}} |f(t)|^p dt \right)^{\frac{1}{p}} \left(\int_{2^k}^{2^{k+1}} dt \right)^{\frac{1}{p'}} \right\}^q \\
&\leq C 2^{q/q'} \sum_{i=-\infty}^{\infty} 2^{i(q\beta+1)} \left(\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^+)} \right)^q \left(\sum_{k=i}^{\infty} 2^{k/p'} \left(\int_{2^k}^{2^{k+1}} 2^{-k(1-\alpha)p} |t^{1-\alpha} f(t)|^p dt \right)^{\frac{1}{p}} \right)^q .
\end{aligned}$$

Notice that $\frac{1}{p} - \frac{1}{q} = \alpha + \beta$,

$$I \leq C 2^{q/q'} \left(\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^+)} \right)^q \sum_{i=-\infty}^{\infty} \left(\sum_{k=i}^{\infty} 2^{(i-k)(\frac{1}{q}+\beta)} \left(\int_{2^k}^{2^{k+1}} |t^{1-\alpha} f(t)|^p dt \right)^{\frac{1}{p}} \right)^q .$$

By Lemma 1.6 $\left(s = \frac{q}{1+q\beta} \right)$,

$$I \leq C \left(\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^+)} \right)^q \left(\int_0^{\infty} |t^{1-\alpha} f(t)|^p dt \right)^{\frac{q}{p}} . \tag{2.1}$$

Now estimate J,

$$\begin{aligned}
J &= 2^{q/q'} \sum_{i=-\infty}^{\infty} \int_{2^i}^{2^{i+1}} \left(\sum_{k=i}^{\infty} \int_{2^k}^{2^{k+1}} \left|(b(t) - b_{(2^i,2^{i+1}]}) f(t)\right| dt \right)^q dx \\
&\leq 2^{2q/q'} \sum_{i=-\infty}^{\infty} \int_{2^i}^{2^{i+1}} \left(\sum_{k=i}^{\infty} \int_{2^k}^{2^{k+1}} \left|(b(t) - b_{(2^k,2^{k+1}]}) f(t)\right| dt \right)^q dx \\
&\qquad + 2^{2q/q'} \sum_{i=-\infty}^{\infty} \int_{2^i}^{2^{i+1}} \left(\sum_{k=i}^{\infty} \int_{2^k}^{2^{k+1}} \left|(b_{(2^k,2^{k+1}]} - b_{(2^i,2^{i+1}]}) f(t)\right| dt \right)^q dx \\
&:= J_1 + J_2.
\end{aligned}$$

Notice that $\frac{1}{p} - \frac{1}{q} = \alpha + \beta$, $\frac{1}{p} + \frac{1}{p'} = 1$, by Lemma 1.5, it can be inferred that

$$\begin{aligned}
J_1 &\leq 2^{2q/q'} \sum_{i=-\infty}^{\infty} \int_{2^i}^{2^{i+1}} \left(\sum_{k=i}^{\infty} \left(\sup_{t\in(2^k,2^{k+1}]} \left|b(t) - b_{(2^k,2^{k+1}]}\right| \right) \int_{2^k}^{2^{k+1}} |f(t)| dt \right)^q dx \\
&\leq C 2^{2q/q'} \left(\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^+)} \right)^q \sum_{i=-\infty}^{\infty} \int_{2^i}^{2^{i+1}} \left(\sum_{k=i}^{\infty} 2^{k\beta} \int_{2^k}^{2^{k+1}} |f(t)| dt \right)^q dx
\end{aligned}$$

$$\leq C2^{2q/q'}\left(\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^+)}\right)^q\sum_{i=-\infty}^{\infty}2^i\left(\sum_{k=i}^{\infty}2^{k(\beta+\frac{1}{p'})}\left(\int_{2^k}^{2^{k+1}}|f(t)|^p dt\right)^{\frac{1}{p}}\right)^q$$

$$\leq C2^{2q/q'}\left(\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^+)}\right)^q\sum_{i=-\infty}^{\infty}2^i\left(\sum_{k=i}^{\infty}2^{k(\beta+\frac{1}{p'})}\left(\int_{2^k}^{2^{k+1}}2^{-k(1-\alpha)p}|t^{1-\alpha}f(t)|^p dt\right)^{\frac{1}{p}}\right)^q$$

$$= C2^{2q/q'}\left(\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^+)}\right)^q\sum_{i=-\infty}^{\infty}\left(\sum_{k=i}^{\infty}2^{(i-k)/q}\left(\int_{2^k}^{2^{k+1}}|t^{1-\alpha}f(t)|^p dt\right)^{\frac{1}{p}}\right)^q.$$

In the third inequality, the Hölder inequality is applied.

By Lemma 1.6 $(s=q)$,

$$J_1 \leq C\left(\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^+)}\right)^q\left(\int_0^{\infty}|t^{1-\alpha}f(t)|^p dt\right)^{\frac{q}{p}}. \tag{2.2}$$

To estimate J_2, for $i>k$, by Lemma 1.5, the following result is obtained.

$$\begin{aligned}\left|b_{(2^k,2^{k+1}]}-b_{(2^i,2^{i+1}]}\right| &\leq \frac{1}{2^i}\int_{2^i}^{2^{i+1}}\left|b(y)-b_{(2^k,2^{k+1}]}\right|dy\\ &\leq \frac{1}{2^i}\frac{1}{2^k}\int_{2^k}^{2^{k+1}}\int_{2^i}^{2^{i+1}}|b(y)-b(z)|\,dydz\\ &\leq \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^+)}\frac{1}{2^i}\frac{1}{2^k}\int_{2^k}^{2^{k+1}}\int_{2^i}^{2^{i+1}}|y-z|^\beta\,dydz\\ &\leq \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^+)}\left(\frac{1}{2^i}\int_{2^i}^{2^{i+1}}y^\beta dy+\frac{1}{2^k}\int_{2^k}^{2^{k+1}}z^\beta dz\right)\\ &\leq C2^{i\beta+1}\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^+)}.\end{aligned}$$

Notice that $\frac{1}{p}-\frac{1}{q}=\alpha+\beta$, $\frac{1}{p}+\frac{1}{p'}=1$; by Lemma 1.5 and the Hölder inequality, we have

$$J_2 \leq C2^{2q/q'+q}\left(\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^+)}\right)^q\sum_{i=-\infty}^{\infty}\int_{2^i}^{2^{i+1}}\left(2^{i\beta}\sum_{k=i}^{\infty}\int_{2^k}^{2^{k+1}}|f(t)|dt\right)^q dx$$

$$\leq C2^{2q/q'+q}\left(\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^+)}\right)^q\sum_{i=-\infty}^{\infty}\left(2^{i(\beta+\frac{1}{q})}\sum_{k=i}^{\infty}2^{k/p'}\left(\int_{2^k}^{2^{k+1}}|f(t)|^p dt\right)^{\frac{1}{p}}\right)^q$$

$$\leq C2^{2q/q'+q}\left(\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^+)}\right)^q\sum_{i=-\infty}^{\infty}\left(2^{i(\beta+\frac{1}{q})}\sum_{k=i}^{\infty}2^{k/p'}\left(\int_{2^k}^{2^{k+1}}2^{-k(1-\alpha)p}|t^{1-\alpha}f(t)|^p dt\right)^{\frac{1}{p}}\right)^q$$

$$= C2^{2q/q'+q}\left(\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^+)}\right)^q\sum_{i=-\infty}^{\infty}\left(\sum_{k=i}^{\infty}2^{(i-k)(\beta+\frac{1}{q})}\left(\int_{2^k}^{2^{k+1}}|t^{1-\alpha}f(t)|^p dt\right)^{\frac{1}{p}}\right)^q.$$

In the second inequality, the Hölder inequality is applied.

By Lemma 1.6 $\left(s=\frac{q}{q\beta+1}\right)$,

$$J_2 \leq C\left(\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^+)}\right)^q\left(\int_0^{\infty}|t^{1-\alpha}f(t)|^p dt\right)^{\frac{q}{p}}. \tag{2.3}$$

Combining (2.1), (2.2) and (2.3), we complete the proof of Theorem 1.4. □

Second Proof. By inequality (1.7) and comparing the size of x and t, we have

$$\begin{aligned}\int_0^\infty |\mathcal{V}_b f(x)|^q dx &= \sum_{i=-\infty}^{\infty} \int_{2^i}^{2^{i+1}} \left| \int_x^\infty f(t)|b(x)-b(x)|dt \right|^q dx \\ &\le \sum_{i=-\infty}^{\infty} \int_{2^i}^{2^{i+1}} \left(\sum_{k=i}^{\infty} \int_{2^k}^{2^{k+1}} |t-x|^\beta \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^+)} |f(t)|\, dt \right)^q dx \\ &\le C \left(\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^+)}\right)^q \sum_{i=-\infty}^{\infty} \int_{2^i}^{2^{i+1}} \left(\sum_{k=i}^{\infty} \int_{2^k}^{2^{k+1}} t^\beta |f(t)|\, dt \right)^q dx \\ &= C \left(\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^+)}\right)^q \sum_{i=-\infty}^{\infty} \int_{2^i}^{2^{i+1}} \left(\sum_{k=i}^{\infty} \int_{2^k}^{2^{k+1}} \frac{t^{1-\alpha} |f(t)|}{t^{1-\alpha-\beta}} dt \right)^q dx \\ &= C \left(\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^+)}\right)^q \sum_{i=-\infty}^{\infty} 2^i \left(\sum_{k=i}^{\infty} \int_{2^k}^{2^{k+1}} \frac{t^{1-\alpha} |f(t)|}{t^{1-\alpha-\beta}} dt \right)^q .\end{aligned}$$

By the Hölder inequality, $\frac{1}{p}+\frac{1}{p'}=1$, the following estimate is obtained.

$$\int_{2^k}^{2^{k+1}} \frac{t^{1-\alpha} |f(t)|}{t^{1-\alpha-\beta}} dt \le \left(\int_{2^k}^{2^{k+1}} |t^{1-\alpha} f(t)|^p dt \right)^{\frac{1}{p}} \left(\int_{2^k}^{2^{k+1}} t^{(\alpha+\beta-1)p'} dt \right)^{\frac{1}{p'}} .$$

Notice that $\frac{1}{p}-\frac{1}{q}=\alpha+\beta$,

$$\int_0^\infty |\mathcal{V}_b f(x)|^q dx \le C \left(\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^+)}\right)^q \sum_{i=-\infty}^{\infty} \left\{ \sum_{k=i}^{\infty} 2^{(i-k)/q} \left(\int_{2^k}^{2^{k+1}} |t^{1-\alpha} f(t)|^p dt \right)^{\frac{1}{p}} \right\}^q .$$

By Lemma 1.6, the desired result is obtained. □

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