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JENSEN'S INEQUALITY FOR CONDITIONAL EXPECTATIONS

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ABSTRACT. We study conditional expectations generated by an abelian C^*-subalgebra in the centralizer of a positive functional. We formulate and prove Jensen's inequality for functions of several variables with respect to this type of conditional expectations, and we obtain as a corollary Jensen's inequality for expectation values.

Key words and phrases: Trace function, Jensen's inequality, Conditional expectation.

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1. PRELIMINARIES

An n-tuple $\underline{x} = (x_1, \ldots, x_n)$ of elements in a C^*-algebra $\mathcal{A}$ is said to be abelian if the elements $x_1, \ldots, x_n$ are mutually commuting. We say that an abelian n-tuple $\underline{x}$ of self-adjoint elements is in the domain of a real continuous function f of n variables defined on a cube of real intervals $\underline{I} = I_1 \times \cdots \times I_n$ if the spectrum $\sigma(x_i)$ of x_i is contained in I_i for each $i = 1, \ldots, n$. In this situation $f(\underline{x})$ is naturally defined as an element in $\mathcal{A}$ in the following way. We may assume that $\mathcal{A}$ is realized as operators on a Hilbert space and let

$$x_i = \int \lambda \, dE_i(\lambda) \qquad i = 1, \ldots, n$$

denote the spectral resolutions of the operators $x_1, \ldots, x_n$. Since the n-tuple $\underline{x} = (x_1, \ldots, x_n)$ is abelian, the spectral measures $E_1, \ldots, E_n$ are mutually commuting. We may thus set

$$E(S_1 \times \cdots \times S_n) = E_1(S_1) \cdots E_n(S_n)$$

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for Borel sets $S_1, \ldots, S_n$ in $\mathbb{R}$ and extend E to a spectral measure on $\mathbb{R}^n$ with support in $\underline{I}$. Setting

$$f(\underline{x}) = \int f(\lambda_1, \ldots, \lambda_n)\, dE(\lambda_1, \ldots, \lambda_n)$$

and since f is continuous, we finally realize that $f(\underline{x})$ is an element in $\mathcal{A}$.

2. CONDITIONAL EXPECTATIONS

Let $\mathcal{C}$ be a separable abelian C^*-subalgebra of a C^*-algebra $\mathcal{A}$, and let φ be a positive functional on $\mathcal{A}$ such that $\mathcal{C}$ is contained in the centralizer

$$\mathcal{A}^\varphi = \{y \in \mathcal{A} \mid \varphi(xy) = \varphi(yx) \quad \forall x \in \mathcal{A}\}.$$

The subalgebra is of the form $\mathcal{C} = C_0(S)$ for some locally compact metric space S.

Theorem 2.1. *There exists a positive linear mapping*

$$\Phi \colon M(\mathcal{A}) \to L^\infty(S, \mu_\varphi) \tag{2.1}$$

on the multiplier algebra $M(\mathcal{A})$ such that

$$\Phi(xy) = \Phi(yx) = \Phi(x)y, \qquad x \in M(\mathcal{A}),\, y \in \mathcal{C}$$

almost everywhere, and a finite Radon measure μ_φ on S such that

$$\int_S z(s)\Phi(x)(s)\, d\mu_\varphi(s) = \varphi(zx), \qquad z \in \mathcal{C},\, x \in M(\mathcal{A}).$$

Proof. By the Riesz representation theorem there is a finite Radon measure μ_φ on S such that

$$\varphi(y) = \int_S y(s)\, d\mu_\varphi(s), \qquad y \in \mathcal{C} = C_0(S).$$

For each positive element x in the multiplier algebra $M(\mathcal{A})$ we have

$$0 \le \varphi(yx) = \varphi(y^{1/2}xy^{1/2}) \le \|x\|\varphi(y), \qquad y \in \mathcal{C}_+.$$

The functional $y \to \varphi(yx)$ on $\mathcal{C}$ consequently defines a Radon measure on S which is dominated by a multiple of μ_φ, and it is therefore given by a unique element $\Phi(x)$ in $L^\infty(S, \mu_\varphi)$. By linearization this defines a positive linear mapping defined on the multiplier algebra

$$\Phi \colon M(\mathcal{A}) \to L^\infty(S, \mu_\varphi) \tag{2.2}$$

such that

$$\int_S z(s)\Phi(x)(s)\, d\mu_\varphi(s) = \varphi(zx), \qquad z \in \mathcal{C},\, x \in M(\mathcal{A}).$$

Furthermore, since

$$\int_S z(s)\Phi(yx)(s)\, d\mu_\varphi(s) = \varphi(zyx) = \int_S z(s)y(s)\Phi(x)(s)\, d\mu_\varphi(s)$$

for $x \in M(\mathcal{A})$ and $z, y \in \mathcal{C}$ we derive $\Phi(yx) = y\Phi(x) = \Phi(x)y$ almost everywhere. Since $\mathcal{C}$ is contained in the centralizer $\mathcal{A}^\varphi$ and thus $\varphi(zxy) = \varphi(yzx)$, we similarly obtain $\Phi(xy) = \Phi(x)y$ almost everywhere. □

Note that $\Phi(z)(s) = z(s)$ almost everywhere in S for each $z \in \mathcal{C}$, cf. [6, 4, 5]. With a slight abuse of language we call Φ a conditional expectation even though its range is not a subalgebra of $M(\mathcal{A})$.

3. JENSEN'S INEQUALITY

Following the notation in [5] we consider a separable C^*-algebra $\mathcal{A}$ of operators on a (separable) Hilbert space $\mathfrak{H}$, and a field $(a_t)_{t\in T}$ of operators in the multiplier algebra

$$M(\mathcal{A}) = \{a \in \mathbb{B}(\mathfrak{H}) \mid a\mathcal{A} + \mathcal{A}a \subseteq \mathcal{A}\}$$

defined on a locally compact metric space T equipped with a Radon measure ν. We say that the field $(a_t)_{t\in T}$ is weak*-measurable if the function $t \to \varphi(a_t)$ is ν-measurable on T for each $\varphi \in \mathcal{A}^*$; and we say that the field is continuous if the function $t \to a_t$ is continuous [4].

As noted in [5] the field $(a_t)_{t\in T}$ is weak*-measurable, if and only if for each vector $\xi \in \mathfrak{H}$ the function $t \to a_t\xi$ is weakly (equivalently strongly) measurable. In particular, the composed field $(a_t^* b_t)_{t\in T}$ is weak*-measurable if both $(a_t)_{t\in T}$ and $(b_t)_{t\in T}$ are weak*-measurable fields.

If for a weak*-measurable field $(a_t)_{t\in T}$ the function $t \to |\varphi(a_t)|$ is integrable for every state $\varphi \in S(\mathcal{A})$ and the integrals

$$\int_T |\varphi(a_t)|\, d\nu(t) \le K, \qquad \forall \varphi \in S(\mathcal{A})$$

are uniformly bounded by some constant K, then there is a unique element (a C^*-integral in Pedersen's terminology [8, 2.5.15]) in the multiplier algebra $M(\mathcal{A})$, designated by

$$\int_T a_t\, d\nu(t),$$

such that

$$\varphi\left(\int_T a_t\, d\nu(t)\right) = \int_T \varphi(a_t)\, d\nu(t), \qquad \forall \varphi \in \mathcal{A}^*.$$

We say in this case that the field $(a_t)_{t\in T}$ is integrable. Finally we say that a field $(a_t)_{t\in T}$ is a unital column field [1, 4, 5], if it is weak*-measurable and

$$\int_T a_t^* a_t\, d\nu(t) = 1.$$

We note that a C^*-subalgebra of a separable C^*-algebra is automatically separable.

Theorem 3.1. *Let $\mathcal{C}$ be an abelian C^*-subalgebra of a separable C^*-algebra $\mathcal{A}$, φ be a positive functional on $\mathcal{A}$ such that $\mathcal{C}$ is contained in the centralizer $\mathcal{A}^\varphi$ and let*

$$\Phi\colon M(\mathcal{A}) \to L^\infty(S, \mu_\varphi)$$

be the conditional expectation defined in (2.1). Let furthermore $f : \underline{I} \to \mathbb{R}$ be a continuous convex function of n variables defined on a cube, and let $t \to a_t \in M(\mathcal{A})$ be a unital column field on a locally compact Hausdorff space T with a Radon measure ν. If $t \to \underline{x}_t$ is an essentially bounded, weak-measurable field on T of abelian n-tuples of self-adjoint elements in $\mathcal{A}$ in the domain of f, then*

$$f(\Phi(y_1), \ldots, \Phi(y_n)) \le \Phi\left(\int_T a_t^* f(\underline{x}_t) a_t\, d\nu(t)\right) \tag{3.1}$$

almost everywhere, where the n-tuple $\underline{y}$ in $M(\mathcal{A})$ is defined by setting

$$\underline{y} = (y_1, \ldots, y_n) = \int_T a_t^* \underline{x}_t a_t\, d\nu(t).$$

Proof. The subalgebra $\mathcal{C}$ is as noted above of the form $\mathcal{C} = C_0(S)$ for some locally compact metric space S, and since the C^*-algebra $C_0(\underline{I})$ is separable we may for almost every s in S define a Radon measure μ_s on $\underline{I}$ by setting

$$\mu_s(g) = \int_{\underline{I}} g(\underline{\lambda})\, d\mu_s(\underline{\lambda}) = \Phi\left(\int_T a_t^* g(\underline{x}_t) a_t\, d\mu(t)\right)(s), \qquad g \in C_0(\underline{I}).$$

Since

$$\mu_s(1) = \Phi\left(\int_T a_t^* a_t \, d\mu(t)\right) = \Phi(1) = 1$$

we observe that μ_s is a probability measure. If we put $g_i(\underline{\lambda}) = \lambda_i$ then

$$\int_{\underline{I}} g_i(\underline{\lambda}) \, d\mu_s(\underline{\lambda}) = \Phi\left(\int_T a_t^* x_{it} a_t \, d\mu(t)\right)(s) = \Phi(y_i)(s)$$

for $i = 1, \ldots, n$ and since f is convex we obtain

$$\begin{aligned} f(\Phi(y_1)(s), \ldots, \Phi(y_n)(s)) &= f\left(\int_{\underline{I}} g_1(\underline{\lambda}) \, d\mu_s(\underline{\lambda}), \ldots, \int_{\underline{I}} g_n(\underline{\lambda}) \, d\mu_s(\underline{\lambda})\right) \\ &\le \int_{\underline{I}} f\left(g_1(\underline{\lambda}), \ldots, g_n(\underline{\lambda})\right) \, d\mu_s(\underline{\lambda}) \\ &= \int_{\underline{I}} f(\underline{\lambda}) \, d\mu_s(\underline{\lambda}) \\ &= \Phi\left(\int_T a_t^* f(\underline{x}_t) a_t \, d\mu(t)\right)(s) \end{aligned}$$

for almost all s in S. □

The following corollary is known as "Jensen's inequality for expectation values". It was formulated (for continuous fields) in the reference [3], where a more direct proof is given.

Corollary 3.2. *Let $f : \underline{I} \to \mathbb{R}$ be a continuous convex function of n variables defined on a cube, and let $t \to a_t \in B(H)$ be a unital column field on a locally compact Hausdorff space T with a Radon measure ν. If $t \to \underline{x}_t$ is a bounded weak*-measurable field on T of abelian n-tuples of self-adjoint operators on H in the domain of f, then*

$$f\big((y_1\xi \mid \xi), \ldots, (y_n\xi \mid \xi)\big) \le \left(\int_T a_t^* f(\underline{x}_t) a_t \, d\nu(t)\xi \mid \xi\right) \tag{3.2}$$

for any unit vector $\xi \in H$, where the n-tuple $\underline{y}$ is defined by setting

$$\underline{y} = (y_1, \ldots, y_n) = \int_T a_t^* \underline{x}_t a_t \, d\nu(t).$$

Proof. The statement follows from Theorem 3.1 by choosing φ as the trace and letting $\mathcal{C}$ be the C^*-algebra generated by the orthogonal projection P on the vector ξ. Then $\mathcal{C} = C_0(S)$ where $S = \{0, 1\}$, and an element $z \in \mathcal{C}$ has the representation

$$z = z(0)P + z(1)(1 - P).$$

The measure $d\mu_\varphi$ gives unit weight in each of the two points, and the conditional expectation Φ is given by

$$\Phi(x)(s) = \begin{cases} (x\xi \mid \xi) & s = 0 \\ \mathrm{Tr}(x - Px) & s = 1. \end{cases}$$

Indeed,

$$\begin{aligned} \varphi(zx) &= \mathrm{Tr}\Big(\big(z(0)P + z(1)(1 - P)\big)x\Big) \\ &= z(0)\Phi(x)(0) + z(1)\Phi(x)(0) \\ &= \int_S z(s)\Phi(x)(s) \, ds \end{aligned}$$

as required. The statement follows by evaluating the functions appearing on each side of the inequality (3.1) at the point $s = 0$. □

Remark 3.3. If we choose ν as a probability measure on T, then the trivial field $a_t = 1$ for $t \in T$ is unital and (3.2) takes the form

$$f\left(\left(\int_T x_{1t}\,d\nu(t)\xi \mid \xi\right),\ldots,\left(\int_T x_{nt}\,d\nu(t)\xi \mid \xi\right)\right) \le \left(\int_T f(\underline{x}_t)\,d\nu(t)\xi \mid \xi\right)$$

for bounded weak*-measurable fields of abelian n-tuples $\underline{x}_t = (x_{1t},\ldots,x_{nt})$ of self-adjoint operators in the domain of f and unit vectors ξ. By choosing ν as an atomic measure with one atom we get a version

$$f\big((x_1\xi \mid \xi),\ldots,(x_n\xi \mid \xi)\big) \le \big(f(\underline{x})\xi \mid \xi\big) \tag{3.3}$$

of the Jensen inequality by Mond and Pečarić [7]. By further considering a direct sum

$$\xi = \bigoplus_{j=1}^m \xi_j \quad \text{and} \quad x = (x_1,\ldots,x_n) = \bigoplus_{j=1}^m (x_{1j},\ldots,x_{nj})$$

we obtain the familiar version

$$f\left(\sum_{j=1}^m (x_{1j}\xi_j \mid \xi_j),\ldots,\sum_{j=1}^m (x_{nj}\xi_j \mid \xi_j)\right) \le \sum_{j=1}^m \Big(f(x_{1j},\ldots,x_{nj})\xi_j \mid \xi_j\Big)$$

valid for abelian n-tuples $(x_{1j},\ldots,x_{nj})$, $j = 1,\ldots,m$ of self-adjoint operators in the domain of f and vectors $\xi_1,\ldots,\xi_m$ with $\|\xi_1\|^2 + \cdots + \|\xi_m\|^2 = 1$.

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