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CONVEX FUNCTIONS IN A HALF-PLANE, II

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I dedicate this paper to the memory of my dear father, Professor Nicolae N. Pascu.

ABSTRACT. In the present paper we obtain new sufficient conditions for the univalence and convexity of an analytic function defined in the upper half-plane. In particular, in the case of hydrodynamically normalized functions, we obtain by a different method a known result concerning the convexity and univalence of an analytic function defined in a half-plane.

Key words and phrases: Univalent function, Convex function, Half-plane.

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1. INTRODUCTION

In the present paper, we continue the work in [4], by obtaining new sufficient conditions for the convexity and univalence for analytic functions defined in the upper half-plane (Theorems 2.3, 2.5 and 2.7). In particular, under the additional hypothesis (1.2) below, they become necessary and sufficient conditions for convexity and univalence in a half-plane (Corollary 2.9), obtaining thus by a different method the results in [5] and [6].

We begin by establishing the notation and with some preliminary results needed for the proofs.

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We denote by $D = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ the upper half-plane in $\mathbb{C}$ and for $\varepsilon \in \left(0, \frac{\pi}{2}\right)$ we let T_ε be the angular domain defined by:

$$T_\varepsilon = \left\{ z \in \mathbb{C}^* : \frac{\pi}{2} - \varepsilon < \arg(z) < \frac{\pi}{2} + \varepsilon \right\}. \tag{1.1}$$

We say that a function $f : D \to \mathbb{C}$ is *convex* if f is univalent in D and $f(D)$ is a convex domain.

For an arbitrarily chosen positive real number $y_0 > 0$ we denote by $\mathcal{A}_{y_0}$ the class of functions $f : D \to \mathbb{C}$ analytic in the upper half-plane D satisfying $f(iy_0) = 0$ and such that $f'(z) \neq 0$ for any $z \in D$. In particular, for $y_0 = 1$ we will denote $\mathcal{A}_1 = \mathcal{A}$.

We will refer to the following normalization condition for analytic functions $f : D \to \mathbb{C}$ as the *hydrodynamic normalization*:

$$\lim_{z \to \infty, z \in D} (f(z) - z) = ai, \tag{1.2}$$

where $a \geq 0$ is a non-negative real number, and we will denote by $\mathcal{H}_1$ the class of analytic functions $f : D \to \mathbb{C}$ satisfying this condition in the particular case $a = 0$.

For analytic functions satisfying the above normalization condition, J. Stankiewicz and Z. Stankiewicz obtained (see [5] and [6]) the following necessary and sufficient condition for convexity and univalence in a half-plane:

Theorem 1.1. *If the function $f \in \mathcal{H}_1$ satisfies:*

$$f'(z) \neq 0, \quad \textit{for all } z \in D \tag{1.3}$$

and

$$\operatorname{Im} \frac{f''(z)}{f'(z)} > 0, \quad \textit{for all } z \in D, \tag{1.4}$$

then f is a convex function.

In order to prove our main result we need the following results from [2]:

Lemma 1.2. *If the function $f : D \to D$ is analytic in D, then for any $\varepsilon \in \left(0, \frac{\pi}{2}\right)$ the following limits exist and we have the equalities:*

$$\lim_{z \to \infty, z \in T_\varepsilon} \frac{f(z)}{z} = \lim_{z \to \infty, z \in T_\varepsilon} f'(z) = c,$$

where $c \geq 0$ is a non-negative real number.

Moreover, for any $z \in D$ we have the inequality

$$\operatorname{Im} f(z) \geq c \operatorname{Im} z, \tag{1.5}$$

and if there exists $z_0 \in D$ such that we have equality in the inequality (1.5), then there exists a real number a such that

$$f(z) = cz + a, \quad \textit{for all } z \in D.$$

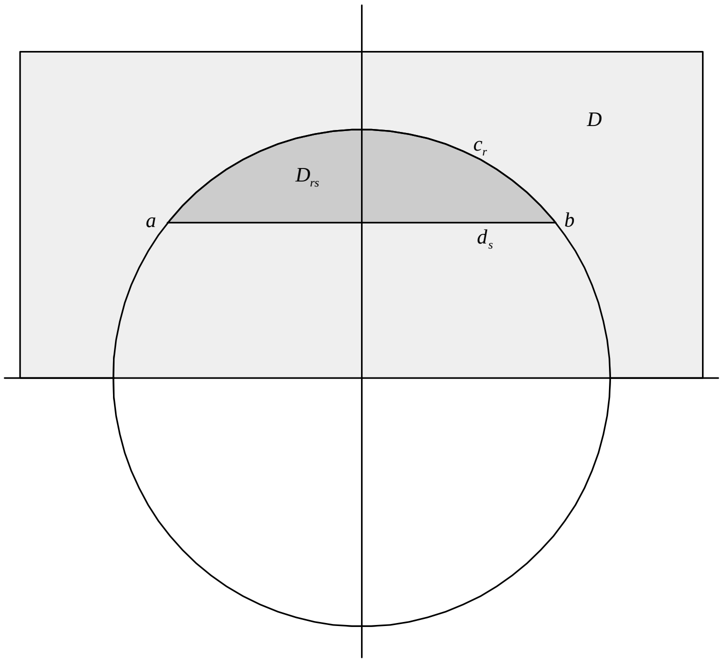

Figure 2.1: The domain D_{rs}.

Lemma 1.3. *If the function $f : D \to D$ is analytic in D and hydrodynamically normalized, then for any $\varepsilon \in (0, \frac{\pi}{2})$ and any natural number $n \geq 2$ we have*

$$\lim_{z\to\infty, z\in T_\varepsilon} \left(z^n f^{(n)}(z)\right) = 0.$$

2. MAIN RESULTS

Let us consider the family of domains $D_{r,s}$ in the complex plane, defined by

$$D_{r,s} = \{z \in \mathbb{C} : |z| < r,\ \operatorname{Im} z > s\},$$

where r and s are positive real numbers, $0 < s < r$ (see Figure 2.1).

Let us note that for any $r > 1$ and $0 < s < 1$ we have the inclusion $D_{r,s} \subset D$, and that for any $z \in D$ arbitrarily fixed, there exists $r_z > 0$ and $s_z > 0$ such that $z \in D_{r,s}$ for any $r > r_z$ and any $0 < s < s_z$ (for example, we can choose r_z and s_z such that they satisfy the conditions $r_z > |z|$ and $s_z \in (0, \operatorname{Im} z)$).

We denote by $\Gamma_{r,s} = c_r \cup d_s$ the boundary of the domain $D_{r,s}$, where c_r and d_s are the arc of the circle, respectively the line segment, defined by:

$$\begin{cases} c_r = \{z \in \mathbb{C} : |z| = r, \quad z \geq s\} \\ d_s = \{z \in \mathbb{C} : |z| \leq r,\ z = s\} \end{cases}.$$

The curve $\Gamma_{r,s}$ has an exterior normal vector at any point, except for the points a and b (with $\arg a < \arg b$) where the line segment d_s and the arc of the circle c_r meet (see Figure 2.1). The exterior normal vector to the curve $f(c_r)$ at the point $f(z)$, with $z = re^{it} \in c_r$, $t \in (\arg a, \arg b)$, has the argument

$$\varphi(t) = \arg(zf'(z)), \tag{2.1}$$

and the exterior normal vector to the curve $f(d_s)$ at the point $f(z)$, with $z = x + is \in d_s$, $x \in (\operatorname{Re} b, \operatorname{Re} a)$, has the argument

$$\psi(x) = -\frac{\pi}{2} + \arg f'(x + is). \tag{2.2}$$

Definition 2.1. We say that the function $f \in \mathcal{A}$ is convex on the curve $\Gamma_{r,s}$ if the argument of the exterior normal vector to the curve $f(\Gamma_{r,s}) - \{f(a), f(b)\}$ is an increasing function.

Remark 2.1. In particular, the condition in the above theorem is satisfied if the functions φ and ψ defined by (2.1)–(2.2) are increasing functions.

Let us note that for $z = re^{it} \in c_r$, we have:

$$\frac{\partial}{\partial t} \log\left(re^{it} f'\left(re^{it}\right)\right) = i\left(\frac{re^{it} f''\left(re^{it}\right)}{f'\left(re^{it}\right)} + 1\right) = \frac{\partial}{\partial t} \ln\left|re^{it} f'\left(re^{it}\right)\right| + i\varphi'(t),$$

and for $z \in d_s$:

$$\frac{\partial}{\partial x} \log f'(x + is) = \frac{f''(x + is)}{f'(x + is)} = \frac{\partial}{\partial x} \ln|f'(x + is)| + i\psi'(x + is).$$

We obtain therefore

$$\varphi'(t) = \frac{re^{it} f''\left(re^{it}\right)}{f'\left(re^{it}\right)} + 1,$$

for $re^{it} \in c_r$, and

$$\psi'(x + is) = \frac{f''(x + is)}{f'(x + is)},$$

for $x + is \in d_s$, and from the previous observation it follows that if the function $f \in \mathcal{A}$ satisfies the inequalities

$$\begin{cases} \dfrac{zf''(z)}{f'(z)} + 1 > 0, & z \in c_r \\ \dfrac{f''(z)}{f'(z)} > 0, & z \in d_s \end{cases}, \tag{2.3}$$

the function f is convex on the curve $\Gamma_{r,s}$, and therefore $f(D_{r,s})$ is a convex domain.

Since the function f has in the domain $D_{r,s}$ bounded by the curve $\Gamma_{r,s}$ a simple zero, from the argument principle it follows that the total variation of the argument of the function f on the curve $\Gamma_{r,s}$ is 2π, and therefore f is injective on the curve $\Gamma_{r,s}$. From the principle of univalence on the boundary, it follows that the function f is univalent $D_{r,s}$.

We obtained the following:

Theorem 2.2. *If the function f belongs to the class $\mathcal{A}$ and there exist real numbers $0 < s < 1 < r$ such that conditions (2.3) are satisfied, then the function f is univalent in the domain $D_{r,s}$ and $f(D_{r,s})$ is a convex domain.*

More generally, we have the following:

Theorem 2.3. *If the function $f : D \to \mathbb{C}$ belongs to the class $\mathcal{A}$ and there exist real numbers $0 < s_0 < 1 < r_0$ such that:*

$$\operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0 \tag{2.4}$$

for any $z \in D$ with $|z| > r_0$, and

$$\operatorname{Im} \frac{f''(z)}{f'(z)} > 0 \tag{2.5}$$

for any $z \in D$ with $\operatorname{Im} z < s_0$, then the function f is convex and univalent in the half-plane D.

Proof. Let z_1 and z_2 be arbitrarily fixed distinct points in the half-plane D. For any $r > r^* = \max\{|z_1|, |z_2|\}$ and any $s \in (0, s^*)$, where $s^* = \min\{\operatorname{Im} z_1, \operatorname{Im} z_2\}$, the points z_1 and z_2 belong to the domain $D_{r,s}$.

From the hypothesis (2.4) and (2.5) and using the Remark 2.1 it follows that for any $r > r_0$ and $s \in (0, s_0)$ the function f is univalent in the domain $D_{r,s}$, and that $f(D_{r,s})$ is a convex domain.

Therefore, choosing $r > \max\{r_0, r^*\}$ and $s \in (0, s_1)$, where $s_1 = \min\{s_0,\ s^*\}$, it follows that the points z_1 and z_2 belong to the domain $D_{r,s}$, and since the function f is univalent in the domain $D_{r,s}$, we obtain that $f(z_1) \neq f(z_2)$.

Since z_1 and z_2 were arbitrarily chosen in the half-plane D, it follows that the function f is univalent in D, concluding the first part of the proof.

In order to show that $f(D)$ is a convex domain, we consider w_1 and w_2 arbitrarily fixed distinct points in $f(D)$, and let $z_1 = f^{-1}(w_1)$ and $z_2 = f^{-1}(w_2)$ be their preimages.

Repeating the above proof it follows that the points z_1 and z_2 belong to the domain $D_{r,s}$ (for any $r > \max\{r_0, r^*\}$ and $s \in (0, s_1)$, where $s_1 = \min\{s_0,\ s^*\}$, in the notation above), and therefore we obtain that $w_1 = f(z_1) \in f(D_{r,s})$ and $w_2 = f(z_2) \in f(D_{r,s})$.

Since $f(D_{r,s})$ is a convex domain, it follows that the line segment $[w_1, w_2]$ is also contained in the domain $f(D_{r,s})$, and since $f(D_{r,s}) \subset f(D)$, we obtain that $[w_1, w_2] \subset f(D)$.

Since $w_1, w_2 \in f(D)$ were arbitrarily chosen, it follows that $f(D)$ is a convex domain, concluding the proof. □

Remark 2.4. The point $z_0 = i$, in which the functions f belonging to the class $\mathcal{A} = \mathcal{A}_1$ are normalized can be replaced by any point $z_0 = iy_0$, with $y_0 > 0$. Repeating the proof of the previous theorem with this new choice for the normalization condition, we obtain the following result which generalizes the previous theorem:

Theorem 2.5. *If the function $f : D \to \mathbb{C}$ belongs to the class $\mathcal{A}_{y_0}$ for some $y_0 > 0$, and there exist real numbers $0 < s_0 < y_0 < r_0$ such that*

$$\begin{cases} \dfrac{zf''(z)}{f'(z)} + 1 > 0, & z \in D,\ |z| > r_0 \\ \dfrac{f''(z)}{f'(z)} > 0, & z \in D,\ z \in (0, s_0) \end{cases},$$

then the function f is univalent and convex in the half-plane D.

Remark 2.6. By noticing that the function $f : D \to \mathbb{C}$ is convex and univalent in D if and only the function $\tilde{f} : D \to \mathbb{C}$, $\widetilde{f}(z) = f(z) - f(iy_0)$ is convex and univalent in D, for any arbitrarily chosen point $y_0 > 0$, and replacing the function f in the previous theorem by $\widetilde{f}(z) = f(z) - f(iy_0)$, we can eliminate from the hypothesis of this theorem the condition $f(iy_0) = 0$, obtaining the following more general result:

Theorem 2.7. *If the function $f : D \to \mathbb{C}$ is analytic in D, satisfies $f'(z) \neq 0$ for all $z \in D$ and there exist real numbers $0 < s_0 < r_0$ such that the following inequalities hold:*

$$\begin{cases} \dfrac{zf''(z)}{f'(z)} + 1 > 0, & z \in D,\ |z| > r_0 \\ \dfrac{f''(z)}{f'(z)} > 0, & z \in D,\ z \in (0, s_0) \end{cases}, \tag{2.6}$$

then the function f is convex and univalent in the half-plane D.

Example 2.1. For $a \in \mathbb{R}$, consider the function $f_a : D \to \mathbb{C}$ defined by

$$f_a(z) = z^a, \quad z \in D,$$

where we have chosen the determination of the power function corresponding to the principal branch of the logarithm, that is:

$$z^a = e^{a \log z}, \quad z \in D,$$

where $\log z$ denotes the principal branch of the logarithm (with $\log i = i\frac{\pi}{2}$).

We have

$$\begin{aligned} f_a'(i) &= a i^{a-1} \\ &= a\left(\cos\frac{(a-1)\pi}{2} + i\sin\frac{(a-1)\pi}{2}\right) \\ &\neq 0, \end{aligned}$$

for any $a \neq 0$.

For an arbitrarily chosen $z \in D$ we have:

$$\begin{aligned} \frac{f_a''(z)}{f_a'(z)} &= (a-1)\,\frac{1}{z} \\ &= -\frac{(a-1)\,z}{|z|^2} \\ &> 0 \end{aligned}$$

for any $a < 1$, and also

$$\begin{aligned} \frac{zf_a''(z)}{f_a'(z)} + 1 &= (a-1) + 1 \\ &= a \\ &> 0 \end{aligned}$$

for any $a > 0$.

It follows that the hypotheses of the previous theorem are satisfied for any $a \in (0, 1)$, and according to this theorem it follows that the function $f_a(z) = z^a$ ($z \in D$) is convex and univalent in the half-plane D for any $a \in (0, 1)$.

It is easy to see that the function $f_a(z) = z^a$, $z \in D$, is convex and univalent for any $a \in (-1, 0) \cup (0, 1)$, and therefore the previous theorem gives only sufficient conditions for the convexity and univalence of an analytic function defined in the upper half-plane D.

Remark 2.8. As shown in [4], the condition

$$\frac{f''(z)}{f'(z)} > 0, \quad z \in D,$$

is a necessary condition (but not also a sufficient one) for an analytic function in D to be convex and univalent in D.

However, in the case of a hydrodynamically normalized function, as shown in Theorem 1.1 (see [5] and [6]), this becomes also a sufficient condition for the convexity and the univalence in the half-plane D. We recall that the hydrodynamic normalization used by Stankiewicz in is given by

(2.7) $$\lim_{z\to\infty, z\in D} (f(z) - z) = 0.$$

In particular, in the case of analytic and hydrodynamically normalized functions in the upper half-plane, from Theorem 2.7 we can obtain as a consequence a new proof of the last cited result, namely a necessary and sufficient condition for the convexity and the univalence of an analytic, hydrodynamically normalized function defined in the half-plane, as follows:

Corollary 2.9. *If the function $f : D \to \mathbb{C}$ is analytic and hydrodynamically normalized by (1.2) in the half-plane D, and it satisfies*

(2.8) $$f'(z) \neq 0 \quad \textit{for all } z \in D$$

and

(2.9) $$\operatorname{Im} \frac{f''(z)}{f'(z)} > 0, \quad \textit{for all } z \in D,$$

then the function f is convex and univalent in the half-plane D.

Proof. Since f satisfies the hydrodynamic normalization condition

$$\lim_{z\to\infty, z\in D} (f(z) - z - ai) = 0,$$

for some $a \geq 0$, it follows that for any $\varepsilon' > 0$ there exists $r > 0$ such that for $z \in D$ with $|z| > r$ we have:

$$|\operatorname{Im}(f(z) - z - ai)| \leq |f(z) - z - ai| < \varepsilon',$$

and therefore we obtain

$$\operatorname{Im} f(z) > \operatorname{Im} z + a - \varepsilon',$$

for any $z \in D$ with $|z| > r$.

Choosing $y_0 = \max\{r, \varepsilon - a\}$ and considering the auxiliary function $g : D \to \mathbb{C}$ defined by

$$g(z) = f(z + 2iy_0)$$

it follows that for all $z \in D$ we have:

$$\begin{aligned} g(z) &= f(z + 2iy_0) \\ &> z + 2y_0 + a - \varepsilon \\ &> y_0 \\ &> 0, \end{aligned}$$

which shows that $g : D \to D$.

Since the function f is hydrodynamically normalized, the function g is also hydrodynamically normalized, and from Lemma 1.2 we obtain

$$\begin{aligned} \lim_{z\to\infty, z\in T_\varepsilon} f'(z + 2iy_0) &= \lim_{z\to\infty, z\in T_\varepsilon} g'(z) \\ &= \lim_{z\to\infty, z\in T_\varepsilon} \frac{g(z)}{z} \\ &= 1, \end{aligned}$$

since from the hydrodynamic normalization condition we have

$$\begin{aligned} \lim_{z\to\infty, z\in D} \frac{g(z)}{z} - 1 &= \lim_{z\to\infty, z\in D} \frac{g(z) - z}{z} \\ &= \frac{\lim\limits_{z\to\infty, z\in D} g(z) - z}{\lim_{z\to\infty, z\in D} z} \\ &= \frac{ai}{\lim\limits_{z\to\infty, z\in D} z} \\ &= 0, \end{aligned}$$

and therefore we obtain $\lim\limits_{z\to\infty, z\in T_\varepsilon} \frac{g(z)}{z} = 1$, for any $\varepsilon \in (0, \frac{\pi}{2})$.

From Lemma 1.3, applied to the function g in the particular case $n = 2$, we obtain:

$$\lim_{z\to\infty, z\in T_\varepsilon} \left[z^2 g''(z)\right] = 0,$$

for any $\varepsilon \in (0, \frac{\pi}{2})$, and therefore we obtain

$$\lim_{z\to\infty, z\in T_\varepsilon} \left[z^2 f''(z)\right] = \lim_{z\to\infty, z\in T_\varepsilon} \left[(z - 2iy_0)^2 g''(z - 2iy_0) \frac{z^2}{(z - 2iy_0)^2}\right] = 0.$$

Since $\lim\limits_{z\to\infty, z\in D} f'(z) = 1$, we obtain

$$\lim_{z\to\infty, z\in T_\varepsilon} \frac{z f''(z)}{f'(z)} = 0,$$

for any $\varepsilon \in (0, \frac{\pi}{2})$.

It follows that for any $\varepsilon \in (0, \frac{\pi}{2})$ arbitrarily fixed, there exists $r_0 > 0$ such that

$$\frac{zf''(z)}{f'(z)} + 1 > 0,$$

for any $z \in T_\varepsilon$ with $|z| > r_0$.

Following the proof Theorem 2.7 it can be seen that this inequality together with the hypotheses (2.8) and (2.9) suffices for the proof, and therefore the function f is convex and univalent in the half-plane D, concluding the proof. □

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