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LUPAŞ-DURRMEYER OPERATORS

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ABSTRACT. In the present paper, we obtain Stechkin-Marchaud-type inequalities for some approximation operators, more precisely for Lupaş-Durrmeyer operators defined as in (1.1).

Key words and phrases: Stechkin-Marchaud-type inequalities, Lupaş Operators, Durrmeyer Operators.

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1. INTRODUCTION

Lupaş proposed a family of linear positive operators mapping $C[0,\infty)$ into $C[0,\infty)$, the class of all bounded and continuous functions on $[0,\infty)$, namely,

$$V_n(f,x) = \sum_{k=0}^{\infty} p_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0,\infty),$$

where $p_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}$.

Motivated by Derriennic [1], Sahai and Prasad [5] proposed modified Lupaş operators defined, for functions integrable on $[0,\infty)$, by

$$B_n(f,x) = (n-1)\sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} p_{n,k}(t) f(t)dt. \tag{1.1}$$

Wicken discussed Stechkin-Marchaud-type inequalities in [2] for Bernstein polynomials and obtained the following results:

$$w_\phi^2\left(f, \frac{1}{\sqrt{n}}\right) \le Cn^{-1} \sum_{k=1}^{n} \left\| \phi^{-\alpha}(B_k f - f)\right\|_\infty.$$

The main object of this paper is to give Stechkin-Marchaud-type inequalities for Lupaş-Durrmeyer operators. In the end of this section we introduce some definitions and notations.

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Definition 1.1. For $0 \leq \lambda \leq 1, \quad 0 < \alpha < 2r, 0 \leq \beta \leq 2r, \quad 0 \leq \alpha(1-\lambda) + \beta \leq 2r$

$$\|f\|_0 = \|f\|_{0,\alpha,\beta,\lambda} = \sup_{x \in I} \left\{ \left| \phi^{\alpha(\lambda-1)-\beta}(x) f(x) \right| \right\}, \tag{1.2}$$

$$C^0_{\alpha,\beta,\lambda} = \{ f \in C_B(I), \|f\|_0 < \infty \}, \tag{1.3}$$

$$\|f\|_r = \|f\|_{r,\alpha,\beta,\lambda} = \sup_{x \in I} \left\{ \left| \phi^{2r+\alpha(\lambda-1)-\beta}(x) f^{(2r)}(x) \right| \right\} \tag{1.4}$$

and

$$C^r_{\alpha,\beta,\lambda} = \left\{ f \in C_B(I), f^{(2r-1)} \in AC_{loc}, \|f\|_r < \infty \right\}, \tag{1.5}$$

where $\phi(x) = \sqrt{x(1+x)}$ and $r = 0, 1, 2, \ldots$.

Definition 1.2. Peetre's K-functional is defined as

$$w^{2r}_{\phi^\lambda}(f,t)_{\alpha,\beta} = \sup_{0<h\leq t} \sup_{x \pm rh\phi^\lambda(x) \in I} \left\{ \left| \phi^{\alpha(\lambda-1)-\beta}(x) \Delta^{2r}_{h\phi^\lambda} f(x) \right| \right\} \tag{1.6}$$

and

$$K_{\phi^\lambda}(f,t^{2r})_{\alpha,\beta} = \inf_{g^{(2r-1)} \in AC_{loc}} \left\{ \|f-g\|_0 + t^{2r} \|g\|_r \right\}, \tag{1.7}$$

where AC_{loc} is the space of real valued absolute continuous and integrable functions on $[0,1]$.

In second section of the paper, we will give some basic results, which will be useful in proving the main theorems; while in Section 3 the main results are given.

2. AUXILIARY RESULTS

Some basic results are given here.

Lemma 2.1. *Suppose that for nonnegative sequences $\{\sigma_n\}, \{\tau_n\}$ with $\sigma_1 = 0$ the inequality $\sigma \leq \left(\frac{k}{n}\right)^p \sigma_k + \tau_k,\ (1 \leq k \leq n)$, is satisfied for $n \in \mathbb{N},\ p > 0$. Then one has*

$$\sigma_n \leq B_p n^{-p} \sum_{k=1}^{n} k^{p-1} \tau_k. \tag{2.1}$$

Lemma 2.2. *For $f^{(2s)} \in C^0_{\alpha,\beta,\lambda},\ s \in N_0$, the following inequalities hold*

$$\left\| B_n^{(2s)} f \right\|_r \leq C_1 n^r \left\| f^{(2s)} \right\|_0, \tag{2.2}$$

and

$$\left\| B_n^{(2s)} f \right\|_r \leq C_2 n^{r + \frac{\alpha(1-\lambda)}{2} + \frac{\beta}{2}} \left\| f^{(2s)} \right\|_\infty. \tag{2.3}$$

Lemma 2.3. *For $f^{(2s)} \in C^r_{\alpha,\beta,\lambda},\ s \in N_0$, the following inequality holds*

$$\left\| B_n^{(2s)} f \right\|_r \leq \left\| f^{(2s)} \right\|_r. \tag{2.4}$$

Lemma 2.4. *Let us suppose that $f^{(2s)} \in C^0_{\alpha,\beta,\lambda},\ s \in N_0$, $0 \leq \alpha(1-\lambda) + \beta \leq 2$, then*

$$\left\| B_n^{(2s)} f \right\|_r \leq C \left(\sum_{k=1}^{n} k^{r-1} \left\| (B_k f - f)^{(2s)} \right\|_0 + \left\| f^{(2s)} \right\|_\infty \right). \tag{2.5}$$

Lemma 2.5. *Suppose that* $r \in \mathbb{N},\ x \pm rt \in I,\ 0 \le \beta \le 2r,\ 0 \le t \le \frac{1}{16r}$*, then*

$$\int_{-\frac{t}{2r}}^{\frac{t}{2r}} \cdots \int_{-\frac{t}{2r}}^{\frac{t}{2r}} \phi^{-\beta}\left(x + \sum_{j=1}^{2r} u_j\right) du_1 \cdots du_{2r} \le C(\beta) t^{2r} \phi^{-\beta}(x). \tag{2.6}$$

3. MAIN RESULTS

We are now ready to prove the main results of this paper.

Theorem 3.1. *For the modulus of smoothness and K-functional*

$$K_{\phi^\lambda}\left(f^{(2s)}, \frac{1}{n^r}\right)_{\alpha,\beta} \le Cn^{-r}\left(\sum_{k=1}^{n} k^{r-1}\left\|(B_k f - f)^{(2s)}\right\|_0 + \left\|f^{(2s)}\right\|_\infty\right), \tag{3.1}$$

$$w_{\phi^\lambda}^{2r}\left(f^{(2s)}, \frac{1}{\sqrt{n}}\right)_{\alpha,\beta} \le Cn^{-\frac{r}{2-\lambda}}\left(\sum_{k=1}^{\left[n^{\frac{1}{2-\lambda}}\right]} k^{-\frac{r-1}{2-\lambda}}\left\|(B_k f - f)^{(2s)}\right\|_0 + \left\|f^{(2s)}\right\|_\infty\right), \tag{3.2}$$

where $\|\cdot\|_\infty$*denotes the supremum norm.*

Proof of (3.1). Taking $\frac{n}{2} \le m \le n$ such that $\left\|(B_m f - f)^{(2s)}\right\|_0 \le \left\|(B_k f - f)^{(2s)}\right\|_0$, $(\frac{n}{2} < k \le n)$, we have

$$\begin{aligned} K_{\phi^\lambda}\left(f^{(2s)}, \frac{1}{n^r}\right)_{\alpha,\beta} &\le \left\|(B_m f - f)^{(2s)}\right\|_0 + n^{-r}\left\|f_m^{(2s)}\right\|_r \\ &\le \frac{2^r}{n^r}\sum_{k=\frac{n}{2}}^{n} k^{r-1}\left\|(B_k f - f)^{(2s)}\right\|_0 \\ &\quad + Cn^{-r}\left(\sum_{k=1}^{m} k^{r-1}\left\|(B_k f - f)^{(2s)}\right\|_0 + \left\|f^{(2s)}\right\|_\infty\right) \\ &\le Cn^{-r}\left(\sum_{k=1}^{n} k^{r-1}\left\|(B_k f - f)^{(2s)}\right\|_0 + \left\|f^{(2s)}\right\|_\infty\right). \end{aligned}$$

□

Proof of (3.2). By definition of K-functional there exists $g \in C^r_{\alpha,\beta,\lambda}$ such that

$$\left\|f^{(2s)} - g\right\|_0 + n^{-\frac{r}{2-\lambda}}\|g\|_r \le K_{\phi^\lambda}\left(f, n^{-\frac{r}{2-\lambda}}\right)_{\alpha,\beta} \tag{3.3}$$

and

$$\left|\Delta^{2r}_{h\phi^\lambda(x)} f^{(2s)}(x)\right| \le C\phi^{\alpha(1-\lambda)+\beta}(x)\left\|f^{(2s)}\right\|_0 \tag{3.4}$$

by Lemma 2.5 for above $g,\ 0 < h\phi^\lambda(x) < \frac{1}{16r},\ x \pm rh\phi^\lambda(x) \in I,$

$$\left|\Delta^{2r}_{h\phi^\lambda(x)} g(x)\right| \le Ch^{2r}\phi^{(-2r+\alpha)(1-\lambda)+\beta}(x)\|g\|_r. \tag{3.5}$$

Using (3.4) and (3.5), again for $0 < h\phi^\lambda(x) < \frac{1}{16r},\ x \pm rh\phi^\lambda(x) \in I$, we get

$$\left|\Delta^{2r}_{h\phi^\lambda(x)} f^{(2s)}(x)\right| \le C\phi^{\alpha(1-\lambda)+\beta}(x)\left\{\left\|f^{(2s)} - g\right\|_0 + h^{2r}\phi^{2r(\lambda-1)}(x)\|g\|_r\right\}. \tag{3.6}$$

For $x \pm rh\phi^{\lambda}(x) \in I$, we obtain

(3.7) $$h^2\phi^{2(\lambda-1)}(x) \le \left[\frac{1}{2}n^{\frac{1}{2-\lambda}}\right]^{-1}.$$

From (3.6) and (3.7) we have

(3.8) $$\begin{aligned}\left|\Delta^{2r}_{h\phi^{\lambda}(x)} f^{(2s)}(x)\right| &\le C\phi^{\alpha(1-\lambda)+\beta}(x) K_{\phi^{\lambda}}\left(f^{(2s)}, \left[\frac{1}{2}n^{\frac{1}{2-\lambda}}\right]^{-1}\right)_{\alpha,\beta} \\ &\le C\phi^{\alpha(1-\lambda)+\beta}(x) n^{-\frac{r}{2-\lambda}}\left(\sum_{k=1}^{\left[n^{\frac{r}{2-\lambda}}\right]} k^{-\frac{r-1}{2-\lambda}}\left\|(B_k f - f)^{(2s)}\right\|_0 + \left\|f^{(2s)}\right\|_{\infty}\right).\end{aligned}$$

□

Corollary 3.2. *If* $0 < \alpha < 2,\ f \in C_B(I)$, *then*

$$|(B_n f)(x) - f(x)| = O\left((n^{-1/2}\phi^{1-\lambda}(x))^{\alpha}\right) \Rightarrow w^2_{\phi^{\lambda}}(f,t) = O(t^{\alpha}),$$

where

$$w^2_{\phi^{\lambda}}(f,t) = \sup_{0<h\le t}\ \sup_{x\pm h\phi^{\lambda}(x)\in I}\left\{\left|\Delta^2_{h\phi^{\lambda}} f(x)\right|\right\}.$$

This is the inverse part in [3].

In (1.4) and (1.5), for $\delta_n(x) = \phi(x) + \frac{1}{\sqrt{n}}$, $\phi(x)$ replaced by $\delta_n(x)$, (3.1) also holds.

Corollary 3.3. *If* $0 < \alpha < 2r,\ f \in C_B(I)$, *then*

$$|(M_n f)(x) - f(x)| = O\left((n^{-1/2}\phi^{1-\lambda}(x))^{\alpha}\right) \Rightarrow w^{2r}_{\phi^{\lambda}}(f,t) = O(t^{\alpha}),$$

where $(M_n f)(x)$ *is linear combination of* $(B_n f)(x)$.

This is the inverse parts in [4].

Remark 3.4. We also propose some other modifications of Lupaş operators as

$$M_n(f,x) = n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} s_{n,k}(t) f(t)\,dt$$

where $s_{n,k}(t) = e^{-nt}\frac{(nt)^k}{k!}$ and $p_{n,k}(x)$is defined in (1.1) for these operators M_n.

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