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LOCAL ESTIMATES FOR JACOBI POLYNOMIALS

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ABSTRACT. It is shown that if $\alpha, \beta \geq -\frac{1}{2}$, then the orthonormal Jacobi polynomials $p_n^{(\alpha,\beta)}$ fulfill the local estimate

$$|p_n^{(\alpha,\beta)}(t)| \leq \frac{C(\alpha,\beta)}{(\sqrt{1-x}+\frac{1}{n})^{\alpha+\frac{1}{2}}(\sqrt{1+x}+\frac{1}{n})^{\beta+\frac{1}{2}}}$$

for all $t \in U_n(x)$ and each $x \in [-1,1]$, where $U_n(x)$ are subintervals of $[-1,1]$ defined by $U_n(x) = [x-\frac{\varphi_n(x)}{n}, x+\frac{\varphi_n(x)}{n}] \cap [-1,1]$ for $n \in \mathbb{N}$ and $x \in [-1,1]$ with $\varphi_n(x) = \sqrt{1-x^2}+\frac{1}{n}$. Applications of the local estimate are given at the end of the paper.

Key words and phrases: Jacobi polynomials, Jacobi weights, Local estimates.

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1. INTRODUCTION

Let $w^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$, $x \in [-1,1]$, be a Jacobi weight with $\alpha, \beta > -1$. Let $p_n(x) = p_n^{(\alpha,\beta)}(x) = \gamma_n^{(\alpha,\beta)}x^n + \dots$, $n \in \mathbb{N}_0$, denote the unique *Jacobi polynomials* of precise degree n, with leading coefficients $\gamma_n^{(\alpha,\beta)} > 0$, fulfilling the orthonormal condition $\int_{-1}^1 p_n(x)p_m(x)w^{(\alpha,\beta)}(x)\,dx = \delta_{n,m}$, $n, m \in \mathbb{N}_0$.

This paper is concerned with local estimates of Jacobi polynomials by means of modified Jacobi weights. By the *modified Jacobi weights* we understand the functions

$$w_n^{(\alpha,\beta)}(x) := \left(\sqrt{1-x}+\frac{1}{n}\right)^{2\alpha}\left(\sqrt{1+x}+\frac{1}{n}\right)^{2\beta}, \quad x \in [-1,1],\ n \in \mathbb{N}. \tag{1.1}$$

We observe that all modified Jacobi weights $w_n^{(\alpha,\beta)}$ are finite and positive. This is in contrast to the fact that the Jacobi weight $w^{(\alpha,\beta)}$ may have singularities and roots in ± 1, depending on whether α and β are negative or positive. The Jacobi polynomials can be estimated by means of modified Jacobi weights as follows (see [3] and Theorem 2.1 below):

$$|p_n^{(\alpha,\beta)}(x)| \le C\,\frac{1}{w_n^{(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4})}(x)}$$

for all $x \in [-1,1]$. If $\alpha, \beta \ge -\frac{1}{2}$, then we will show that this estimate can be further extended, namely

$$|p_n^{(\alpha,\beta)}(t)| \le C\,\frac{1}{w_n^{(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4})}(x)}$$

for all $t \in U_n(x)$ and each $x \in [-1,1]$, where $U_n(x)$ are subintervals of $[-1,1]$ defined by

$$U_n(x) := \left\{ t \in [-1,1] \,\middle|\, |t-x| \le \frac{\varphi_n(x)}{n} \right\} = \left[x - \frac{\varphi_n(x)}{n}, x + \frac{\varphi_n(x)}{n} \right] \cap [-1,1] \tag{1.2}$$

for $n \in \mathbb{N}$ and $x \in [-1,1]$ with

$$\varphi_n(x) := \sqrt{1-x^2} + \frac{1}{n}. \tag{1.3}$$

Thus $U_n(x)$ is located around x and is *small*, i.e., $|U_n(x)| = O(1/n)$. In our case of Jacobi weights on $[-1,1]$ we need intervals around x with radius $\frac{\varphi_n(x)}{n}$ instead of $\frac{1}{n}$. In this case the radius varies together with x and becomes smaller if x tends to 1 or -1.

2. THEOREMS

The following theorem provides a useful local estimate of the orthonormal Jacobi polynomials by means of the modified weights w_n. The estimate can also be found in the paper [3] by Lubinsky and Totik. Here we will give an explicit proof. The proof is essentially based on an estimate taken from Szegö [4].

Theorem 2.1. *Let $\alpha, \beta > -1$ and $n \in \mathbb{N}$. Then*

$$|p_n^{(\alpha,\beta)}(x)| \le C\,\frac{1}{w_n^{(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4})}(x)} \tag{2.1}$$

for all $x \in [-1,1]$ with a positive constant $C = C(\alpha,\beta)$ being independent of n and x.

Proof. First let $x \in [0,1]$, and let $t \in [0, \frac{\pi}{2}]$ such that $x = \cos t$. Moreover, let $P_n = P_n^{(\alpha,\beta)} = (h_n^{(\alpha,\beta)})^{\frac{1}{2}} p_n^{(\alpha,\beta)}(x)$, $n \in \mathbb{N}$, be the polynomials normalized by the factor $(h_n^{(\alpha,\beta)})^{\frac{1}{2}}$, namely $P_n^{(\alpha,\beta)} = (h_n^{(\alpha,\beta)})^{\frac{1}{2}} p_n^{(\alpha,\beta)}(x)$, as can be found in Szegö [4, eq. (4.3.4)]. According to Szegö's book [4, Theorem 7.32.2] the estimate

$$|P_n^{(\alpha,\beta)}(\cos t)| \le C \begin{cases} n^{\alpha}, & \text{if } 0 \le t \le \frac{c}{n} \\ t^{-(\alpha+\frac{1}{2})} n^{-\frac{1}{2}}, & \text{if } \frac{c}{n} \le t \le \frac{\pi}{2} \end{cases} \tag{2.2}$$

is valid, where c and C are fixed positive constants being independent of n and t. We substitute $t = \arccos x \in [0, \frac{\pi}{2}]$ and $P_n^{(\alpha,\beta)}(x) = (h_n^{(\alpha,\beta)})^{\frac{1}{2}} p_n^{(\alpha,\beta)}(x)$ in (2.2) and obtain, using $(h_n^{(\alpha,\beta)})^{-\frac{1}{2}} \le \tilde{C} \cdot n^{\frac{1}{2}}$ (resulting from [4, eq. (4.3.4)]),

$$|p_n^{(\alpha,\beta)}(x)| \le C_1 \begin{cases} n^{\alpha+\frac{1}{2}}, & \text{if } 0 \le \arccos x \le \frac{c}{n} \\ (\arccos x)^{-(\alpha+\frac{1}{2})}, & \text{if } \frac{c}{n} \le \arccos x \le \frac{\pi}{2} \end{cases} \tag{2.3}$$

with $C_1 = C_1(\alpha, \beta) > 0$ independent of n and x. Below we will make use of the estimates

$$\frac{\pi}{2}\sqrt{1-x} = \frac{\pi}{\sqrt{2}}\sqrt{\frac{1-x}{2}} = \frac{\pi}{\sqrt{2}}\sin\frac{t}{2}$$
$$\geq \frac{\pi}{\sqrt{2}}\left(\frac{2}{\pi}\cdot\frac{t}{\sqrt{2}}\right) = t = \arccos x \tag{2.4}$$

and

$$\sqrt{2}\sqrt{1-x} = 2\sqrt{\frac{1-x}{2}} = 2\sin\frac{t}{2} \leq 2\cdot\frac{t}{2} = t = \arccos x. \tag{2.5}$$

The cases $-1 < \alpha \leq -\frac{1}{2}$ and $\alpha > -\frac{1}{2}$ are considered separately in the following.

Case $-1 < \alpha \leq -\frac{1}{2}$: In this case it follows that $-\left(\alpha + \frac{1}{2}\right) \geq 0$. If $0 \leq \arccos x \leq \frac{c}{n}$, then

$$\left|p_n^{(\alpha,\beta)}(x)\right| \overset{(2.3)}{\leq} C_1\, n^{\alpha+\frac{1}{2}} = C_1 \left(\frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)} \leq C_1 \left(\sqrt{1-x} + \frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}.$$

If $\frac{c}{n} \leq \arccos x \leq \frac{\pi}{2}$, then

$$\left|p_n^{(\alpha,\beta)}(x)\right| \overset{(2.3)}{\leq} C_1\,(\arccos x)^{-\left(\alpha+\frac{1}{2}\right)} \overset{(2.4)}{\leq} C_2\,(\sqrt{1-x})^{-\left(\alpha+\frac{1}{2}\right)}$$
$$\leq C_2 \left(\sqrt{1-x} + \frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}.$$

Case $\alpha > -\frac{1}{2}$: In this case we obtain $-\left(\alpha + \frac{1}{2}\right) < 0$. If $0 \leq \arccos \leq \frac{c}{n}$, then from (2.5) we obtain $\frac{c}{n} \geq \sqrt{2}\sqrt{1-x}$ and hence

$$\left|p_n^{(\alpha,\beta)}(x)\right| \overset{(2.3)}{\leq} C_1\, n^{\alpha+\frac{1}{2}} = C_2 \left(\frac{c}{n} + \frac{c}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}$$
$$\leq C_3 \left(\sqrt{1-x} + \frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}.$$

If $\frac{c}{n} \leq \arccos x \leq \frac{\pi}{2}$, then

$$\left|p_n^{(\alpha,\beta)}(x)\right| \overset{(2.3)}{\leq} C_1\,(\arccos x)^{-\left(\alpha+\frac{1}{2}\right)} = C_4\,(\arccos x + \underbrace{\arccos x}_{\geq \frac{c}{n}})^{-\left(\alpha+\frac{1}{2}\right)}$$
$$\overset{(2.5)}{\leq} C_5 \left(\sqrt{1-x} + \frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}.$$

With both previous cases we have proved

$$\left|p_n^{(\alpha,\beta)}(x)\right| \leq C_6(\alpha,\beta) \left(\sqrt{1-x} + \frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)} \cdot \left(\sqrt{1+x} + \frac{1}{n}\right)^{-\left(\beta+\frac{1}{2}\right)}$$

for all $x \in [0,1]$, $n \in \mathbb{N}$ and $\alpha, \beta > -1$. Since $p_n^{(\alpha,\beta)}(x) = (-1)^n p_n^{(\beta,\alpha)}(-x)$, we obtain

$$\left|p_n^{(\alpha,\beta)}(x)\right| \leq C_6(\beta,\alpha) \left(\sqrt{1+x} + \frac{1}{n}\right)^{-\left(\beta+\frac{1}{2}\right)} \cdot \left(\sqrt{1-x} + \frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}$$

for all $x \in [-1,0)$, $n \in \mathbb{N}$ and $\alpha, \beta > -1$. This furnishes the validity of (2.1). ☐

Estimate (2.1) of Theorem 2.1 cannot hold true for $n = 0$ since the modified weight w_n is not defined for $n = 0$. However, if $n = 0$, then

$$\left|p_0^{(\alpha,\beta)}(x)\right| \le C(\alpha,\beta)\,\frac{1}{w_1^{\left(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}\right)}(x)}, \tag{2.6}$$

since $p_0^{(\alpha,\beta)}(x)$ is a constant and $C_1(\alpha,\beta) \le w_1^{\left(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}\right)}(x) \le C_2(\alpha,\beta)$ with positive constants $C_1(\alpha,\beta)$ and $C_2(\alpha,\beta)$.

Next, we will see that the local estimate of Theorem 2.1 can be further extended. We will show that $\left|p_n^{(\alpha,\beta)}(x)\right|$ in (2.1) can be replaced by $\left|p_n^{(\alpha,\beta)}(t)\right|$, whenever t is not too far away from x, namely if t is in the interval $U_n(x) = \left[x - \frac{\varphi_n(x)}{n}, x + \frac{\varphi_n(x)}{n}\right] \cap [-1,1]$. However, for this estimate we will need the assumption $\alpha, \beta \ge -\frac{1}{2}$. The result is stated in the following

Theorem 2.2. *Let $\alpha, \beta \ge -\frac{1}{2}$ and $n \in \mathbb{N}$. Then*

$$|p_n^{(\alpha,\beta)}(t)| \le C\,\frac{1}{w_n^{\left(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}\right)}(x)} \tag{2.7}$$

for all $t \in U_n(x)$ and each $x \in [-1,1]$, where the interval $U_n(x)$ has been given in (1.2) *and $C = C(\alpha,\beta)$ is a positive constant independent of n, t and x.*

It must be mentioned that Theorem 2.2 cannot be extended to hold true even for all $\alpha, \beta > -1$. This is due to the fact that $1/w_n^{\left(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}\right)}(x) \to 0$ as $n \to \infty$, if x is a boundary point $x = 1$ or $x = -1$ and $\frac{\alpha}{2} + \frac{1}{4} < 0$ or $\frac{\beta}{2} + \frac{1}{4} < 0$ respectively.

First, we need an auxiliary lemma.

Lemma 2.3. *Let $a, b \le 0$, $n \in \mathbb{N}$ and $x \in [-1,1]$. Then*

$$w_n^{(a,b)}(t) \le 16^{-(a+b)}\, w_n^{(a,b)}(x) \tag{2.8}$$

for all $t \in U_n(x)$.

Proof. First, let $a \le 0$. We will prove that

$$16^a \left(\sqrt{1-t} + \frac{1}{n}\right)^{2a} \le \left(\sqrt{1-x} + \frac{1}{n}\right)^{2a} \tag{2.9}$$

holds true for all $t \in U_n(x)$ with $x \in [-1,1]$ and $n \in \mathbb{N}$. There is nothing to prove for $a = 0$. Let $a < 0$. Then inequality (2.9) is equivalent to

$$4\left(\sqrt{1-t} + \frac{1}{n}\right) \ge \sqrt{1-x} + \frac{1}{n}$$

and

$$4\sqrt{1-t} \ge \sqrt{1-x} - \frac{3}{n} \tag{2.10}$$

respectively. In order to prove (2.10) for $t \in U_n(x)$ we will discuss below the cases $x \in \left[1 - \frac{9}{n^2}, 1\right]$ and $x \in \left[-1, 1 - \frac{9}{n^2}\right)$ separately. We must note that the latter interval is empty for $n = 1, 2, 3$.

Case $\mathbf{x} \in \left[\mathbf{1} - \frac{\mathbf{9}}{\mathbf{n^2}}, \mathbf{1}\right]$: In this case we obtain $\sqrt{1-x} - \frac{3}{n} \le \frac{3}{n} - \frac{3}{n} = 0$, which immediately gives (2.10).

Case $\mathbf{x} \in \left[-\mathbf{1}, \mathbf{1} - \frac{\mathbf{9}}{\mathbf{n^2}}\right)$: In this case we obtain $\sqrt{1-x} - \frac{3}{n} > 0$. Therefore inequality (2.10) is equivalent to (squaring both sides of (2.10))

$$16(1-t) \geq 1 - x - \frac{6}{n}\sqrt{1-x} + \frac{9}{n^2}$$

or, rewritten,

$$15 + x + \frac{6}{n}\sqrt{1-x} - \frac{9}{n^2} \geq 16t. \tag{2.11}$$

Since $t \in U_n(x) \subset \left[x - \frac{\varphi_n(x)}{n}, x + \frac{\varphi_n(x)}{n}\right]$, we obtain

$$\begin{aligned} x + \frac{6}{n}\sqrt{1-x} - \frac{9}{n^2} &= \left(x + \frac{2}{n}\sqrt{1-x} + \frac{1}{n^2}\right) + \left(\frac{4}{n}\sqrt{1-x} - \frac{10}{n^2}\right) \\ &\geq x + \frac{\varphi_n(x)}{n} + \frac{4}{n}\sqrt{1-x} - \frac{10}{n^2} \\ &\geq t + \frac{4}{n}\sqrt{1-x} - \frac{10}{n^2}. \end{aligned}$$

Hence, inequality (2.11) holds true if

$$15 + \frac{4}{n}\underbrace{\sqrt{1-x}}_{\geq \frac{3}{n}} - \frac{10}{n^2} \geq 15t$$

or if

$$15 + \frac{2}{n^2} \geq 15t. \tag{2.12}$$

Since $t \leq 1$, inequality (2.12) is fulfilled. Hence inequality (2.10) is also proved. This completes the proof of (2.9) for all $x \in [-1, 1]$ and $t \in U_n(x)$.

Now, let $b \leq 0$, $x \in [-1, 1]$ and $t \in U_n(x)$. Then $-t \in U_n(-x)$. From (2.9) we obtain

$$\begin{aligned} 16^b \left(\sqrt{1+t} + \frac{1}{n}\right)^{2b} &= 16^b \left(\sqrt{1-(-t)} + \frac{1}{n}\right)^{2b} \\ &\overset{(2.9)}{\leq} \left(\sqrt{1-(-x)} + \frac{1}{n}\right)^{2b} = \left(\sqrt{1+x} + \frac{1}{n}\right)^{2b}, \end{aligned}$$

which proves the validity of (2.8). □

Proof of Theorem 2.2. Since $\alpha, \beta \geq -\frac{1}{2}$, it follows that $\frac{\alpha}{2} + \frac{1}{4}, \frac{\beta}{2} + \frac{1}{4} \geq 0$. Therefore we can apply Lemma 2.3 with $a = -\frac{\alpha}{2} - \frac{1}{4}$ and $b = -\frac{\beta}{2} - \frac{1}{4}$, obtaining

$$\frac{1}{w_n^{(\frac{\alpha}{2}+\frac{1}{4}, \frac{\beta}{2}+\frac{1}{4})}(t)} = w_n^{(-\frac{\alpha}{2}-\frac{1}{4}, -\frac{\beta}{2}-\frac{1}{4})}(t) \overset{\text{Lem. 2.3}}{\leq} \frac{4^{\alpha+\beta+1}}{w_n^{(\frac{\alpha}{2}+\frac{1}{4}, \frac{\beta}{2}+\frac{1}{4})}(x)}$$

for all $t \in U_n(x)$. Application of Theorem 2.1 therefore yields inequality (2.2) for all $t \in U_n(x)$ as claimed. □

3. APPLICATIONS

In this section we will give some applications of the local estimates of the Jacobi polynomials. We apply Theorem 2.2 and obtain

$$\int_{U_n(x)} \left|p_n^{(\alpha,\beta)}(t)\right|^2 w^{(\alpha,\beta)}(t)\, dt \leq C \frac{1}{w_n^{(\alpha+\frac{1}{2}, \beta+\frac{1}{2})}(x)} \int_{U_n(x)} w^{(\alpha,\beta)}(t)\, dt.$$

Using

$$\int_{U_n(x)} w^{(\alpha,\beta)}(t)\,dt \le C\,\frac{1}{n}\,w_n^{(\alpha+\frac{1}{2},\beta+\frac{1}{2})}(x)$$

(see [2]) we find that

$$\int_{U_n(x)} \left|p_n^{(\alpha,\beta)}(t)\right|^2 w^{(\alpha,\beta)}(t)\,dt \le C(\alpha,\beta)\,\frac{1}{n}, \quad x\in[-1,1], \tag{3.1}$$

is valid for all $n\in\mathbb{N}$ with $\alpha,\beta\ge -\frac{1}{2}$. Estimate (3.1) shows that the intervals $U_n(x)$ are appropriate for measuring the growth of the orthonormal polynomials on subintervals of $[-1,1]$: $U_n(x)$ is located around x, $|U_n(x)| = O(1/n)$, the radius $\frac{\varphi_n(x)}{n}$ varies together with x and becomes smaller if x tends to 1 or -1 and the weighted integration of $(p_n^{(\alpha,\beta)}(t))^2$ on $U_n(x)$ is $O(1/n)$, whereas the weighted integral on $[-1,1]$ equals 1, i.e.,

$$\int_{-1}^{1} \left|p_n^{(\alpha,\beta)}(t)\right|^2 w^{(\alpha,\beta)}(t)\,dt = 1, \quad x\in[-1,1].$$

Let $a,b>-\frac{1}{2}$ and $C_1,C_2>0$. Let $m\colon [1,\infty)\to\mathbb{R}$ be a differentiable function fulfilling the Hormander conditions

$$0\le m(t)\le C_1 \quad\text{and}\quad |m'(t)|\le C_2\,t^{-1}$$

for $t\ge 1$. It was proved in [1] that

$$\sum_{k=1}^{n} \frac{m(k)}{w_k^{(a,b)}(x)} \le C\,\frac{n}{w_n^{(a,b)}(x)} \tag{3.2}$$

for all $x\in[-1,1]$ and $n\in\mathbb{N}$ with a positive constant $C = C(a,b,C_1,C_2)$ being independent of n and x.

Let $\alpha,\beta\ge-\frac{1}{2}$. Now, we will apply Theorem 2.2 and the above estimate (3.2) with $a = \alpha+\frac{1}{2}\ge 0$ and $b=\beta+\frac{1}{2}\ge 0$, to obtain

$$\sum_{k=1}^{n} m(k)\,(p_k^{(\alpha,\beta)}(t))^2 \underset{(3.2)}{\overset{\text{Theorem 2.2}}{\le}} C\,\frac{n}{w_n^{(\alpha+\frac{1}{2},\beta+\frac{1}{2})}(x)} \tag{3.3}$$

for all $t\in U_n(x)$ and each $x\in[-1,1]$ with a constant $C=C(\alpha,\beta,C_1,C_2)>0$ being independent of n and x.

In particular, if we let $m(k)=1$, then estimate (3.3) shows that the Christoffel function, defined by

$$\lambda_n^{(\alpha,\beta)}(t) := \left\{\sum_{k=1}^{n}(p_k^{(\alpha,\beta)}(t))^2\right\}^{-1},$$

fulfills the estimate

$$(\lambda_n^{(\alpha,\beta)}(t))^{-1}\le C(\alpha,\beta)\,\frac{n}{w_n^{(\alpha+\frac{1}{2},\beta+\frac{1}{2})}(x)}$$

for $t\in U_n(x)$ and $x\in[-1,1]$ and $n\in\mathbb{N}$.

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